ELSEVIER

Discrete Applied Mathematics 116 (2002) 17–36

DISCRETE APPLIED MATHEMATICS

Measures on monotone properties of graphs ☆

Jòzsef Balogh[a], Béla Bollobás[b,c], David Weinreich[d,*]

[a] Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA
[b] Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA
[c] Trinity College, Cambridge CB2 1TQ, England, UK
[d] Department of Mathematics, Gettysburg College, Box 402, Gettysburg, PA 17325-1486, USA

Received 10 July 2000; accepted 22 December 2000

Abstract

Given a monotone property $\mathscr{P}$ of graphs, write $\mathscr{P}^n$ for the set of graphs with vertex set $[n]$ having property $\mathscr{P}$. Building on recent results in the enumeration of graphical properties, we prove numerous results about the structure of graphs in $\mathscr{P}^n$ and the functions $|\mathscr{P}^n|$. We also examine the measure $e_{\mathscr{P}}(n)$, the maximum number of edges in a graph of P^n. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Extremal graph theory; Graph properties; Monotone; Hereditary; Speed; Size.

1. Introduction

A *property* of graphs $\mathscr{P}$ is a (infinite) class of labeled graphs closed under isomorphism, that is, if $H \in \mathscr{P}$ and $G \cong H$ then $G \in \mathscr{P}$. A property $\mathscr{P}$ is called *monotone* if it is closed under taking subgraphs and *hereditary* if it is closed under taking only induced subgraphs. Clearly, each monotone property is hereditary, but the converse is not true. For example, the property of planar graphs is both hereditary and monotone, while the property of perfect graphs is hereditary but not monotone.

A valuable tool to study graph properties is graphical enumeration. Often one cannot directly say how one property is related to another, but counting the number of graphs or examining the types of graphs in a property gives an indication of the information one needs.

There are two reasonable ways to count monotone properties. The first is the *speed* (of growth) of a property, defined as follows. Given a property $\mathscr{P}$, write $\mathscr{P}^n$ for the

☆ Research partially supported by OTKA grant F026049 and NSF Grant DMS 9971788.

* Corresponding author.

E-mail addresses: balogj@msci.memphis.edu (J. Balogh), bollobas@msci.memphis.edu (B. Bollobás), dweinrei@gettysburg.edu (D. Weinreich).

PII: S0166-218X(01)00175-5

n-level of $\mathscr{P}$, the set of graphs in $\mathscr{P}$ with vertex set $[n]=\{1,\ldots,n\}$. Then the speed of $\mathscr{P}$, $|\mathscr{P}^n|$, is the number of graphs in the property on n vertices. For example, if $\mathscr{P}$ is the trivial property containing all graphs, then $|\mathscr{P}^n|=2^{\binom{n}{2}}$. If $\mathscr{P}=\{G\colon G=K_{1,r}\dot\cup\overline{K_s}\}$, then $|\mathscr{P}^n|=\sum_{i=3}^n i\binom{n}{i}+\binom{n}{2}+1=n2^{n-1}\binom{n}{2}-n+1$. An important but trivial fact to note is that if $G\in\mathscr{P}^n$, all graphs isomorphic to G are in $\mathscr{P}^n$, so $|\mathscr{P}^n|$ is at least as large as the number of non-isomorphic labelings of G.

Another useful measure in the case of monotone properties is its *size*. Again, this measure actually considers how the property grows. The size of a monotone property $\mathscr{P}$ at level n is the maximum number of edges in a graph of $\mathscr{P}^n$. That is, the size $e_{\mathscr{P}}(n)=\max\{|E(G)|\colon G\in\mathscr{P}^n\}$. As a monotone property contains every subgraph of each graph achieving this number of edges, the size is a good measure of how large a property is. In fact, this measure played a large part in the development of the field of extremal graph theory, as shall be discussed in Section 7.

Certainly there is some correspondence between the size and the speed of the growth of a property. In particular,

$$2^{e_{\mathscr{P}}(n)}\leqslant|\mathscr{P}^n|\leqslant\binom{\binom{n}{2}}{e_{\mathscr{P}}(n)}.$$

For some properties $\mathscr{P}$, this inequality is asymptotically sharp. For example if $\mathscr{P}=\{G\colon \omega(G)\leqslant k\}$, then $e_{\mathscr{P}}(n)=t_k(n)$ and $|\mathscr{P}^n|=2^{t_k(n)+o(n^2)}$, where the kth Turán number, $t_k(n)=\binom{n}{2}-\sum_{i=0}^{k-1}\binom{\lfloor (n+i)/k\rfloor}{2}\sim(1-1/k)\binom{n}{2}$. However, in other cases the speed and the size can be arbitrarily far apart. For example, if $\mathscr{P}=\{G\colon G=H\dot\cup\overline{K_r}$ for some H with $|V(H)|\leqslant k\}$, $e_{\mathscr{P}}(n)=\binom{k}{2}$ for all n but $\mathscr{P}^n\geqslant\binom{n}{k}\gg 2^{\binom{k}{2}}$. Hence the measures are in fact fundamentally different.

Every monotone property may be described in terms of its *forbidden subgraphs*. Given a collection $\mathscr{H}$ of graphs, **Forb**$\mathscr{H}$ is defined to be the property of all graphs having no subgraph isomorphic to any graph of $\mathscr{H}$. This is clearly a monotone property, and it is also clear that any monotone property has a set of forbidden subgraphs. All early work on measures on properties of graphs was thought of in terms of a set of forbidden subgraphs (see [12,18-21]), but recent works have viewed the measures and properties under consideration more broadly.

In particular, there has been a great deal of work recently on the speed of general hereditary properties of graphs (e.g. [23,5,2]), characterizing the possible speeds and structures that such properties have. In some sense, this gives a broad characterization for monotone properties as well, since all monotone properties are hereditary. However, as the condition of monotonicity is more restrictive than that of being hereditary, one might expect a more restrictive characterization of speeds and structures for monotone properties. This is in fact the case, and we produce results and descriptions which are much more elegant than those possible for hereditary properties.

Previous work on hereditary properties (see [2–5,23]) produced the results consolidated in the theorem below. Note that $b(n)$ is the nth Bell number, and $b(n)=n^{(1+o(1))n}$.

Theorem 1. *Let $\mathscr{P}$ be a hereditary property of graphs. Then, for all sufficiently large n, one of the following holds*:

(i) $|\mathscr{P}^n|$ *is identically zero, one or two.*

(ii) *There is an integer $k > 0$ such that $|\mathscr{P}^n|$ is a polynomial in n of degree k.*

(iii) *There is an integer $k > 0$ such that $|\mathscr{P}^n|$ has exponential order of the form $\sum_{i=0}^{k} p_i(n)i^n$, where p_i is a polynomial in n.*

(iv) *There is an integer $k > 1$ such that $|\mathscr{P}^n| = n^{(1-1/k+o(1))n}$.*

(v) $b(n) \leqslant |\mathscr{P}^n| \leqslant n^{o(n^2)}$.

(vi) *There is an integer $k > 1$ such that $|\mathscr{P}^n| = 2^{(1-1/k+o(1))n^2/2}$.*

Thus, putting it somewhat vaguely, the growth of $|\mathscr{P}^n|$ can be constant, polynomial, exponential, or in one of three factorial ranges.

In the rest of the paper, we use definitions of these terms (as in [2]) which are non-standard, describing the dominant factor of growth rather than the whole function. Keep in mind also that all of our functions are defined as $f\colon \mathbb{N} \to \mathbb{R}$, but our speed descriptions act as if they are defined everywhere. A *constant* function is one which, for sufficiently large n, is constant. A *polynomial* function is one which, for sufficiently large n, is polynomial. Our notation for both polynomial and constant speeds is standard. An *exponential* function is one which, for sufficiently large n, acts like the sum of exponential terms with polynomial coefficients. For $k > 1$, the notation $\Omega(k^n)$ thus has its usual meaning, but we shall write $f(n) = O(k^n)$ if the fastest growing term in the expansion of $f(n)$ has the form $cn^t k^n$ for some c, t. We define $\Theta(k^n)$ similarly. A *factorial* function is one which is at least n^{cn} for some $c > 0$.

Otherwise our notation and terminology is standard and may be found in any graph theory text. In particular, $V(G)$ is the vertex set of G, $E(G)$ is the edge set of G, and $v(G)$, $e(G)$ are their respective cardinalities. We write $H \subseteq G$ if H is a subgraph of G, $H \leqslant G$ if H is an induced subgraph of G, and use $G[X]$ to denote the induced subgraph of G on vertex set $X \subseteq V(G)$. The graph $H \dot{\cup} G$ is the disjoint union of two graphs H and G, and the graph $H \oplus G$ is their *join*, where each vertex of H is adjacent to each vertex of G.

As noted earlier, Theorem 1 applies equally well to monotone properties, and the methods of earlier research (in particular [2]) could be modified to make the appropriate sharpening of the theorem obtained in the current work. However, the nature of monotonicity allows for a more streamlined methodology than in the previous work. In studying hereditary properties, a great deal of machinery is necessary to deal with the difficulties inherent in allowing induced subgraphs. We shall allude to Theorem 1 throughout this paper, but each result is proven independently of any prior work. Throughout the paper, but particularly in the last sections, we shall also explore the other measure $e_{\mathscr{P}}(n)$, its properties, and its relation to the speed. This measure has been studied extensively for properties with very large sizes (i.e. Turán's Theorem) but the full range has not been studied before in detail.

2. Bounded growth

In the case of hereditary properties, Scheinerman and Zito [23] showed that when $|\mathscr{P}^n|$ is bounded, for sufficiently large n, $\mathscr{P}^n=\emptyset$ or $\{K_n\}$ or $\{\overline{K_n}\}$, or $\{K_n,\overline{K_n}\}$. Hence the result that for large enough values of n, $|\mathscr{P}^n|\in\{0,1,2\}$.

Recalling that every monotone property is hereditary, we know that we can get no other speeds if $|\mathscr{P}^n|$ is bounded. However, not every hereditary property is monotone. In particular, if $\mathscr{P}$ is monotone and $\{K_n\colon n\in\mathbb{N}\}\subset\mathscr{P}$, then $\mathscr{P}$ is the trivial property and $|\mathscr{P}^n|=2^{\binom{n}{2}}$. This highlights the point that the speeds of monotone properties will have a different pattern than those of hereditary properties. The following result could be obtained as a direct corollary of that mentioned above. We prove it independently here in order to set the stage for our further work.

Theorem 2. *Let $\mathscr{P}$ be a monotone property. If there is some K such that $|\mathscr{P}^n|\leqslant K$ for all n, then for sufficiently large n, $|\mathscr{P}^n|\in\{0,1\}$ and $\mathscr{P}$ is either the empty property or $\{\overline{K_n}\colon n\in\mathbb{Z}\}$.*

Proof. If, for all n, there is a $G\in\mathscr{P}^n$ with $e(G)>0$, then $\{K_2\dot\cup\overline{K_n}\colon n\in\mathbb{N}\}\subset\mathscr{P}$ by monotonicity. In such a case, $|\mathscr{P}^n|\geqslant\binom{n}{2}$, contradicting the bound on the speed. Hence there exists N such that if $n>N$ and $G\in\mathscr{P}^n$, then $e(G)=0$. So for sufficiently large n, $\mathscr{P}=\emptyset$ or $\{\overline{K_n}\}$. □

In fact, if the growth of $|\mathscr{P}^n|$ is not bounded, the proof implies that $|\mathscr{P}^n|\geqslant\binom{n}{2}+1$, as $\mathscr{P}^n$ contains the empty graph and every graph with a single edge. The property $\mathscr{P}=\{G\colon e(G)\leqslant 1\}$ achieves equality. In fact, this gives the following corollary relating size and speed.

Corollary 3. *Let $\mathscr{P}$ be a monotone property. The speed $|\mathscr{P}^n|$ is bounded if and only if $e_{\mathscr{P}}(n)=0$ for sufficiently large n.*

Another way of looking at monotone properties with bounded speed is that every graph of the property has maximal component order 1. This characterization will be similar to the framework we develop for properties with higher speeds.

3. Polynomial growth

We now understand all properties whose speed is bounded above by a constant. If a property has speed greater than any constant, then its speed is bounded below by a polynomial. What sorts of monotone properties have speeds that are bounded both above and below by a polynomial function? The structure of such properties which are hereditary is well described in [2]. However, some of those structures and speeds do not occur for monotone properties.

We begin our study of this range by proposing a collection of properties exhibiting the proper speeds. We shall then show that these are the only monotone properties possible at this speed.

The work in [2] relied on defining equivalence classes of twins in the graphs of the property. For monotone properties, can we proceed in a simpler way? First, we observe that properties with bounded growth are defined by a (trivial) bound on the number of edges which may appear in the graph. Hence it would be reasonable to consider properties with a constant bound on $e_{\mathscr{P}}(n)$. This is a good approach, but may be strengthened by noting that constraining the maximum number of edges in a graph also constrains the number of vertices that may appear in non-trivial components.

Given a graph G, let G^* be the graph that remains after removing all isolated vertices. Let $v^*(G) = v(G^*)$ and, given a monotone property $\mathscr{P}$, let $v^*(\mathscr{P}) = \limsup_{n\to\infty} \{v^*(G)\colon G \in \mathscr{P}^N, N > n\}$. If $v^*(\mathscr{P}) < \infty$, then every graph in $\mathscr{P}$ consists of the disjoint union of some graph on at most $v^*(\mathscr{P})$ vertices and a collection of isolated vertices. Such a property has polynomial speed.

Theorem 4. *Let $\mathscr{P}$ be a monotone property. If $v^*(\mathscr{P}) = k \leqslant \infty$ and $k > 1$, then*

$$|\mathscr{P}^n| = \sum_{i=0}^{k} a_i \binom{n}{i},$$

where $0 \leqslant a_j \leqslant 2^{\binom{j}{2}}$ is an integer for all j.

Proof. If $v^*(\mathscr{P}) = k \leqslant \infty$, then for sufficiently large n the vertices of each $G \in \mathscr{P}^n$ can be uniquely decomposed into two sets A and B such that $|A| \leqslant k$ and $\deg(v) = 0$ if and only if $v \in B$. We will call $H = G[A]$ the *head* of G and $G[B]$ the *body* of G.

Let $\mathscr{H} = \{H\colon$ for all n there is a graph $G \in \mathscr{P}^n$ with head $H\}$. Since $k \geqslant 2$, $\mathscr{H}$ is not empty. As there are only finitely many graphs on $\leqslant k$ vertices, $\mathscr{H}$ is finite. Finally, $\mathscr{H}$ is a monotone property and contains graphs on every number of vertices up to and including k.

Let n be sufficiently large that the graphs in $\mathscr{H}$ are the only graphs that appear as heads of graphs in $\mathscr{P}^n$. For each $H \in \mathscr{H}$, let $h(H)$ be the number of automorphisms of H. Then there are $(v(H)!/h(H))\binom{n}{v(H)}$ graphs in $\mathscr{P}$ with head H. Summing over all $H \in \mathscr{H}$, we get the result.

The bounds on a_j come from the number of labeled graphs on j vertices, and it must be an integer since $|\mathscr{P}^n|$ is an integer-valued function at integers. □

It is easy to see that these are in fact the only monotone properties with a polynomial bound on their speeds.

Theorem 5. *Let $\mathscr{P}$ be a monotone property. If $v^*(\mathscr{P}) = \infty$, then*

$$|\mathscr{P}^n| \geqslant (1 + o(1))2^{n/2}.$$

Proof. If $v^*(\mathscr{P}) = \infty$, then there are arbitrarily large graphs in $\mathscr{P}$ with no isolated vertices. A graph on n vertices that has no isolated vertices has at least $n/2$ edges. Hence $e_{\mathscr{P}}(n) \geqslant n/2$ and as $|\mathscr{P}^n| \geqslant 2^{e_{\mathscr{P}}(n)}$, we have our result. □

Thus the following assertion completely characterizes graphs of polynomial growth.

Corollary 6. *A monotone property* $\mathscr{P}$ *has polynomial speed if and only if* $v^*(\mathscr{P})$ *is finite.*

Noting that $v^*(\mathscr{P})$ restricts not only the order of the non-trivial part of each graph in $\mathscr{P}$ but also the number of edges in each graph, we obtain a corollary which strongly relates size and speed in this range.

Corollary 7. *A monotone property* $\mathscr{P}$ *has polynomial speed if and only if* $e_{\mathscr{P}}(n)$ *is bounded.*

The actual polynomials that occur in the polynomial range are restricted as well, as described in the following corollary to Theorem 4.

Corollary 8. *Let* $L_k = \{K_{1,k-1} \dot\cup \overline{K_n} : n \in \mathbb{Z}\}$ *and* $U_k = \{A \dot\cup \overline{K_n} : |V(A)| \leqslant k$ *and* $n \in \mathbb{Z}\}$. *If* $|\mathscr{P}^n| = \Theta(n^k)$, *then, for sufficiently large values of k,* $|L_k^n| \leqslant |\mathscr{P}^n| \leqslant |U_k^n|$, *where* $|L_k^n| = \sum_{i=3}^{k}(i)\binom{n}{i} + \binom{n}{2} + 1$ *and* $|U_k^n| \leqslant \binom{n}{k}2^{\binom{k}{2}}$.

Proof. We first consider the smallest property in the collection of properties with speeds following $\Theta(n^k)$. From the proof of Theorem 4, it is clear that we would like to maximize the number of automorphisms of the graphs in $\mathscr{A}$. We would also like to minimize the number of graphs on i vertices in $\mathscr{A}$ for each $i \in [k]$, given that $\mathscr{A}$ is monotone. Clearly the family of stars on at most k vertices achieves both of these, and no other family does so.

The upper bound is trivial. □

Our main result of this section, Theorem 4, provides a simple description of monotone properties with polynomial speeds: polynomial properties are precisely those in whose graphs all but a finite number of vertices are isolates. Put another way to be consistent with a possible characterization of bounded speed properties, we can say polynomial properties are those in whose graphs all but a finite number of components have order 1. We shall obtain a similar characterization for properties at the next level of speed, where the speed is at least exponential.

4. Exponential growth

Theorem 5 tells us that if there is no polynomial bound on the speed of a monotone property, then its speed is at least exponential. What sorts of properties are bounded both above and below by an exponential function? Again, such hereditary properties are described in detail in [2], but the characterization is quite complicated. We seek a simple description of exponential monotone properties.

Thus far, our main tool has been to guarantee a large number of isolated vertices. The properties we now wish to consider, however, will have graphs without a large number of isolates. In order to consider something similar, we turn instead to independent vertices. Given an independent set I in a graph G, the removal of $G-I$ from G leaves only isolates, so perhaps considering an independent set would be a good approach.

Given a graph G and a monotone property $\mathscr{P}$, recall that the covering number $\beta(G) = n - \alpha(G)$, where $\alpha(G)$ is the independence number of G. Let $\beta_n(\mathscr{P}) = \max_{G\in\mathscr{P}^n}\beta(G)$ and $\beta(\mathscr{P}) = \limsup_{n\to\infty}\{\beta_n(\mathscr{P})\}$. A simple counting argument shows that if $\beta(\mathscr{P})$ is finite, then the speed of $\mathscr{P}$ is bounded by an exponential function.

Lemma 9. *Let $\mathscr{P}$ be a monotone property. If $\beta(\mathscr{P}) = k < \infty$, then $|\mathscr{P}^n| = O((2^k)^n)$.*

Proof. For sufficiently large n, every graph $G \in \mathscr{P}$ has a set of $k = \beta(\mathscr{P})$ vertices whose removal yields an empty graph. Every such graph is a subgraph of $K_k \oplus \overline{K_{n-k}}$. The largest such property therefore consists of $K_k \oplus \overline{K_{n-k}}$ for each n and all of its subgraphs. The graph $K_k \oplus \overline{K_{n-k}}$ has fewer than $\binom{n}{k}(2^n)^k$ labeled subgraphs on n vertices. Hence $|\mathscr{P}^n| < \binom{n}{k}(2^n)^k = O((2^k)^n)$. □

In fact, every monotone property with exponential bound on its speed has finite $\beta(\mathscr{P})$.

Lemma 10. *Let $\mathscr{P}$ be a monotone property. If $\beta(\mathscr{P}) = \infty$, then $|\mathscr{P}^n| \geqslant n^{(1+o(1))n/2}$.*

Proof. For every t there is a graph in $\mathscr{P}$ which, upon removing any $2t$ vertices, is not an empty graph. But then, by monotonicity, every property contains the graph tK_2, a matching of arbitrary size (remove two adjacent vertices at a time, and there will remain an edge). There are $(2t)!/t!2^{-t} > t!$ ways to label the graph tK_2, so this implies $|\mathscr{P}^{2t}| > t! \geqslant t^{(1+o(1))t}$. Thus a monotone property containing a matching of order n has speed $|\mathscr{P}^n| \geqslant n^{(1+o(1))n/2}$. □

Hence we have the following corollary.

Corollary 11. *Let $\mathscr{P}$ be a monotone property. The speed $|\mathscr{P}^n| = O(K^n)$ if and only if $\beta(\mathscr{P})$ is finite.*

We would like to know exactly what types of functions are achievable for monotone properties with exponential bounds. We can show that the speeds that occur in this range are precisely exponential functions with polynomial coefficients.

In order to do so, we shall again split the graph into two parts, a "head" with fewer constraints on its structure and a "body" whose structure is tightly constrained. We then use this partition to control the structure of the graph as a whole. Note that although the definitions given in the proof below for head and body are different than in the previous section, using the new definitions for a property with polynomial speed would still yield the polynomial head and body as described.

Theorem 12. *Let $\mathscr{P}$ be a monotone property. Suppose there is a $K < \infty$ such that $|\mathscr{P}^n| = O(K^n)$. Then there is some $k \leqslant K$ such that $|\mathscr{P}^n| = \Theta(k^n)$ and further $|\mathscr{P}^n| = \sum_{i=0}^{k} p_i(n)i^n$, for some collection of polynomials $\{p_i(n)\}$.*

Proof. Lemmas 9 and 10 imply that $\beta(\mathscr{P}) = K' \leqslant \log_2 K$. Hence for sufficiently large n each graph $G \in \mathscr{P}^n$ can be partitioned into a maximal independent set (which we shall call a "body" of G) and "head" of at most K' vertices. The vertices of a head induce a partition of its corresponding body into at most $2^{K'}$ classes according to the $2^{K'}$ possible neighborhoods each vertex of the body may have.

For each graph $G \in \mathscr{P}$, choose a head $H(G)$ that partitions its body $B(G)$ into as few classes as possible. Let $k(G)$ be this number of classes. Let $k_n = \max_{G \in \mathscr{P}^n} k(G)$ and $k = \limsup_{n\to\infty} k_n$. We claim $|\mathscr{P}^n| = \Theta(k^n)$.

Given a graph G and a head $H(G)$, let $B_1, \ldots, B_k$ be the induced partition of $B(G)$. For each vertex v of $H(G)$, label v according to its adjacencies to the parts of the partition. That is, label v with a vector $\{x_1, \ldots, x_k\}$ such that $x_i = 1$ if and only if v is adjacent to the vertices of B_i, else $x_i = 0$. Call this labeled graph $H^*(G)$ the "augmented head" of G. This is well defined up to a permutation of the coordinates of the vectors. Given an augmented head graph H^*, let $k(H^*)$ be the number of classes distinguished by the labeling. Note that $k(G) = k(H^*(G))$ and that G can be reconstructed from H^* by determining how many vertices are in each of the distinguished classes.

Let $\mathscr{H} = \{H^*\colon$ for all n there exists $G \in \mathscr{P}^n$ with $H^* = H^*(G)\}$. There are easily fewer than $(2^k)^{K'} 2^{\binom{K'}{2}}$ possible augmented head graphs in $\mathscr{H}$, the first term reflecting the possible vector labelings and the last reflecting the possible head graphs.

For sufficiently large n, the family $\mathscr{H}$ describes $\mathscr{P}^n$, since there are a finite number of augmented head graphs. For each $H^* \in \mathscr{H}$, there are $\binom{n}{|H^*|} k(H^*)^{n-|H^*|}$ graphs in $\mathscr{P}^n$ with that augmented head. Hence, for sufficiently large n,

$$|\mathscr{P}^n| = \sum_{H^* \in \mathscr{H}} \binom{n}{|H^*|} k(H^*)^{n-|H^*|} \frac{|H^*|!}{h(H^*)}, \tag{1}$$

where $h(H^*)$ is again the number of label-respecting automorphisms of H^*. Approximating the various terms of (1) and grouping them according to the number of vertices in H^*, we obtain the bound

$$|\mathscr{P}^n| \leqslant \sum_{i=\log_2 k}^{K'} \binom{n}{i} 2^{\binom{i}{2}} (2^k)^i k^{n-i} + p(n) = O(k^n),$$

where $p(n)$ is a polynomial in n.

Also, since k is minimal, for all n there is a $G \in \mathscr{P}^n$ such that $H^*(G)$ partitions $B(G)$ into k classes. This $H^*(G)$ has at least $\log_2(k)$ vertices and at most K' vertices, so

$$|\mathscr{P}^n| \geqslant \binom{n}{\log_2(k)} k^{n-K'} = \Omega(k^n).$$

Finally, it is not hard to see that (1) will have the desired form when expanded. □

The leading base in the exponential speed function is based on the partitions a head may induce on the body. Hence for polynomial properties, where no vertex of the head is adjacent to the body, the body is always a single partition and the leading base is

1. Let us make this more clear by providing another description of the structure of monotone properties with exponential speed.

But first a definition: given a graph G, a collection of disjoint vertex sets $\{V_1,\ldots,V_s\}$ *is distinguished* by a collection of vertices $\{v_1,\ldots,v_t\}$ if for each i,j either $(v,v_i)\in E(G)$ for all $v\in V_j$ or $(v,v_i)\notin E(G)$ for all $v\in V_j$ and if $v\in V_i$, $w\in V_j$, $i\neq j$ means v and w have different neighborhoods in $\{v_1,\ldots,v_t\}$. If $\{V_1,\ldots,V_s\}$ is maximal in this respect, then $\{v_1,\ldots,v_t\}$ *distinguishes* $\{V_1,\ldots,V_s\}$.

An infinite graph is an *exponential graph with parameter k* if it has a decomposition into three parts A, B, and C (as shown in Fig. 1) such that

- $|V(G)\setminus B|$ is finite,
- A is not empty,
- the set B, the *body*, is independent in G,
- the set A, the *head*, distinguishes infinite sets $B_1,\ldots,B_k$ in B such that if B_i has neighborhood $A_i\subset A$, then for each $A'\subset A_i$, there is some distinguished set B_j with neighborhood A',
- the set C, the *trash*, has no edges to B.
- there is no restriction on $G[A]$, $G[C]$, or $G[A,C]$.

Note that the head as defined in the proof of Theorem 12 consists of the head together with the trash of an exponential graph.

If G is an exponential graph let $\mathscr{P}_G$ be the *monotone closure* of G, that is, the property consisting of all finite subgraphs of G. Clearly $\mathscr{P}_G$ is a monotone property and $|\mathscr{P}_G^n|=\Theta(k^n)$. In fact, if $|\mathscr{P}^n|=\Theta(k^n)$, then the proof of Theorem 12 tells us that, with the possible exception of a finite number of levels, $\mathscr{P}$ is the monotone closure of a collection of exponential graphs, each of which has parameter at most k and at least one of which has parameter k. We say that a property which is the monotone closure of exponential graphs has *exponential structure*.

If $k=1$, we obtain an infinite graph in which the infinite, independent class is isolated from the rest of the graph. The property based on this graph is either polynomial or has constant speed.

This description of exponential structure for monotone properties is very similar to that given for hereditary properties in [2], but much more restrictive.

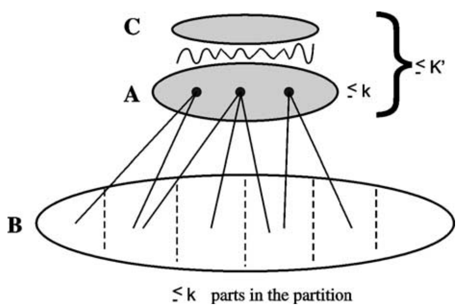

Fig. 1. The form of graphs in exponential properties.

Note that a graph of exponential structure has a finite number of non-trivial components. Let h be the maximal order of a graph in $\mathscr{H}$. Then if $|\mathscr{P}^n| = \Theta(k^n)$, every $G \in \mathscr{P}$ has $k + (h-k)/2$ components, since the head (as defined above) can have at most k vertices and give rise to at most k components, while the trash can have at most $(h-k)$ vertices and thus have at most half that many non-trivial components. This is a characteristic that clearly distinguishes exponential properties from superexponential properties, as shown in Lemma 10.

Before we utilize these facts to explore the next range of speeds, let us again note the relationship between speed and size for properties of exponential speed. This is a direct result of the restriction on structure noted above and in Theorem 12.

Corollary 13. *Let $\mathscr{P}$ be a monotone property. If the speed of $\mathscr{P}$ is exponential in n, then the size of $\mathscr{P}$ is linear in n.*

Note that this goes only in one direction, unlike Corollary 7 for the size of polynomial properties. To see why this is true, consider the monotone property $\mathscr{P}$ of being a path forest. Clearly $e_{\mathscr{P}}(n) = n-1$, but $|\mathscr{P}^n| > (1/2)n!$, the number of labelings of a path on n vertices. This is a factorial, not an exponential function.

Thus we see that for exponential properties the number of edges is not the critical factor: what is critical is that we can cover the edges by a bounded number of vertices. Once we can no longer do that, we are ensured of a property at the next level of speed.

5. Factorial growth

We know from Lemma 10 that a lower bound on the next level of speed is $n^{(1+o(1))n/2}$. What upper bound should we consider for the next range of speed? In [2], it was shown that if a hereditary property $\mathscr{P}$ has superexponential speed, either there exists a k such that $|\mathscr{P}^n| = n^{(1-1/k+o(1))n}$ or $|\mathscr{P}^n| \geqslant n^n$. Hence we should consider functions of the first form for our next candidate properties.

In the previous sections, we have developed structural constraints for our properties based on finding a nice structure that guarantees the desired count, then removing vertices from our graphs so that the desired structure appears. Here we shall do the same.

A basic property developed in [2] that has speed $n^{(1-1/k+o(1))n}$ is the property of being an induced subgraph of an infinite collection of disjoint k-cliques. This is one of the smallest hereditary properties with the desired speed, but would be a rather large property if considered as a monotone property. However, its speed is still in the desired range. Let $\mathscr{P}_{k\text{-}cl}$ be the property of being a subgraph of $\dot{\bigcup}_\infty K_k$. Then

$$\binom{n}{k,\ldots,k}\frac{1}{(n/k)!} \leqslant |\mathscr{P}^n_{k\text{-}cl}| \leqslant \binom{n}{k,\ldots,k}\frac{1}{(n/k)!}\left(2^{\binom{k}{2}}\right)^{n/k}.$$

Both upper and lower bounds are $n^{(1-1/k+o(1))n}$, hence we have our result.

Thus $\mathscr{P}_{k\text{-}cl}$ is a property with the desired speed, and many subproperties of $\mathscr{P}_{k\text{-}cl}$ also have this speed. As the calculation above implies, what is important is that the graphs of the property have arbitrarily many components of order k. However, these are clearly not the only monotone properties with this speed. Let us at this point at least describe a collection of properties inspired by this formulation and by our previous work with removing vertices that do not fit in our scheme. Once again we shall use the term "head" in a new way to fit our current needs, but the new meaning could be seen to subsume the previous meaning.

Theorem 14. *Let $\mathscr{P}$ be a monotone property. Suppose there exist k and c such that for all $G \in \mathscr{P}$ the removal of some set of c vertices from G leaves a graph with maximum component order k. Then $|\mathscr{P}^n| \leqslant n^{(1-1/k+o(1))n}$. Furthermore, if k is minimal, then we have equality.*

Proof. Given a graph $G \in \mathscr{P}$, let H be a set of c vertices such that $G \setminus H$ leaves a graph with maximum component order k. We shall call H a "*head*" of G and the components $B_1, \dots, B_s$ of $G \setminus H$ shall be called "*pseudocomponents*" of G. There are fewer than $2^{\binom{k}{2}}$ graphs which may appear as pseudocomponents of graphs in $\mathscr{P}$, and each of these may be related to the head in at most $(2^c)^k$ different ways. Hence there are a bounded number of structures that occur in the pseudocomponents. Let $A_1, \dots, A_l$ be the possible structures.

In order to simplify notation below, we suppose below that $k \mid n$, but the calculations go through similarly if $k \nmid n$ as well.

Given $G \in \mathscr{P}^n$, decompose it as above. Let $b_1, \dots, b_n$ be the orders of the pseudocomponents (some b_i may be 0) and a_i be the multiplicity of structure A_i as a pseudocomponent. Then there are $\binom{n}{c}\binom{n-c}{b_1,\dots,b_n}\frac{1}{a_1!\cdots a_l!}$ ways to pick labels for the parts of the decomposition. Allowing for the different graphs and structures that occur, we obtain the bound

$$|\mathscr{P}^n| \leqslant \binom{n}{c} 2^{\binom{c}{2}} \sum_{\substack{b_i \leqslant k \\ \sum b_i = n-c}} \binom{n-c}{b_1,\dots,b_n} \prod_{i=1}^{n} 2^{\binom{b_i}{2}} (2^c)^{b_i} \frac{1}{a_1! \cdots a_l!}.$$

Noting that

$$a_1! \cdots a_l! \geqslant \left(\left(\left(\sum a_i\right) \Big/ l\right)!\right)^l \geqslant ((n/kl)!)^l \geqslant ((n/kle)^{n/kl})^l = n^{n/k}(1/kle)^{n/k}$$

and replacing each b_i with its upper bound k, we get

$$|\mathscr{P}^n| \leqslant \binom{n}{c} 2^{\binom{c}{2}} \sum_{\substack{b_i \leqslant k \\ \sum b_i = n-c}} \binom{n-c}{b_1,\dots,b_n} (2^{\binom{k}{2}}(2^c)^k)^n \frac{(kle)^{n/k}}{n^{n/k}}$$

$$\leqslant n^c C^n \sum_{\substack{b_i \leqslant k \\ \sum b_i = n-c}} \binom{n-c}{b_1,\ldots,b_n} \frac{1}{n^{n/k}}.$$

As the multinomial coefficient above is maximized when the b_i are equal, we get

$$|\mathscr{P}^n| \leqslant C^n \binom{n}{k,\ldots,k} \frac{1}{n^{n/k}} = n^{(1-1/k+o(1))n},$$

giving the desired upper bound.

To obtain the lower bound, we choose k minimal. In particular, let k be minimal such that there is a c such that for all $G \in \mathscr{P}$ there exists $T \subset V(G)$ with $|T| \leqslant c$ such that $G - T$ has maximal component order k. Note that k is the largest number such that there are graphs in $\mathscr{P}$ with arbitrarily many components of order k. In fact, since there are a finite number $(2^{\binom{k}{2}})$ graphs on k vertices, and there are graphs in $\mathscr{P}$ with arbitrarily many components of order k, there must be graphs in $\mathscr{P}$ that have as components arbitrarily many copies of a particular graph, say L_k, on k vertices. Since $\mathscr{P}$ is monotone, this means that $rL_k \in \mathscr{P}$ for every r. Let $G = (n/k)L_k$. Again we assume that $k \mid n$, but the calculations are similar in other cases as well. We can label G in at least $\binom{n}{k,\ldots,k}1/(n/k)!$ ways, with $1/(n/k)!$ appearing because the components are isomorphic. Hence

$$|\mathscr{P}^n| \geqslant \binom{n}{k,\ldots,k} \frac{1}{(n/k)!} = \frac{n!}{(k!)^{n/k}(n/k)!} = n^{(1-1/k+o(1))n}.$$

Note that we may use the same k for both the upper and lower bounds, yielding the desired result. □

Recall the formulation of exponential properties: every graph has a bounded number of vertices whose removal leaves components of order 1. The formulation in Theorem 14 is similar: every graph has a bounded number of vertices whose removal leaves components of order k. In fact, these are the only monotone properties with speed $n^{(1-1/k+o(1))n}$. If this were not the case, the speed would jump to the next highest level.

Theorem 15. *Let $\mathscr{P}$ be a monotone property. If $|\mathscr{P}^n| \leqslant n^{(1-1/k+o(1))n}$, then there exists c such that every $G \in \mathscr{P}$ has a collection of c vertices whose removal leaves a graph every component of which has order at most k.*

Proof. Suppose not. Then for all N there is a $G \in \mathscr{P}$ such that for all $W \subset V(G)$ with $|W| \leqslant N$, the graph $G - W$ has a component of order $k+1$. If we remove the vertices of some such component and another $N-(k+1)$ vertices, we may obtain two vertex disjoint connected subgraphs of G, each of order $k+1$. Continuing in this way, we can see that G must contain at least $\lfloor N/(k+1) \rfloor$ vertex disjoint connected subgraphs on $k+1$ vertices. Hence $\mathscr{P}$ contains graphs with arbitrarily many components of order $k+1$ and Theorem 14 implies that $|\mathscr{P}^n| \geqslant n^{(1-1/(k+1)+o(1))n}$. □

Thus we have described all monotone properties with speeds less than $n^{(1+o(1))n}$. The structure that we have placed on the properties forms a clear hierarchy, and the speeds are all well-behaved functions. Unfortunately, this does not continue into the next range.

6. Superfactorial growth

In [3], it was shown that there is a monotone property which infinitely often has speed n^{cn} (for some $c > 1$) and infinitely often has speed $2^{n^{2-1/c}}$. Hence we know that if $\mathscr{P}$ is a monotone property with $|\mathscr{P}^n| = n^{(c+o(1))n}$, we can have no further characterization than that given in Theorem 14. In fact, this is quite a lovely description of properties with speeds less than $n^{(1+o(1))n}$, but if $c \geqslant 1$ we can hope for no theorem giving speeds and structures. For details of why, see [3]. However, can we get a characterization for speeds above $2^{n^{2-1/c}}$?

Recent work of Bollobás and Thomason [6,7] gave a partial characterization for both monotone and hereditary properties.

Theorem 16. *If $\mathscr{P}$ is a monotone or hereditary property and there exists a constant c such that $|\mathscr{P}^n| \geqslant 2^{cn^2}$ infinitely often, then $|\mathscr{P}^n| = 2^{(1-1/k+o(1))n^2/2}$ for some integer k.*

To obtain this result, they define a coloring number for a set of graphs as follows: Given a collection of $\mathscr{H}$ of graphs, we say that $\mathscr{H}$ can be r colored if there is an integer $s \leqslant r$ and a graph $G \in \mathrm{Forb}(\mathscr{H})$, such that $V(G)$ can be partitioned into r sets, s of them inducing cliques and $r - s$ inducing empty graphs. The minimal such r is the coloring number $r(\mathscr{H})$ of the set of the graphs. Their result says precisely that the speed of the corresponding property is $2^{(1-1/r+o(1))n^2/2}$.

We wish to examine monotone properties which have speeds which fall in the gap between $2^{n^{2-c}}$ and $2^{(1/2+o(1))n^2/2}$. That is, we wish to improve the result in Theorem 16 by considering the case when $r = 1$. Our results in the previous sections do precisely that, but we would like to show that there is in fact a gap between the two speeds mentioned above; no property has speed both above $2^{n^{2-c}}$ and below $2^{(1/2+o(1))n^2/2}$. The following theorem from [3] does just that.

Theorem 17. *Let $\mathscr{P}$ be a monotone property. If $|\mathscr{P}^n| = 2^{o(n^2)}$, then there is a $t \geqslant 1$ such that $|\mathscr{P}^n| \leqslant 2^{n^{2-1/t+o(1)}}$.*

Furthermore, the upper bound given in this theorem is nearly sharp for some properties. If $\mathscr{P} = \mathrm{Forb}(\{K_{t,t}, K_3\})$, then $|\mathscr{P}^n| \geqslant 2^{n^{2-2/t}}$. Since this property is of the type described, we cannot hope for an improvement.

However, Theorem 17 does provide the following slight improvement of Theorem 16 for monotone properties.

Corollary 18. *Let $\mathscr{P}$ be a monotone property. Suppose $\omega(n)=n^{o(1)}$. If $|\mathscr{P}^n| \geqslant 2^{n^2/w(n)}$ infinitely often, then $|\mathscr{P}^n| = 2^{(1-1/k+o(1))n^2/2}$ for some integer k.*

Note that we can obtain no similar improvement for hereditary properties, as Theorem 17 does not hold for hereditary properties.

7. The size of a monotone property

We now turn our attention to the size of a property, that is to the function $e_{\mathscr{P}}(n)$. This is really where the study of monotone properties originated. As with speeds of properties, the largest values are relatively well understood. Unlike speeds, however, the largest sizes have been well understood since almost the beginning, when Erdős [10,11] and Simonovits [22] (see also [13–16]) presented their work in extending the Erdős–Stone Theorem.

In particular, the following was known, in slightly different forms.

Theorem 19. *If $\mathscr{P}$ is a monotone property, then there is an $r \in \mathbb{N}$ such that*

$$e_{\mathscr{P}}(n) = (1 - 1/r)\binom{n}{2} + O(n^{2-\varepsilon})$$

for some ε.

Note especially the lower order term in this statement. While the theorem does seem to echo the result of Bollobás and Thomason (Theorem 16), neither implies the other. However, this does cover all of the largest sizes, in a very strong sense we shall address towards the end of this section.

For now we turn our attention to the sizes of properties that are contained in the asymptotic term of Theorem 19, starting with the very smallest sizes possible. As in our study of the function $|\mathscr{P}^n|$, we shall consider several cases according to the magnitude of the function. Our goal will be to describe the types of sizes that may occur for monotone properties, as well as to discuss the relationship of size and speed. Our first result, a slightly more detailed form of Corollary 7, follows from Theorem 4.

Theorem 20. *If $e_{\mathscr{P}}(n)$ is bounded for a property $\mathscr{P}$, then $\lim e_{\mathscr{P}}(n) = k$ for some integer k. The speed of this property is a polynomial with degree at most $2k$.*

From now on, we shall suppose that $e_P(n)$ is unbounded. Since $\mathscr{P}$ is monotone, for every t there is a $G \in \mathscr{P}$ with exactly t edges. As t edges can cover at most $2t$ vertices, for every t there is an $H \in \mathscr{P}$ with $|V(H)| \leqslant 2t$ and $|E(G)| = t$. This shows that $e_{\mathscr{P}}(n) \geqslant \lfloor n/2 \rfloor$.

This inequality is sharp, as can be seen, for example, in the property $\mathscr{P}=\{G\colon \Delta(G) \leqslant 1\}$. In fact one can prove considerably more.

Theorem 21. *If* $\limsup e_{\mathscr{P}}(n)/n < 1$, *then there are integers* $a \leqslant b$ *and* k *such that* $e_{\mathscr{P}}(n) = \lfloor (1-1/k)(n-a) \rfloor + b$ *for large enough* n.

Proof. If there are graphs in a property $\mathscr{P}$ containing arbitrarily large components, then there are arbitrarily large connected graphs in $\mathscr{P}$ and $e_{\mathscr{P}}(n) \geqslant n-1$ which implies $\limsup e_{\mathscr{P}}(n)/n \geqslant 1$. Hence if $\limsup e_{\mathscr{P}}(n)/n < 1$, there is an l such that no graph in $\mathscr{P}$ has a component of order greater than l. Let $k \leqslant l$ be the maximal integer such that there are graphs in $\mathscr{P}$ with arbitrarily many components with order k. This gives $e_{\mathscr{P}}(n) \geqslant \lfloor (1-1/k)n \rfloor$. As the order of the components is bounded, the large graphs in $\mathscr{P}$ have many components.

Note that we have equality above if and only if all of the components are trees of order k. We can obtain a strict inequality above from two sources: if we have components with order greater than k (but at most l) and if we have components that are not trees. We call either of these types of components "bad". How many edges can we have in "bad" components? There are only boundedly many (by a constant, say c, depending on $\mathscr{P}$) components of order greater than k and each has order less than l, so there are fewer than $c\binom{l}{2}$ edges arising from large components. The number of edges from non-tree components is bounded as well. If there are graphs in $\mathscr{P}$ with unboundedly many non-tree components, then for all n there is an n-graph in $\mathscr{P}$ in which all but at most one component is not a tree. This gives $e_{\mathscr{P}}(n) \geqslant n-1$. But since it is almost always true that $e_{\mathscr{P}}(n) < n-1$, the number of non-tree components in the graphs of $\mathscr{P}$ is bounded. Hence the number of "extra" edges due to non-tree components is bounded as well.

Thus there exist $b \geqslant a \geqslant 0$ depending only on $\mathscr{P}$ such that each graph in $\mathscr{P}$ has at most a vertices in "bad" components and b edges in components with more than $k-1$ edges. Then, for sufficiently large n, $e_{\mathscr{P}}(n) \leqslant \lfloor (1-1/k)(n-a) \rfloor + b$. As there are only a finite number of choices for the "bad" components (they have order $\leqslant l$), for sufficiently large n there exist choices of a and b to obtain equality. □

Given a property $\mathscr{P}$ with $e_{\mathscr{P}}(n) = \lfloor (1-1/k)(n-a) \rfloor + b$, the proof of Theorem 21 tells us the structure of a maximal size graph in $\mathscr{P}^n$. In particular, each graph has all but a bounded number of vertices in components that are trees with order at most k.

Even more is implied by the proof. If the size of $\mathscr{P}$ is above the range given in Theorem 21, that is, if $\liminf e_{\mathscr{P}}/n \geqslant 1$, then one of the following holds.

1. For each n, there is a maximal graph $G \in \mathscr{P}^n$ with components of bounded order such that at most one component is a tree.
2. The order of the components of the graphs in $\mathscr{P}$ is unbounded. Hence, for each n, $\mathscr{P}$ contains a connected graph on n vertices.

In either case, $e_{\mathscr{P}}(n) \geqslant n-1$. Thus we have the following result.

Corollary 22. *If* $\limsup e_{\mathscr{P}}(n)/n \geqslant 1$, *then* $e_{\mathscr{P}}(n) \geqslant n-1$.

This result is tight, since when $\mathscr{P} = \{G \text{ is acyclic}\}$ equality holds. However, after this the size of a property does not behave as nicely. The authors have shown in [3] that there are properties $\mathscr{P}$ for which $e_{\mathscr{P}}(n)$ oscillates between significantly different values when $e_{\mathscr{P}}(n)/n > 1$. The proofs and theorems in that work imply directly this result by presenting specific properties with oscillating speeds that also happen to have similarly oscillating sizes. In particular, the following is shown about the oscillation of sizes. Similar results are known about the oscillation of speeds.

Theorem 23 (Balogh et al. [3]). *For any $c, d > 1$ and any $\varepsilon > 0$, there is a monotone property $\mathscr{P}$ with $e_{\mathscr{P}}(n) = cn$ infinitely often and $e_{\mathscr{P}}(n) = dn$ infinitely often. Also, there is a property $\mathscr{R}$ with $e_{\mathscr{R}}(n) = cn$ infinitely often and $e_{\mathscr{R}}(n) = n^{2-1/c-\varepsilon}$ infinitely often.*

Here we present a different property and proof than that in [3], which in some sense allows us to show more about properties with oscillating size. The following theorem shows that even if we keep the size of the property only slightly above n we can have oscillation of size.

Theorem 24. *For any sequence* with $\varepsilon_n > 0, \varepsilon_n \to 0$, *there is a property $\mathscr{P}$ with $e_{\mathscr{P}}(n) \geqslant n^{1+\varepsilon_n}$ infinitely often, and $e_{\mathscr{P}}(n) = n - 1$ infinitely often.*

Proof. The idea of constructing this property $\mathscr{P}$ comes from the fact that there are "dense" graphs with large girth. We can construct a sequence $\{v_i\}$ (depending on ε_n), such that $\mathscr{P}^{v_i} = \{G\colon G \text{ is acyclic, with } |V(G)| = v_i\}$ and if $v_i < n < v_{i+1}$ then $\mathscr{P}^n = \{G\colon G$ contains no C_j for any $j\}$. It is clear that for any sequence $\{v_i\}$, the size $e_{\mathscr{P}}(v_i) = v_i - 1$ for all i. We shall prove that we can construct a sequence such that $e_{\mathscr{P}}(n) \geqslant n^{1+\varepsilon_n}$ for $n = v_{i+1} - 1$.

Suppose that we have constructed the sequence $(v_i)_{i=1}^{s-1}$ so that it satisfies the required size function up to $n = v_{s-1}$ and we wish to choose a value for v_s. Let $l = v_{s-1}$ and choose some $c < 1/(l-1)$. Consider $\mathscr{G}_{n,p}$ with $p = n^{-1+c}$. Let G be a graph in $\mathscr{G}_{n,p}$ and X be the number of cycles C_i of length $i \leqslant l$ in G. $\mathbb{E}(|E(G)|) = 1/2n^{1+c}$ and $\mathbb{E}(X) = \sum_{i=3}^{l} \binom{n}{i}(i!/2)p^i \leqslant l/2n^{lc} = o(n^{1+c})$. In particular $\Pr(X \geqslant n^{1+c}/4) = o(1)$. So there is a specific G with at least $n^{1+c}/2$ edges and at most $o(n^{1+c})$ cycles with length at most l. Deleting from each cycle an edge we get that there is a graph with at least $n^{1+c}/4$ edges and no cycle with length at most l. As $\varepsilon_n \to 0$ when $n \to \infty$, if we choose v_s large enough so that $\varepsilon_{v_s - 1} < c$ then the conditions hold. □

One of the questions raised in [3] relates to the optimality of the results summarized in Theorem 23. There we presented for each $k > 1$ a monotone property $\mathscr{P}$ that has $e_{\mathscr{P}}(n) = kn$ for infinitely many values of n and $e_{\mathscr{P}}(n) = n^{2-1/k}$ for infinitely many values of n as well. We wish to consider whether there are properties with a wider range of oscillation. While for hereditary properties the sharpness of this range is unknown, for monotone properties, this result is nearly sharp as demonstrated by the following theorem. It shows that if $\mathscr{P}$ is monotone, we will not be able to get a significantly larger range of oscillation for size than that of the property cited. Note that this theorem is in some sense a corollary of the main result of [17].

Theorem 25. *Let k be an integer, $\varepsilon > 0$, and $\mathscr{P}$ be a monotone property. If $e_{\mathscr{P}}(n) > n^{2-1/k+\varepsilon}$ infinitely often, then $e_{\mathscr{P}}(n) \geqslant kn + o(n)$ for all n.*

Proof. Fix some $m \in \mathbb{N}$. Let n be such that $e_{\mathscr{P}}(n) > n^{2-\varepsilon}$ and $n^{1-k\varepsilon+k} > (m-k)\binom{n}{k}$, which may be done as $k\varepsilon < 1$. Let $G \in \mathscr{P}^n$ be a graph with $|E(G)| > n^{2-\varepsilon}$. Without loss of generality, the minimum degree of G is at least $n^{1-\varepsilon}$. Consider pairs of the type (x, S) where $S \subset \Gamma(x)$, $|S| = k$, and $x \notin S$. The number of such pairs is

$$n\binom{n^{1-\varepsilon}}{k} \sim Cn^{1-k\varepsilon+k}$$

for some constant C. The choice of n gives that there are at least $(m-k)\binom{n}{k}$ such pairs. Hence there is a set S which appears in at least $m-k$ different pairs. The set S together with the k vertices it is paired with induces a graph on m vertices with at least $k(m-k)$ edges, so $e_{\mathscr{P}}(m) \geqslant k(m-k) = km + o(m)$. □

Theorem 25 implies a lower bound on the size of properties which infinitely often have very large size. We can also give a lower bound when the property takes on sizes in a middle range. The following is similar to Theorem 24, but here we gain more by having the exponent maintain a positive distance from 1.

Theorem 26. *Let k be an integer, $c > 1/k$, and $\mathscr{P}$ be a monotone property. If $e_{\mathscr{P}}(n) \geqslant n^{1+c}$ infinitely often, then $e_{\mathscr{P}}(n) \geqslant (1 + 1/k)n - 3$ holds for all sufficiently large n.*

Proof. For $k=2$, Theorem 25 implies the desired result. So assume $k > 2$. The hypothesis tells us that for arbitrarily large n such that there is a $G \in \mathscr{P}^n$ with $|E(G)| \geqslant n^{1+c}$. We wish to show that for any large m, if n is large enough and $|E(G)| \geqslant n^{1+c}$, there is an $H \subset G$ with $|V(H)| = m$ and $|E(H)| \geqslant (1 + 1/k)m - 3$. This would prove the theorem.

Fix $m \gg k$ and let $n \gg m$. Let G be such that $|V(G)| = n$ and $|E(G)| \geqslant n^{1+c}$. Without loss of generality, the minimum degree of G is n^c, as we may otherwise remove vertices and create a subgraph with such a minimum degree and greater density. Let v be any vertex of G and let $B_i = \{u \in V(G)\colon d(u, v) = i\}$. We start with two cases in which we can easily show that a subgraph with the desired edge density exists.

Case 1: For some $1 < i \leqslant k+1$, there is a vertex $w \in B_i$ such that $|\Gamma(w) \cap B_{i-1}| \geqslant m$. Then there are m paths (not necessarily disjoint) of length i between v and w. We will construct a subgraph H as follows. Arbitrarily order the paths. Start with the first path. It has i edges and $i+1$ vertices. Add in the vertices of each path in turn, in order from v to w, until you obtain a graph on m vertices. As you add in whole paths from v to w, you add some number a of vertices and $a+1$ edges. The last path you add may not reach all the way to w, but will add some number of vertices and the same number of edges. When you obtain a graph on m vertices in this way, it will have at least $(1 + 1/(i-1))m - 3$ edges, and $(1 + 1/(i-1))m - 3 \geqslant (1 + 1/k)m - 3$.

Case 2: For some $1 \leqslant i \leqslant k$, $|E(B_i)| \geqslant |B_i| - 1$. We first show that if a graph G has the same number of edges as a tree, then there is a nested sequence of induced subgraphs $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_{|V(G)|} = G$ such that each A_j has at least as many edges as a tree. If G is connected, then we are done, as a search of a spanning tree would suffice. Otherwise, label the components of G as $C_1, C_2, \ldots, C_l$ so that $|E(C_j)|/|V(C_j)| \geqslant |E(C_{j+1})|/|V(C_{j+1})|$. Build a spanning tree on C_1, then on C_2, etc., and build the graphs A_i according to this ordering.

So let $A_1 \subset A_2 \subset \cdots \subset A_{|B_i|} = B_i$ be sets of vertices with induced graphs as above. Then consider a collection of paths from the vertices in A_m to v. As in Case 1, order the paths, only this time order them according to the sequence (A_i). Take vertices of the paths in order as before to obtain a graph H on m vertices. We will have the fewest number of edges if $H - B_i$ is a tree. In this case, $H - B_i$ has $\lceil m/i \rceil$ leaves and $h = \lfloor m/i \rfloor$ of the paths from v to the leaves have length i. Furthermore $H \cap B_i = A_h$ and $|E(A_h)| \geqslant h - 1 \geqslant m/i - 2$. Hence $|E(H)| = m - 1 + |E(A_h)| \geqslant (1 + 1/i)m - 3 \geqslant (1 + 1/k)m - 3$.

Suppose that neither Case 1 or Case 2 holds, that is, there is no subgraph H with the proper density. Then Case 2 implies that for all $1 \leqslant i \leqslant k$, $|E(B_i)| < |B_i| - 1$. Further Case 1 implies that for all i if $u \in B_{i+1}$ then $|\Gamma(u) \cap B_i| < m$ and if $w \in B_i$ then $|\Gamma(w) \cap B_{i-1}| < m$. Note that this last fact is true trivially for $i = 1$, and the implications of the failure of Case 1 tell us that for all $1 \leqslant i \leqslant k$, $|E(B_{i-1}, B_i)| < m|B_i|$ and $|E(B_i, B_{i+1})| < m|B_{i+1}|$. Recalling that each vertex has degree at least n^c, we obtain for all $1 \leqslant i \leqslant k$,

$$\begin{aligned} |B_i|n^c &= 2|E(B_i)| + |E(B_{i-1}, B_i)| + |E(B_i, B_{i+1})| \\ &\leqslant |B_i| - 1 + m|B_i| + m|B_{i+1}|, \end{aligned}$$

and hence,

$$|B_{i+1}| > |B_i|(n^c - m - 1) \geqslant |B_i|n^{1/k},$$

where the last inequality holds if n is sufficiently large.

Hence $|B_i| > n^{i/k}$ for all $1 \leqslant i \leqslant k$, in particular for $i = k$. But then $|B_k| > n$, which is not possible. Hence one of the two cases holds and we have our result. □

The last three theorems taken together tell us that if the size of a property is in the range $(n - 1, n^{2-\varepsilon})$ the size can oscillate within certain ranges. However, if $\mathscr{P}$ is a monotone property, its size cannot go above $n^{2-\varepsilon_n}$ if $\varepsilon_n \to 0$ and still oscillate. This is implied by Theorem 19 of Erdős and Simonovits [13–16]. That is, if $e_{\mathscr{P}}(n) > n^{2-\varepsilon}$ for all $\varepsilon > 0$, then, by Theorem 19 there is an integer $k > 1$ and number $\varepsilon > 0$ such that $e_{\mathscr{P}}(n) = (1 - 1/k)n^2/2 + O(n^{2-\varepsilon})$.

In the other direction, we can show, just as we did in Theorem 17, that if a property has a size below the highest range then its size must drop to a lower level. Although this is also implied by Theorem 19, here we show it independently. As with many of our results on size, this is easier to prove than the corresponding result on speed (Theorem 17), but also relies on the work of Kővári, Sós, and Turán in [17].

Theorem 27. *If for a property* $\mathscr{P}$, $e_{\mathscr{P}}(n) < \lfloor n^2/4 \rfloor$ *infinitely often, then there is an* $\varepsilon > 0$ *such that* $e_{\mathscr{P}}(n) = O(n^{2-\varepsilon})$.

Proof. Assume there is no $\varepsilon > 0$ such that $e_{\mathscr{P}}(n) = O(n^{2-\varepsilon})$. Then $e_{\mathscr{P}}(n) > n^{2-\varepsilon}$ for all ε and $|\mathscr{P}^n| > 2^{n^{2-\varepsilon}}$ for all ε. But then by the Kővári–Sós–Turán Theorem [17], for sufficiently large n, the graph $K_{t,t} \in \mathscr{P}$ for all $t \geqslant 1$. But then $e_{\mathscr{P}}(n) \geqslant n^2/4$, a contradiction. □

The theorems we have presented provide a great deal of information about the ranges into which sizes and speeds may fall for monotone properties. Oscillation is possible for the size function much earlier than for the speed function, which hinders the development of any more detailed picture of allowable sizes. However, these results give us further insight into the speed functions of monotone properties as well and will hopefully lead to a complete picture with additional work.

8. Towards a complete picture

What would a complete picture consist of? It would include exact results on the full range of oscillation for both speed and size of monotone properties. For size the results presented here are perhaps as detailed as we can hope; however it may be possible to remove some of the asymptotics that are factored into the present work. In particular, a clearer picture of the lower order terms that occur would be nice. These lower order terms are even less understood in the case of the speed functions; there is still significant improvement possible of those results. We know, for example, that there are large gaps within the polynomial range: i.e. the only allowable quadratic speed for monotone properties is $\binom{n}{2} + 1$. What other jumps occur?

A complete picture would also include analogous results for hereditary properties. Theorem 1 is also missing lower order terms, and the oscillation of speed for hereditary properties is even less understood than for monotone properties. Although the definition of size given here does not make sense for hereditary properties, one may define a hereditary size ($eh_{\mathscr{P}}(n)$) consistent with the current definition that satisfies the relation $|\mathscr{P}^n| \geqslant 2^{eh_{\mathscr{P}}(n)}$. Perhaps results on this measure would yield insights into the speed as well.

Uncited references

[1,8,9,24].

References

[1] V.E. Alekseev, On the entropy values of hereditary classes of graphs, Discrete Math. Appl. 3 (1993) 191–199.

[2] J. Balogh, B. Bollobás, D. Weinreich, The size of hereditary properties of graphs, J. Combin. Theory B 79 (2000) 131–156.

[3] J. Balogh, B. Bollobás, D. Weinreich, The penultimate range of growth for graph properties, European J. Combin. 22 (2001) 277–289.

[4] J. Balogh, B. Bollobás, D. Weinreich, The structure of posets and an application to hereditary properties, preprint.

[5] B. Bollobás, A. Thomason, Projections of bodies and hereditary properties of hypergraphs, J. London Math. Soc. 27 (1995) 417–424.

[6] B. Bollobás, A. Thomason, Hereditary and monotone properties of graphs, in: R.L. Graham, J. Nešetřil (Eds.), The mathematics of Paul Erdős II, Algorithms and Combinatorics, Vol. 14, Springer, Berlin, 1997, pp. 70–78.

[7] B. Bollobás, A. Thomason, Generalized chromatic numbers of random graphs, Random Structures Algorithms 6 (1995) 353–356.

[8] B. Bollobás, A. Thomason, Colouring random graphs by hereditary properties, to appear.

[9] B. Bollobás, P. Erdős, On the structure of edge graphs, Bull. London Math. Soc. 5 (1973) 317–321.

[10] P. Erdős, Some recent results on extremal problems in graph theory, Results, 1967 Theory of Graphs, International Symposium, ed. P. Rosenstiehl, Gordon & Breach, NY, Rome, 1966, pp. 117–123.

[11] P. Erdős, On some new inequalities concerning extremal properties of graphs, 1968 Theory of Graphs, Proceedings Colloqium, Tihany, Academic Press, NY, 1966, pp. 77–81.

[12] P. Erdős, P. Frankl, V. Rödl, The asymptotic enumeration of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, Graphs Combin. 2 (1986) 113–121.

[13] P. Erdős, M. Simonovits, On a valence problem in extremal graph theory, Discrete Math. 5 (1973) 323–334.

[14] P. Erdős, M. Simonovits, Some extremal problems in graph theory, Combinatorial theory and its applications, Vol. I, Proceedings Colloqium, Balatonfűred, 1969, North-Holland, Amsterdam, 1970, pp. 377–390.

[15] P. Erdős, M. Simonovits, An extremal graph problem, Acta Math. Acad. Sci. Hungar. 22 (1971/72) 275–282.

[16] P. Erdős, M. Simonovits, A limit theorem in graph theory, Studia Sci. Math. Hungar. 1 (1966) 51–57.

[17] T. Kővári, V.T. Sós, P. Turán, On a problem of K. Zarankiewicz., Colloq. Math. 3 (1954) 50–57.

[18] H.J. Prömel, A. Steger, Excluding induced subgraphs: quadrilaterals, Random Struct. Algorithms 2 (1991) 55–71.

[19] H.J. Prömel, A. Steger, Excluding induced subgraphs II: extremal graphs, Discrete Appl. Math. 44 (1993) 283–294.

[20] H.J. Prömel, A. Steger, Excluding induced subgraphs III: a general asymptotic, Random Struct. Algorithms 3 (1992) 19–31.

[21] H.J. Prömel, A. Steger, The asymptotic structure of H-free graphs, in: N. Robertson, P. Seymour (Eds.), Graph Structure Theory, Contemporary Mathematics, Vol. 147, Amer. Math. Soc., Providence, RI, 1993, pp. 167–178.

[22] M. Simonovits, A method for solving extremal problems in graph theory, stability problems,1968 Theory of Graphs, Proceedings Colloqium, Tihany, Academic Press, NY, 1966, pp. 279–319.

[23] E.R. Scheinerman, J. Zito, On the size of hereditary classes ofgraphs, J. Combin. Theory B 61 (1994) 16–39.

[24] E. Szemerédi, Regular partitions of graphs, in: J.-C. Bermond, J.-C. Fournier, M. las Vergnas, D. Sotteau (Eds.), Proceedings Colloque Inter. CNRS, 1978.