ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Applied Mathematics Letters 19 (2006) 758–766

Applied Mathematics Letters

www.elsevier.com/locate/aml

Oscillation and asymptotic behavior for a class of delay parabolic differential[☆]

Qisheng Wang[a,b,*], Zigen Ouyang[a], Jiding Liao[a]

[a] *School of Mathematics and Physical Science, Nanhua University, Hengyang Hunan, 421001, PR China*

[b] *School of Mathematics and Computing Science, Xiangtan University, Xiangtan Hunan, 411105, PR China*

Received 14 May 2004; received in revised form 19 August 2005; accepted 27 September 2005

Abstract

Some comparative theorems are given for the oscillation and asymptotic behavior for a class of high order delay parabolic differential equations of the form

$$\frac{\partial^n (u(x,t) - p(t)u(x,t-\tau))}{\partial t^n} - a(t)\triangle u + c(x,t,u) + \int_a^b q(x,t,\xi) f(u(x,g_1(t,\xi)),\ldots,u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) = 0, \qquad (x,t) \in \Omega \times R_+ \equiv G,$$

where n is an odd integer, Ω is a bounded domain in R^m with a smooth boundary $\partial\Omega$, and $\triangle$ is the Laplacian operation with three boundary value conditions. Our results extend some of those of [P. Wang, Oscillatory criteria of nonlinear hyperbolic equations with continuous deviating arguments, Appl. Math. Comput. 106 (1999), 163–169] substantially.

Keywords: Oscillation; High order; Delay parabolic differential equation; Asymptotically positive solution; Asymptotic behavior

1. Introduction

Consider the following odd-order delay parabolic differential equation:

$$\frac{\partial^n (u(x,t) - p(t)u(x,t-\tau))}{\partial t^n} - a(t)\triangle u + c(x,t,u) + \int_a^b q(x,t,\xi) f(u(x,g_1(t,\xi)),\ldots,u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) = 0, \qquad (x,t) \in G, \tag{1.1}$$

where n is an odd integer, τ is a positive constant, $R_+ = [0, +\infty)$; we will assume throughout this work that

☆ Supported by the Natural Science Foundation of China under Grant No. 10471086.
* Corresponding author at: School of Mathematics and Physical Science, Nanhua University, Hengyang Hunan, 421001, PR China.
E-mail address: wangqisheng001@yahoo.com.cn (Q. Wang).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2005.09.003

(H$_1$) *$a(t) \in C(R_+, R_+)$, $g_i(t,\xi) \in C(R_+ \times [a,b], R)$, $g_i(t,\xi) \le t$, $\xi \in [a,b]$ and $\lim_{t\to\infty} g_i(t,\xi) = \infty$, for $i = 1, \dots, l$.*

(H$_2$) *$q(x,t,\xi) \in C(\overline{\Omega} \times R_+ \times [a,b], R_+)$, $Q(t,\xi) = \min_{x\in\overline{\Omega}}\{q(x,t,\xi)\}$, $f \in C(R^m \times R_+ \times [a,b], R)$, $u_i f \ge 0$ when each $u_i > 0$, $i = 1, \dots, l$, f is convex, $-f = f(-u_1, \dots, -u_l)$, in which $u_i = u(x, g_i(t,\xi))$, for $i = 1, \dots, l$.*

(H$_3$) *$p(t) \in C(R_+, R_+)$, and $\lim_{t\to\infty} p(t) = p < 1$.*

(H$_4$) *$c(x,t,u) \in C(\overline{\Omega} \times R_+ \times R, R)$; $h_1(t)\varphi_1(\xi) \le c(x,t,\xi) \le h_2(t)\varphi_2(\xi)$ for $\xi > 0$, in which $h_1(t) \in C(R_+, R_+)$, $\varphi_1(\xi) \in C([a,b], R)$, $\varphi_1(\xi)$ is a positive and convex function in $(0,\infty)$, and $c(x,t,-\xi) = -c(x,t,\xi)$, $\varphi_1(-\xi) = -\varphi_1(\xi)$.*

We consider the following boundary conditions:

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times R+, \tag{B$_1$}$$

$$\frac{\partial u(x,t)}{\partial N} = 0, \qquad (x,t) \in \partial\Omega \times R+, \tag{B$_2$}$$

$$\frac{\partial u(x,t)}{\partial N} + \nu u = 0, \qquad (x,t) \in \partial\Omega \times R+, \tag{B$_3$}$$

where N is the unit exterior vector normal to $\partial\Omega$, and $\nu(x,t)$ is a nonnegative continuous function on $\partial\Omega \times R+$.

Recently, many authors [1–8] have studied oscillations for the solutions of parabolic differential equations; they obtained some comparative theorems for the oscillations of these equations. In more recent times, Wang [10] has investigated the following nonlinear hyperbolic equations:

$$\frac{\partial^2(u(x,t) - p(t)u(x,t-\tau))}{\partial t^2} - a(t)\triangle u + c(x,t,u) + \int_a^b q(x,t,\xi)u(x,g(t,\xi))\,\mathrm{d}\sigma(\xi) = 0, \qquad (x,t) \in G, \tag{1.2}$$

where $a(t)$, $g(t)$, $c(x,t,u)$, $p(t)$ are defined as above. Some comparative oscillation for the solutions of the boundary value problem (1.2)–(B$_3$) were obtained. But few authors [9] have studied oscillation of the high order parabolic differential equation.

In this work, we investigate Eq. (1.1) with the boundary conditions (B$_1$), (B$_2$) and (B$_3$). This work is organized as follows: in Section 2, we discuss the comparative oscillation for the solutions of Eq. (1.1) with boundary conditions (B$_1$), (B$_2$) and (B$_3$), and some comparative results will be obtained; in Section 3, we will investigate the asymptotic behavior of non-oscillatory solutions of (1.1), and we shall give one example to explain the oscillation and asymptotic behavior of Eq. (1.1) with the above boundary conditions.

We first recall some definitions as follows.

Definition 1.1. A function $u(x,t) \in C^2(\Omega) \times C^n(R_+)$ is said to be a solution to the problem (1.1)–(B$_i$) ($i = 1,2,3$) if it satisfies (1.1) in the domain G and satisfies the boundary condition (B$_i$) ($i = 1,2,3$).

Definition 1.2. The solution $u(x,t)$ of problem (1.1) is said to be oscillatory in the domain G if for any positive number μ, there exists a point $(x_1,t_1) \in \Omega \times [\mu,\infty)$ such that the equality $u(x_1,t_1) = 0$ holds. If every solution of Eq. (1.1) is oscillatory, then Eq. (1.1) is called oscillatory.

Definition 1.3. A function $u(x,t)$ is called eventually positive (negative) if there exists a number $T \ge 0$ such that $u(x,t) > 0$ (< 0) for every $(x,t) \in \Omega \times [T,\infty)$.

Remark. If a solution is non-oscillatory, then it is eventually positive or eventually negative.

2. Comparative oscillation of (1.1) with the boundary value conditions (B_1), (B_2), (B_3)

For convenience, we consider the following Dirichlet boundary value problem in the domain Ω:

$$\triangle u + \alpha u = 0, \qquad \text{in } (x,t) \in G, \tag{2.1}$$

$$u = 0, \qquad \text{on } (x,t) \in \partial\Omega \times R_+, \tag{2.2}$$

in which α is a constant.

It is well known from [11] that the smallest eigenvalue λ_1 of problem (2.1) is positive and that the corresponding eigenfunction $\psi(x) \geq 0$ for $x \in \Omega$.

Let $u(x,t)$ be a solution of problem (1.1)–(B_1); we define throughout this section

$$U(t) = \frac{\int_\Omega u(x,t)\psi(x)\,dx}{\int_\Omega \psi(x)\,dx}. \tag{2.3}$$

Let $u(x,t)$ be a solution of problem (1.1)–(B_i), $i = 2, 3$; we always define

$$V(t) = \frac{\int_\Omega u(x,t)\,dx}{\int_\Omega dx}. \tag{2.4}$$

To obtain our results, we first introduce a lemma as follows:

Lemma 2.1. *Suppose that* (H_1)–(H_4) *hold; then*

$$(z(t) - p(t)z(t-\tau))^{(n)} + h_1(t)\varphi_1(z) + \int_a^b Q(t,\xi) f(z(g_1(t,\xi)),\ldots,z(g_l(t,\xi)))d\sigma(\xi) \leq 0, \qquad t \in R_+, \tag{2.5}$$

has an eventually positive solution if and only if

$$(z(t) - p(t)z(t-\tau))^{(n)} + h_1(t)\varphi_1(z) + \int_a^b Q(t,\xi) f(z(g_1(t,\xi)),\ldots,z(g_l(t,\xi)))\,d\sigma(\xi) = 0, \qquad t \in R_+ \tag{2.6}$$

has an eventually positive solution.

Proof. The sufficiency is obvious. We only need to prove the necessity. Assume that (2.5) has an eventually positive solution $z(t)$; this means that there is a number $T > 0$ such that $z(t) > 0, z(t-\tau) > 0, z(g_i(t,\xi)) > 0$ for $t > T, i = 1,\ldots,l$. It follows that there exists a nonnegative integer $n^* \leq n-1$ (if n is odd, then n^* is even; if n is even, then n^* is odd), such that

$$\begin{aligned} &z^{(i)}(t) > 0, \qquad i = 0,1,\ldots,n^* \\ &(-1)^i z^{(i)}(t) > 0, \qquad i = n^*,\ldots,n-1, \end{aligned} \tag{2.7}$$

By using (2.7), (H_1)–(H_3) and integrating (2.5) from t to ∞, $n - n^*$ times, we obtain

$$(z(t) - p(t)z(t-\tau))^{(n^*)} \geq \int_t^\infty \frac{(s-t)^{n-n^*-1}}{(n-n^*-1)!}\left[h_1(s)\varphi_1(z(s)) + \int_a^b Q(s,\xi) f(z(g_1(s,\xi)),\ldots,z(g_l(s,\xi)))\,d\sigma(\xi)\right] ds. \tag{2.8}$$

Let $T > 0$ is large enough that (2.8) holds and $z(t-\tau) > 0, z(g_i(t,\xi)) > 0, i = 1,\ldots,l$. Integrating (2.8) from T to t and using (2.8), we have

$$z(t) \geq p(t)z(t-\tau) + \int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!}\int_s^\infty \frac{(r-s)^{n-n^*-1}}{(n-n^*-1)!}\left[h_1(r)\varphi_1(z(r)) + \int_a^b Q(r,\xi) f(z(g_1(r,\xi)),\ldots,z(g_l(r,\xi)))\,d\sigma(\xi)\right] dr\,ds. \tag{2.9}$$

We define a set of functions as follows: $K = C([T-\tau,\infty),[0,1])$, and define an operator S on K:

$$(Sy)(t) = \begin{cases} (Sy)(T), & T-\tau \le t < T, \\ \dfrac{1}{z(t)}\left\{p(t)y(t-\tau)z(t-\tau) + \displaystyle\int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!}\int_s^\infty \frac{(r-s)^{n-n^*-1}}{(n-n^*-1)!} \right. & \\ \quad \times \Big[h_1(r)\varphi_1(y(r)z(r)) & \\ \quad + \displaystyle\int_a^b Q(r,\xi)\, f(y(g_1(r,\xi))z(g_1(r,\xi)),\ldots,y(g_l(r,\xi))z(g_l(r,\xi))) & \\ \quad \left. \times\, \mathrm{d}\sigma(\xi)\Big]\,\mathrm{d}r\,\mathrm{d}s\right\}, & t \ge T. \end{cases}$$

It is easy to see by (2.9) that S maps K into itself, and for any $y \in K$, we have $(Sy)(t) > 0$, for $T-\tau \le t < T$. Now, we define the sequences $y_k(t)$ in K:

$$y_0(t) \equiv 1, \qquad t \ge T-\tau,$$

and

$$y_{k+1}(t) = (Sy_k)(t), \qquad \text{for } t \ge T-\tau,\ k = 0,1,\ldots,$$
$$\lim_{k\to\infty} y_k(t) = y(t), \qquad t \ge T-\tau.$$

This means, from Lebesgue's dominated convergence theorem, that there exists a function $y(t)$ that satisfies

$$y(t) = \frac{1}{z(t)}\left\{p(t)y(t-\tau)z(t-\tau) + \int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!}\int_s^\infty \frac{(r-s)^{n-n^*-1}}{(n-n^*-1)!}\Big[h_1(r)\varphi_1(y(r)z(r)) \right.$$
$$\left. + \int_a^b Q(r,\xi)\, f(y(g_1(r,\xi))z(g_1(r,\xi)),\ldots,y(g_l(r,\xi))z(g_l(r,\xi)))\,\mathrm{d}\sigma(\xi)\Big]\,\mathrm{d}r\,\mathrm{d}s\right\}, \qquad t \ge T.$$

and

$$y(t) = (Sy)(T), \qquad T-\tau \le t < T.$$

This being the case, define

$$w(t) = y(t)z(t).$$

It is obvious that $w(t) > 0$ for $T-\tau \le t < T$ and

$$w(t) = p(t)w(t-\tau) + \int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!}\int_s^\infty \frac{(r-s)^{n-n^*-1}}{(n-n^*-1)!}\Big[h_1(r)\varphi_1(w(r))$$
$$+ \int_a^b Q(r,\xi)\, f(w(g_1(r,\xi)),\ldots,w(g_l(r,\xi)))\,\mathrm{d}\sigma(\xi)\Big]\,\mathrm{d}r\,\mathrm{d}s \qquad t \ge T.$$

Thus, $w(t)$ is a nonnegative solution of Eq. (2.6) for $t \ge T$. Finally, it remains to show that $w(t) > 0$ for $t > T-\mu$. Assume that there exists a $t^* > T-\mu$ such that $w(t) > 0$ for $T-\tau < t < t^*$ and $w(t^*) = 0$. Then $t^* > T$, and

$$0 = w(t^*) = p(t^*)w(t^*-\tau) + \int_T^{t^*} \frac{(t^*-s)^{n^*-1}}{(n^*-1)!}\int_s^\infty \frac{(r-s)^{n-n^*-1}}{(n-n^*-1)!}\Big[h_1(r)\varphi_1(w(r))$$
$$+ \int_a^b Q(r,\xi)\, f(w(g_1(r,\xi)),\ldots,w(g_l(r,\xi)))\,\mathrm{d}\sigma(\xi)\Big]\,\mathrm{d}r\,\mathrm{d}s, \qquad t^* \ge T.$$

which contradicts the assumption. Thus, $w(t)$ is an eventually positive solution of Eq. (2.6); this is a contradiction to the supposition too, and the proof is completed. □

If we let $y(t) = -z(t)$, then Lemma 2.1 changes into the following:

Corollary 2.1. *Suppose that* (H_1)–(H_4) *hold; then*

$$(z(t) - p(t)z(t-\tau))^{(n)} + h_1(t)\varphi_1(z) + \int_a^b Q(t,\xi) f(z(g_1(t,\xi)),\dots,z(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) \geq 0, \qquad t \in R_+, \tag{2.10}$$

has an eventually negative solution if and only if (2.6) *has an eventually negative solution.*

Theorem 2.1. *Suppose that* (H_1)–(H_4) *hold; if every solution of the differential equation*

$$(z(t) - p(t)z(t-\tau))^{(n)} + \lambda_1 a(t)z(t) + h_1(t)\varphi_1(z) + \int_a^b Q(t,\xi) f(z(g_1(t,\xi)),\dots,z(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) = 0, \qquad t \in R_+, \tag{2.11}$$

oscillates, then every solution of (1.1)–(B_1) *oscillates.*

Proof. Assume that there is a non-oscillatory solution $u(x,t)$ of the problem (1.1)–(B_1). Without loss of generally, let $u(x,t)$ be an eventually positive solution of problem (1.1)–(B_1); then there exists a number $T > 0$ such that $u(x,t) > 0$, $u(x,t-\tau) > 0$, $u(x,g_k(t,\xi)) > 0$ for $t > T$, $k = 1,\dots,l$.

Multiplying both sides of Eq. (1.1) by $\psi(x)$, and integrating both sides over the domain Ω with respect to x, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_\Omega (u(x,t) - p(t)u(x,t-\tau))\psi(x)\,\mathrm{d}x - a(t)\int_\Omega \triangle u\psi(x)\,\mathrm{d}x + \int_\Omega c(x,t,u)\psi(x)\,\mathrm{d}x + \int_\Omega \psi(x)\int_a^b q(x,t,\xi) f(u(x,g_1(t,\xi)),\dots,u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi)\,\mathrm{d}x = 0, \qquad t \geq T. \tag{2.12}$$

Using (2.1) and (2.2), and Green's formula, we have

$$\int_\Omega \triangle u\psi(x)\,\mathrm{d}x = -\lambda_1\int_\Omega u\psi(x)\,\mathrm{d}x, \qquad t \geq T. \tag{2.13}$$

Combining (2.12), (2.13), (H_3), (H_4), using Jensen's inequality and changing the order of integration, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_\Omega (u(x,t) - p(t)u(x,t-\tau))\psi(x)\,\mathrm{d}x + \lambda_1 a(t)\int_\Omega u\psi(x)\,\mathrm{d}x + h_1(t)\int_\Omega \varphi_1(u)\psi(x)\,\mathrm{d}x + \int_a^b Q(t,\xi) f\left(\frac{\int_\Omega \psi(x)u(x,g_1(t,\xi))\,\mathrm{d}x}{\int_\Omega \psi(x)\,\mathrm{d}x},\dots,\frac{\int_\Omega \psi(x)u(x,g_l(t,\xi))\,\mathrm{d}x}{\int_\Omega \psi(x)\,\mathrm{d}x}\right)\mathrm{d}\sigma(\xi)\int_\Omega \psi(x)\,\mathrm{d}x \leq 0, \quad t \geq T.$$

Using (2.3), this means

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(U(t) - p(t)U(t-\tau)) + \lambda_1 a(t)U(t) + h_1(t)\varphi_1(U) + \int_a^b Q(t,\xi) f(U(g_1(t,\xi)),\dots,U(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) \leq 0, \qquad t \geq T. \tag{2.14}$$

Because $u(x,t)$ is positive, from (2.3) again, we have that $U(t)$ is eventually positive. It follows that $U(t)$ is an eventually positive solution of (2.5); according to Lemma 2.1, Eq. (2.6) has an eventually positive solution, and this contradicts the supposition.

If $u(x,t)$ is an eventually negative solution of problem (1.1)–(B_1), then there exists a number $T > 0$ such that $u(x,t) < 0$, $u(x,t-\tau) < 0$, $u(x,g_k(t,\xi)) < 0$ for $t > T$, $k = 1,\dots,l$; then (2.12) changes into

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_\Omega ((-u(x,t)) - p(t) - (u(x,t-\tau)))\psi(x)\,\mathrm{d}x - a(t)\int_\Omega \triangle(-u)\psi(x)\,\mathrm{d}x + \int_\Omega c(x,t,-u)\psi(x)\,\mathrm{d}x + \int_\Omega \psi(x)\int_a^b q(x,t,\xi) f(-u(x,g_1(t,\xi)),\dots,-u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi)\,\mathrm{d}x = 0, \qquad t \geq T.$$

Let

$$v(x,t) = -u(x,t),$$

Using a method similar to the above, we can derive a contradiction too. The proof is completed. □

If $p(t) = 0$, $c(x,t,u) = 0$, then (1.1) has the following special form:

$$\frac{\partial^n u(x,t)}{\partial t^n} - a(t)\triangle u + \int_a^b q(x,t,\xi) \times f(u(x,g_1(t,\xi)),\ldots,u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) = 0, \qquad (x,t)\in G. \tag{2.15}$$

We consider the following differential equation:

$$z^{(n)}(t) + \alpha_1 a(t) z(t) + \int_a^b Q(t,\xi) f(z(g_1(t,\xi)),\ldots,z(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) = 0, \qquad t\in R_+. \tag{2.16}$$

Using Theorem 2.1, we have:

Corollary 2.2. *Suppose that* (H_1)–(H_4)*hold, and every solution of differential equation* (2.16) *oscillates; then every solution of problem* (2.15)–(B_1) *oscillates in* G.

Now, we investigate the oscillation of problem (1.1) with boundary value condition (B_2).

Theorem 2.2. *Suppose that* (H_1)–(H_4) *hold; if every solution of the differential equation*

$$(z(t) - p(t)z(t-\tau))^{(n)} + h_1(t)\varphi_1(z) + \int_a^b Q(t,\xi) f(z(g_1(t,\xi)),\ldots,z(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) = 0, \qquad t\in R_+ \tag{2.17}$$

oscillates, then every solution of (1.1)–(B_2) *oscillates.*

Proof. Assume that there is a non-oscillatory solution $u(x,t)$ of the problem (1.1)–(B_2). Without loss of generally, we assume $u(x,t)$ is an eventually positive solution of (1.1)–(B_2)(if it is an eventually negative solution, the proof is similar). This means that there exists a number $T \geq 0$ such that $u(x,t) > 0$, $u(x,g_k(t,\xi)) > 0$,for $t > T$, $k = 1,\ldots,l$.

Integrating (1.1) on both sides over the domain Ω with respect to x, it follows that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_\Omega (u(x,t) - p(t)u(x,t-\tau))\,\mathrm{d}x - a(t)\int_\Omega \triangle u\,\mathrm{d}x + \int_\Omega c(x,t,u)\,\mathrm{d}x + \int_\Omega\int_a^b q(x,t,\xi) f(u(x,g_1(t,\xi)),\ldots,u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi)\,\mathrm{d}x = 0, \qquad t\geq T. \tag{2.18}$$

Using Green's formula and (B_2), we obtain

$$\int_\Omega \triangle u\,\mathrm{d}x = \int_{\partial\Omega}\frac{\partial u(x,t)}{\partial N}\,\mathrm{d}s = 0, \qquad t\geq T. \tag{2.19}$$

Combining (2.18), (2.19), (H_2), (H_3) and using Jensen's inequality, we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_\Omega (u(x,t) - p(t)u(x,t-\tau))\,\mathrm{d}x + h_1(t)\int_\Omega \varphi_1(u)\,\mathrm{d}x + \int_a^b Q(t,\xi) f\left(\frac{\int_\Omega u(g_1(t,\xi))\,\mathrm{d}x}{\int_\Omega \mathrm{d}x},\ldots,\frac{\int_\Omega u(g_l(t,\xi))\,\mathrm{d}x}{\int_\Omega \mathrm{d}x}\right)\mathrm{d}\sigma(\xi)\int_\Omega \mathrm{d}x \leq 0, \qquad t\geq T.$$

According to (2.4), it follows that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(V(t) - p(t)V(t-\tau)) + h_1(t)\varphi_1(V(t)) + \int_a^b Q(t,\xi) f(V(g_1(t,\xi)),\ldots,V(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) \leq 0, \qquad t\geq T. \tag{2.20}$$

Because $u(x,t)$ is positive, from (2.4) again, we have that $V(t)$ is positive eventually. According to Lemma 2.1, $V(t)$ is an eventually positive solution of Eq. (2.17); this contradicts the supposition. The proof is completed. □

From Theorems 2.1 and 2.2, Lemma 2.1, it is easy to prove:

Corollary 2.3. *Suppose that* (H_1)–(H_4)*hold; then every solution of the boundary value problem* (2.15)–(B_2) *oscillates if and only if every solution of*

$$z^{(n)}(t)+\int_a^b Q(t,\xi)f(z(g_1(t,\xi)),\ldots,z(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi)=0,\qquad t\in R_+ \tag{2.21}$$

is oscillatory.

Using Theorem 2.2 and Lemma 2.1, it is easy to show.

Theorem 2.3. *Suppose that* (H_1)–(H_4) *hold; if every solution of the differential equation*

$$\begin{aligned}&(z(t)-p(t)z(t-\tau))^{(n)}+h_1(t)\varphi_1(z)\\&\quad+\int_a^b f(z(g_1(t,\xi)),\ldots,z(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi)=0,\qquad t\in R_+,\end{aligned} \tag{2.22}$$

oscillates, then every solution of (1.1)–(B_3) *oscillates.*

3. Asymptotic behavior of non-oscillatory solutions

In this section, we will establish the asymptotic behavior of the non-oscillatory solutions (1.1)–(B_1), (B_2), (B_3). For convenience, we let

$$y(t)=z(t)-p(t)z(t-\tau), \tag{3.1}$$

and

$$v(x,t)=u(x,t)-p(t)u(x,t-\tau). \tag{3.2}$$

Theorem 3.1. *Suppose that* (H_1)–(H_4) *hold; if* $z(t)$ *is an eventually positive solution of*

$$(z(t)-p(t)z(t-\tau))^{(n)}+h_2(t)\varphi_2(z)+\int_a^b Q(t,\xi)f(z(g_1(t,\xi)),\ldots,z(g_l(t,\xi)))\,\mathrm{d}\sigma(\xi)=0,\quad t\in R_+, \tag{3.3}$$

then there exists a non-oscillatory solution of the boundary value problem (1.1)–(B_1), (B_2), (B_3) *such that:*

(1) *If* $\lim_{t\to\infty}z(t)=0$, *then* $\lim_{t\to\infty}u(x,t)=0$, $\lim_{t\to\infty}v(x,t)=0$, $\frac{\partial^i v(x,t)}{\partial t^i}$ *is monotonic and* $\lim_{t\to\infty}\frac{\partial^i v(x,t)}{\partial t^i}=0$, $\frac{\partial^i v(x,t)}{\partial t^i}\frac{\partial^{i+1}v(x,t)}{\partial t^{i+1}}<0$, *for* $i=1,\ldots,n-1$.
(2) *If* $z(t)\neq 0$ *as* $t\to\infty$, *then* $u(x,t)>0$, $v(x,t)>0$, *or* $u(x,t)<0$, $v(x,t)<0$, *where* $y(t)$, $v(x,t)$ *are defined by* (3.1) *and* (3.2) *respectively.*

Proof. (1) We only prove that there exists a solution of (1.1)–(B_1) which satisfies (1), (2) in Theorem 3.1; the rest is similar. Without loss of generality, assume (3.3) has an eventually positive solution $z(t)$ (if it has an eventually negative solution, the proof is similar) and let $u(x,t)=z(t)$; then it is an eventually positive solution of

$$\begin{aligned}&\frac{\partial^n(u(x,t)-p(t)u(x,t-\tau))}{\partial t^n}-a(t)\triangle u+h_2(t)\varphi_2(u)\\&\quad+\int_a^b q(x,t,\xi)f(u(x,g_1(t,\xi)),\ldots,u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi)\le 0,\qquad (x,t)\in G,\end{aligned}$$

From (H_4), it follows that $u(x,t)$ satisfies

$$\begin{aligned}&\frac{\partial^n(u(x,t)-p(t)u(x,t-\tau))}{\partial t^n}-a(t)\triangle u+c(x,t,u)\\&\quad+\int_a^b q(x,t,\xi)f(u(x,g_1(t,\xi)),\ldots,u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi)\le 0,\qquad (x,t)\in G,\end{aligned} \tag{3.4}$$

by Lemma 2.1, $u(x,t)$ is an eventually positive solution of (1.1)–(B_1), and $\triangle u = 0$, that is

$$\frac{\partial^n(u(x,t)-p(t)u(x,t-\tau))}{\partial t^n} = -c(x,t,u) - \int_a^b q(x,t,\xi) f(u(x,g_1(t,\xi)),\ldots,u(x,g_l(t,\xi)))\,\mathrm{d}\sigma(\xi) \le 0, \qquad (x,t)\in G. \tag{3.5}$$

This means that $\frac{\partial^n v(x,t)}{\partial t^n} \le 0$, and $v(x,t) \ge 0$; it follows that $\frac{\partial^i v(x,t)}{\partial t^i}$ is monotone, and $\frac{\partial^{n-1} v(x,t)}{\partial t^{n-1}}$ is decreasing. Thus it follows that $\lim_{t\to\infty} \frac{\partial^{n-1} v(x,t)}{\partial t^{n-1}}$ exists. Assume that $\lim_{t\to\infty} \frac{\partial^{n-1} v(x,t)}{\partial t^{n-1}} = L < 0$; then there exists a number $L_1 < 0$, and a number $T > t_0$, such that

$$\lim_{t\to\infty} \frac{\partial^{n-1} v(x,t)}{\partial t^{n-1}} \le L_1, \qquad t > T.$$

This contradicts $\lim_{t\to\infty} v(x,t) = 0$.

On the other hand, if $L > 0$, then $\frac{\partial^{n-1} v(x,t)}{\partial t^{n-1}} \ge L > 0$, which contradicts $\lim_{t\to\infty} v(x,t) = 0$ also. Thus, we have $\lim_{t\to\infty} \frac{\partial^{n-1} v(x,t)}{\partial t^{n-1}} = 0$. Moreover, $\frac{\partial^{n-1} v(x,t)}{\partial t^{n-1}} > 0$, $t > T$; this means that $\frac{\partial^{n-2} v(x,t)}{\partial t^{n-2}}$ is increasing for $n > 2$. Let $\lim_{t\to\infty} \frac{\partial^{n-2} v(x,t)}{\partial t^{n-2}} = L_2$. It is obvious that $L_2 = 0$; by the similar proof, we can easily prove that

$$\lim_{t\to\infty} \frac{\partial^i v(x,t)}{\partial t^i} = 0, \qquad i = 1,\ldots,n-1,$$

and

$$\frac{\partial^i v(x,t)}{\partial t^i} \frac{\partial^{i+1} v(x,t)}{\partial t^{i+1}} < 0, \qquad i = 1,\ldots,n-1.$$

(2) If $\lim_{t\to\infty} z(t) \neq 0$, then $\limsup_{t\to\infty} z(t) > 0$, because $u(x,t) = z(t)$ is eventually positive and $\limsup_{t\to\infty} u(x,t) = U(x) > 0$. Assume that $v(x,t)$ is not eventually positive; then it is eventually negative. If $z(t)$ is unbounded, then there exists a sequence $t_k \to \infty$, and a $x_0 \in \Omega$ such that $v(x_0,t_k) = max_{t\le t_k} v(x_0,t)$, and $\limsup_{t_k\to\infty} z(t_k) = \limsup_{t_k\to\infty} u(x,t_k) = \infty$. By (3.2), and taking $x_0 \in \Omega$, we have

$$\begin{aligned} v(x_0,t_k) &= u(x_0,t_k) - p(t_k)u(x_0,t_k-\tau) \\ &\ge u(x_0,t_k)[1-p(t_k)] \ge u(x_0,t_k)[1-p]. \end{aligned} \tag{3.6}$$

Taking the limits of both sides of (3.6) as $t_k \to \infty$, we can get $\limsup v(x_0,t) = \infty$; this contradicts $v(x,t) < 0$. If $v(x,t)$ is bounded, then there exists a sequence $t_k \to \infty$ such that

$$\lim_{t_k\to\infty} v(x,t_k) = \limsup_{t\to\infty} v(x,t) = V(x) > 0.$$

Taking $x_0 \in \Omega$, we have $V(x_0) > 0$. Combining (3.2) and (3.6) and taking the limit as $t_k \to \infty$, we have

$$\begin{aligned} 0 &\ge \lim_{t_k\to\infty} v(x_0,t_k) \\ &= \lim_{t_k\to\infty} u(x_0,t_k) - \lim_{t_k\to\infty} (p(t_k)u(x_0,t_k-\tau)) \\ &\ge \lim_{t_k\to\infty} u(x_0,t_k)[1-p(t_k)] \\ &\ge \lim_{t_k\to\infty} u(x_0,t_k)[1-p] > 0. \end{aligned} \tag{3.7}$$

This is also a contradiction. The proof is completed. □

Example 3.1. Consider the following boundary value problem:

$$\frac{\partial^n(u(x,t) - \frac{1}{3e^4}u(x,t-4))}{\partial t^n} - u_{xx} + \frac{1}{3e^4}u(x,t-4) + \frac{2}{3\pi^2 e^{-e^4}} \int_0^\pi e^{-(t-e^{-t})}\xi e^{-u(x,t-4)}\,\mathrm{d}\xi = 0, \qquad (x,t)\in(0,\pi)\times R_+ \equiv G, \tag{3.8}$$

with the following boundary value condition:

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(\pi,t)}{\partial x} = 0, \qquad t \geq 0, \tag{3.9}$$

where n is an odd number, $f = \frac{2}{3\pi^2 e^{-e^4}} \int_0^\pi e^{-(t-e^{-t})} \xi e^{-u(x,t-4)} \mathrm{d}\xi = \frac{1}{3e^{-e^4}} e^{-(t-e^{-t})} e^{-u(x,t-4)}$, $p = \frac{1}{3e^4}, c = \frac{1}{3e^4} u(x, t-4)$; its corresponding delay differential equation is as follows:

$$\left(z(t) - \frac{1}{3e^4} z(t-4)\right)^{(n)} + \frac{1}{3e^4} z(t-4) + \frac{1}{3e^{-e^4}} e^{-(t-e^{-t})} e^{-z(t-4)} = 0, \qquad t \in [t_0, +\infty), t_0 \in R, \tag{3.10}$$

and it is obvious that $z = e^{-t}$ is an eventually positive solution of (3.10), and $\lim_{t\to\infty} z(t) = 0$. It follows that $u(x,t) = z(t) = e^{-t}$ is an eventually positive solution of problem (3.8) and (3.9), and $\lim_{t\to\infty} u(x,t) = 0$, $v(x,t) = \frac{2}{3} e^{-t}$, $\lim_{t\to\infty} v(x,t) = 0$, $\frac{\partial^k v(x,t)}{\partial t^k} = (-1)^k \frac{2}{3} e^{-t}$; it is easy to show that $\frac{\partial^k v(x,t)}{\partial t^k} \frac{\partial^{k+1} v(x,t)}{\partial t^{k+1}} < 0$, $\lim_{t\to\infty} \frac{\partial^k v(x,t)}{\partial t^k} = 0$, $k = 1, \ldots, n-1$.

Acknowledgements

The author thanks the referees and Professor Yunqing Huang who read the manuscript carefully and put forward some helpful opinions.

References

[1] D.P. Mishev, Oscillation properties of the hyperbolic differential equations with “maximum”, Hiroshima Math. J. 16 (1986) 77–83.
[2] D.P. Mishev, D.D. Bainov, Oscillation properties of the solutions of a class of hyperbolic equations of neutral type, Funkcial Ekvac. 29 (2) (1986) 213–218.
[3] N. Yoshida, On the zeros of solutions of hyperbolic equations of neutral types, Differ. Integral Equ. 3 (1990) 155–160.
[4] Y.K. Li, Oscillation of systems of hyperbolic differential equations with deviating argument, Acta Math. Sinica. 40 (1) (1997) 100–105.
[5] X.L. Fu, Oscillation of a class of parabolic differential equations, Acta Math. Sci. 13 (3) (1993) 316–322.
[6] B.T. Cui, B.S. Lalli, Y.H. Yu, Oscillation of hyperbolic differential equations with deviating arguments, Acta Math. Appl. Sinica. 11 (4) (1995) 369–377.
[7] B.T. Cui, Oscillation properties of the solutions of hyperbolic equations with deviating arguments, Demonstratio Math. 29 (1996) 61–68.
[8] Y.H. Yu, X.L. Fu, oscillations of second order nonlinear neutral equations with continuous distributed deviating arguments, Rad Mat. 7 (1) (1991) 167–176.
[9] Z.G. Ouyang, Necessary and sufficient conditions for oscillation of odd order neutral delay parabolic differential equations, Appl. Math. Lett. 16 (2003) 1039–1045.
[10] P. Wang, Oscillatory criteria of nonlinear hyperbolic equations with continuous deviating arguments, Appl. Math. Comput. 106 (1999) 163–169.
[11] V.S. Vladimirov, Equations of Mathematics Physics, Nauka, Moscow, 1981.