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A theorem on paths in locally planar triangulations

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Abstract

In this note, we show that every 5-connected triangulation in a surface with sufficiently large representativity is Hamiltonian-connected.

Keywords: Hamiltonian-connected; Large representativity; Surface

1. Introduction

The basic notation and terminology in this paper is the same as in [10] and [5].

A *closed surface* means a connected compact 2-dimensional manifold without boundary. For a closed surface F^2, let $\varepsilon(F^2)$ denote the Euler characteristic of F^2. The number $k = 2 - \varepsilon(F^2)$ is called the *Euler genus* of F^2. We denote an orientable and nonorientable closed surface of genus g by S_g and N_g, respectively. It is well-known that for every even $k \geq 0$, it is either $S_{\frac{k}{2}}$ or N_k, and for every odd k, N_k.

Let G be a graph on a nonspherical closed surface F^2. The *representativity* of G on F^2, denoted by $r(G)$, is the minimum number of intersecting points of G and ℓ, where ℓ ranges over all essential closed curves on F^2. We say that G is *r-representative* if $r(G) \geq r$. Many researchers pointed out that graphs on a closed surface with sufficiently large representativity have similar properties to plane graphs, for example, with respect to chromatic number and hamiltonicity, etc., cf. [1, 3, 4, 10, 11].

Thomassen [9] conjectured that large representativity of a 5-connected triangulation implies it is Hamiltonian, and this was proved by Yu [11]. Thomassen [9] pointed out that 5-connectivity is the best possible because, no matter how large the representativity is,

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there are 4-connected triangulations which are not 1-tough. In this note, we will prove the following theorem, which is a generalization of Yu's result.

Theorem 1. *For every nonnegative integer k, there exists an integer R such that if G is a 5-connected triangulation in a surface with Euler genus k and $r(G) \geq R$, then it is Hamiltonian-connected.*

But perhaps the condition "triangulation" is not necessary. We refer the reader to [1] or [5] on this issue.

We introduce some notation. If G is a 2-connected plane graph with outer cycle C_1 and another facial cycle C_2, then we call G a *cylinder* with outer cycle C_1 and inner cycle C_2. In our proof of Theorem 1, we will use the following lemma due to Yu [11].

Lemma 1. *Let G be a cylinder with outer cycle C_1 and inner cycle C_2, $x, y \in V(C_1)$, and $u, v \in V(C_2)$ be four distinct vertices. Suppose that* (1) *any simple closed curve in the plane separating C_1 from C_2 intersects G at least 7 times, and* (2) *any simple curve in the plane from C_1 to C_2 intersects G at least 8 times. Then G has two disjoint paths P and Q with P from x to y and Q from u to v such that any $(P \cup Q)$-bridge not containing vertices in $C_1 \cup C_2$ has at most 4 attachments and any $(P \cup Q)$-bridge containing a vertex in $C_1 \cup C_2$ has at most 2 attachments.*

2. Proof

Suppose $a, b \in V(G)$ are given. We want to prove that there exists a Hamiltonian path from a to b. Let us first consider the orientable case.

Suppose H is a cylinder with outer cycle C_1 and inner cycle C_2. If H' is a graph on a surface of Euler genus k with disjoint facial cycles C_1', C_2' of the same lengths as C_1, C_2 (respectively), then we can identify C_1' and C_1 into a cycle C_1'', and C_2' and C_2 into a cycle C_2''. Let M be the graph obtained from the union of H' and H after identifications. Then M has Euler genus $k + 2$. Conversely, we can say that H' is obtained from M by *cutting* C_1'', C_2'', and by *deleting* the cylinder H. The *cylinder-width* of H is the largest integer a such that G has a pairwise disjoint cycles $R_1, \ldots, R_a$ such that $C_1 \subseteq \mathrm{int}(R_1) \subseteq \mathrm{int}(R_2) \subseteq \cdots \subseteq \mathrm{int}(R_a)$. (See the definition of int in [1, 10], say.) Using the argument in [1, 3, 4, 6, 10], the following is not difficult to prove. For any k and c, there exists a number R such that any 3-connected graph on an orientable surface with Euler genus k and $r(G) \geq R$ contains k pairwise disjoint cylinders $Q_1, \ldots, Q_k$ of cylinder width at least c whose cutting and deletion results in a 2-connected plane graph. Let us fix $c = 33$ and focus on one of the k cylinders, say Q_j. Then, we can take two disjoint noncontractible cycles D_1 and D_2 in Q_j such that a, b are not in the cylinder with the outer cycle D_1 and the inner cycle D_2, and any curve from D_1 to D_2 intersects the cylinder at least 8 times. Let D_1', D_2' be the nontriangulated facial cycles after cutting and deleting the cylinder. Also, let D_1'', D_2'' be the nontriangulated facial cycles after cutting and deleting two cycles D_1' and D_2'. Note that $D_1'', D_1', D_1, D_2, D_2', D_2''$ occur in this order in the handle. Since we take $c = 33$, we can also choose D_1', D_1'', D_2', D_2'' such that a, b are in neither the cylinder with the outer cycle D_1' and the inner cycle D_1'' nor the cylinder with the outer cycle D_2' and the inner cycle D_2''.

We claim that we can choose these four cycles D_1, D_2, D'_1, D'_2 such that all of them are chordless, any curve from D_1 to D_2 intersects the cylinder at least 8 times and a, b are not in the cylinder. This means the cylinder is 3-connected since the cylinder is a subgraph of the 5-connected triangulation G and has no 2-cuts.

Suppose there exists a chord xy in D_1, say. Let A' and A'' be the two segments of D_1 bounded by $\{x, y\}$ such that $A' + xy$ is noncontractible. We assume that A'' is chordless (we just take the smallest "chord" segment). Let $u_1, \dots, u_t$ and $v_1, \dots, v_s$ be the neighbors of x, y, respectively, such that $u_1, \dots, u_t, v_s, \dots, v_1$ is the path inside the disk $A'' \cup xy$ with u_1, v_1 being on A''. Since G is a triangulation, $u_t = v_s$. Let u and v be the neighbor of u_1, v_1 in D'_1, respectively such that the path uD'_1v is as long as possible. By the path uD'_1v, we mean the path between u and v along D'_1 such that all of vertices on it have at least one neighbor to A''. By the path vD'_1u, we mean the path obtained from D'_1 by removing $V(uD'_1v) - \{u, v\}$. Set $D = vD'_1uu_1, \dots, u_tv_s, \dots, v_1v$. Now we consider D_1 as $A' + xy$ and D'_1 as D. There are no 2-cuts in the segment $uu_1, \dots, u_tv_s, \dots, v_1v$ of D'_1 since $\{x, y, u, v\}$ is not a cutset in G. This "replacement" of D_1, D'_1 does not destroy the assumption that any curve from D_1 to D_2 intersects the cylinder at least 8 times. (Because we just replace the "chord" part, which does not destroy the assumption.) By continuing this procedure, we can get D_1, D_2, D'_1, D'_2 such that all of them are chordless and any curve from D_1 to D_2 intersects the cylinder at least 8 times and both a and b are not in the cylinder.

By doing this procedure to each cylinder in each handle, we can get the cylinders which are 3-connected, internally 5-connected (a cylinder H with outer cycle C_1 and inner cycle C_2 is said to be *internally k-connected* if $H - X$ does not contain a component which has no vertex in $V(C_1 \cup C_2)$ for any $X \subset V(H)$ with $|X| < k$. The definition of internally k-connected for planar graph with outer cycle C is similar to that of a cylinder.).

Let G' be the plane graph after cutting and deleting all the cylinders. Then G' is also 3-connected since all nontriangulated faces are chordless. There are now $2k$ nontriangulated faces $F_1, \dots, F_{2k}$ such that F_{2i-1}, F_{2i} correspond to D'_1, D'_2 in each cylinder for $1 \le i \le k$. We add a vertex r_i and edges to F_i for $1 \le i \le 2k$ such that for any $r \in V(F_i)$, if r has at least two neighbors to the cylinder, then we add the edge rr_i. Let G'' be the resulting graph. We claim G'' is 4-connected. Suppose there is a 3-cut $\{x_1, x_2, x_3\}$. Since G' is 3-connected, none of r_i are in $\{x_1, x_2, x_3\}$. But in this case, we can easily find a 4-cut in G, a contradiction. Hence G'' is 4-connected. Then we use the result of Thomassen [8]. We can find a Hamiltonian path P from a to b in G''. The Hamiltonian path P passes through each r_i. Let r'_i and r''_i be two vertices in F_i which is just before r_i, and just after r_i in P, respectively. We extend P to a Hamiltonian path in G. In each cylinder, we can take four distinct vertices $s_{2i-1}, s'_{2i-1}, s_{2i}, s'_{2i}$ such that s_{2i-1}, s'_{2i-1} are adjacent to r'_{2i-1}, r''_{2i-1}, respectively, and s_{2i}, s'_{2i} are adjacent to r'_{2i}, r''_{2i}, respectively, for $1 \le i \le k$. By applying Lemma 1 to each i, we can find two disjoint paths P'_{2i-1}, P'_{2i} such that P'_{2i-1} is from s_{2i-1} to s'_{2i-1} and P'_{2i} is from s_{2i} to s'_{2i}, and all vertices in the cylinder are either on P'_{2i-1} or P'_{2i} for $1 \le i \le k$. Then we can get the Hamiltonian path from a to b using $P - \{r_1, \dots, r_k\}$, P'_{2i-1}, P'_{2i} for $1 \le i \le k$, and $s_{2i-1}r'_{2i-1}, s'_{2i-1}r''_{2i-1}, s_{2i}r'_{2i}, s'_{2i}r''_{2i}$ for $1 \le i \le k$. This completes the proof of the orientable case.

Let us briefly sketch a proof for the nonorientable case N_k. If k is even, then using the theorem of Robertson and Seymour [6] and the above argument, we can find k pairwise disjoint cylinders whose removal results in the 3-connected planar “nearly” triangulation, and the cylinder is also 3-connected internally 5-connected “nearly” planar triangulation (the word “nearly” means that all faces except for at most $2k$ faces are triangulated). So we can find the Hamiltonian path from a to b by the same argument of orientable case. If k is odd, we can also find $k - 1$ pairwise disjoint cylinders whose removal results in the 3-connected projective planar “nearly” triangulation, and the cylinder in each handle is also 3-connected internally 5-connected “nearly” planar triangulation. By the theorem of Fielder et al. [2], the large representativity, and the above argument, we can find a cycle W in the projective plane graph such that if all chords of nonplanar crossings are deleted, the resulting graph is the 3-connected planar “nearly” triangulation with outer cycle W. Then we use the theorem of Sanders [7] and the argument above to construct a Hamiltonian path from a to b. This completes the proof. □

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