Three Moves on Signed Surface Triangulations

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We consider finite triangulations of surfaces with signs attached to the faces. Such a signed triangulation is said to have the Heawood property if, at every vertex $x$, the sum of the signs of the faces incident to $x$ is divisible by 3. For a triangulation $G$ of the sphere, Heawood signings are essentially equivalent to proper 4-vertex-colorings of $G$. We introduce three moves on signed surface triangulations which preserve the Heawood property. We then prove that every Heawood signed triangulation of the sphere can be obtained from a Heawood signed triangle by a suitable sequence of our moves.

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1. INTRODUCTION

Let $G = (V, E)$ be a triangulation of a closed surface $S$. We mean that $G$ is a finite graph, loop-free but possibly with multiple edges, embedded in the surface $S$ and subdividing it into triangular faces (i.e., all faces have exactly three incident edges). In this paper we are interested in the triangulations of the sphere $S^2$. We denote by $\mathcal{F}(G)$ the set of faces of a triangulation $G$ of the sphere, and by $\mathcal{F}_v$ the subset of faces incident to some vertex $v \in V$. A polygon $T$ is an induced subgraph of $G$ where all faces are faces of $G$ except one, the boundary face which is a cycle of $G$ (see Fig. 1).

By Euler’s formula, we have that a triangulation $G$ of the sphere has $3n - 6$ edges (where $n$ denotes the number of vertices in $G$) and so:
**Proposition 1.1.** If $G$ is not a triangle then there exist at least 4 vertices of degree $\leq 5$ in any triangulation of the sphere.

If $G$ is a triangulation of the sphere then the neighborhood $N(x)$ of a vertex $x$ induces a cycle $x_1, \ldots, x_t$ (with edges $x_ix_{i+1}$ and $t \geq 2$) in $G$. We will denote $F_{x_ix_{i+1}}$ the triangle formed by $x_i, x_{i+1}$. 

- A coloring of $G$ is a mapping $c: V(G) \to \{1, \ldots, 4\}$ such that for all adjacent vertices $u, v$ we have $c(u) \neq c(v)$. A strict 4-coloring of $G$ is a 4-coloring of $G$ which uses the four colors.

- A signing of $G$ is a mapping $s: \mathcal{F}(G) \to \{-1, +1\}$ (in the figures we will represent the sign as in $\{+, -\}$). We denote by $\bar{s}$ the signing obtained from $s$ by changing all signs and $s(v)$ the sum of the signs of the faces $F$ incident to $v$, i.e.,

$$s(v) = \sum_{F \in \mathcal{F}} s(F).$$

A signing $s$ is a Heawood signing if at every vertex $v$ of $G$, one has $s(v) \equiv 0 \mod 3$. This notion is dual to the notion of “Heawood vertex characters” for cubic graphs on $S$, where signs are attached to the vertices of the graph, with the condition that the sum of the signs of the vertices around every region is divisible by 3.

- A valuation of $G$ is a mapping $v: E \to \{0, 1\}$. Let $s$ be a Heawood signing of a triangulation $G$ of the sphere. We define the Heawood valuation $v$ of $G$ associated to $s$ by: for all edge $xy \in E$, set $v(xy) = 0$ iff the two triangles of $G$ containing $xy$ have the same sign under $s$ (see Fig. 2).

Now, from a Heawood valuation $v$ of $G$, we may construct a unique coloring $c$ of $G$ (up to permutation of colors). First color with 3 colors an
arbitrary triangle of $G$. Now apply the following rule: for all quadrilateral $xyzt$ composed from two triangles of $G$ adjacent to the edge $yt$, color the vertices $x$ and $z$ with the same color iff $v(yt) = 1$.

Observe that to a 3-coloring of $G$ there corresponds a Heawood valuation where all edges are valuated 1; the Heawood signing corresponding to this valuation will be called *alternating* (this is a 2-coloring of the faces of $G$).

Moreover, Heawood discovered (see [5]) that, for a triangulation $G$ of the sphere $S^2$, every 4-coloring yields a Heawood signing, and conversely. Hence, Heawood signings, Heawood valuations and 4-colorings are equivalent notions for a triangulation of the sphere. So in the sequel, if $G$ is a triangulation of the sphere then $(G, e)$ means as well a Heawood signing $s(G, e)$, the valuation $v(G, e)$ associated to $s(G, e)$ and the coloring $c(G, e)$ associated to $v(G, e)$. Such a pair will be called a *Heawood triangulation*; moreover if $c(G, e)$ is a strict 4-coloring of $G$ then we specify a *strict Heawood triangulation*.

Let $(G, e)$ be a Heawood triangulation of the sphere. We set $s = s(G, e)$ and $v = v(G, e)$. We consider the following four moves (see Fig. 3):

(I) Let $Q = xyzt$ be a quadrilateral of $G$ formed by the two faces $F_1$, $F_2$ of $G$ adjacent to $yt$. The *flip* is only defined if $s(F_1) = s(F_2) = \mu$. Removing $yt$ and replacing it by the opposite edge $xz$, we obtain a new triangulation $G'$. Let $F'_1$, $F'_2$ be the two faces adjacent to the new edge $xt$ of $G'$. Set $s'(F'_1) = s'(F'_2) = -\mu$, whereas for the other inner faces $F$ of $G'$, which are also inner faces of $G$, there is no sign change, i.e. $s'(F) = s(F)$. As observed in [3], $s'$ is still a Heawood signing of $G'$.
(II) Let $F = xyz$ be a face of $G$ and $s(F) = \mu$. The add of a vertex $a$ in $F$ consists to replace the face $F$ by the three faces $F_{xy} = axy$, $F_{yz} = ayz$ and $F_{xz} = axz$ and to set $s'(F_{xy}) = s'(F_{yz}) = s'(F_{xz}) = -\mu$, $s'(A) = s(A)$ for every face $A \neq F$. The resulting signing $s'$ is clearly still a Heawood signing of the triangulation $G'$. We will denote $G'$ by $G + a$.

(III) The deletion of a vertex $a$ of degree 3 is the converse of move (II).

(IV) Let $x$ be a vertex of a triangulation of the sphere and let $x_1, \ldots, x_t$ be the cycle induced by the neighborhood of $x$. If the faces $F_{x_{i}x_{i+1}}$
have alternating signs then the *switch* at \( x \) consists to modify \( s \) by changing the sign of \( F_{x_{i+1} y} \) for every \( i \). Again, he resulting signing is clearly still a Heawood signing of \( G \).

Observe that the move (I) does not modify the coloring of the vertices of \( G \), and modifies the valuation as follows: \( v'(xy) = 1 - v(xy) \), \( v'(yz) = 1 - v(yz) \), \( v'(zt) = 1 - v(zt) \), \( v'(tx) = 1 - v(tx) \), \( v'(xz) = v(yt) = 0 \) and \( v'(ab) = v(ab) \) for all other edges of \( G \). The move (II) consists to add a new vertex with a fourth color distinct from the three colors of \( x, y, z \). For the valuation this consists to change all the valuations of the edges \( xy, yz, zt \) and to assign 0 to the edges \( ax, ay \) and \( az \) (conversely for move (III)). The move (IV) consists to change the color of \( x \) and to change the valuation of edges \( x_i x_{i+1} \).

The link between signed flips and colorings of a triangulation was first introduced independently by Eliahou [3] and in dual form by Kryuchkov [6]. In [4], the authors proved the following theorem which settled a conjecture proposed in [3]:

**Theorem 1.2 [4].** Let \( n \geq 3 \), and let \( T_1, T_2 \) be two triangulations of a convex plane polygon \( P \). There is a sequence of signed flips from \( T_1 \) to \( T_2 \) if and only if there is a 4-coloring of \( P \) which induces a proper coloring of \( T_1 \) and \( T_2 \).

Let \((G, e), (G', e')\) be two Heawood triangulations of the sphere. We write \((G, e) \sim_0 (G', e')\), if we can obtain \((G', e')\) from \((G, e)\) by applying moves (I) and/or (II). We write \((G, e) \sim_1 (G', e')\), if we can obtain \((G', e')\) from \((G, e)\) by applying moves (I), (II) and once (III) or (IV). Notice that \( \sim_0 \) is a transitive relation.

Let \((T_0, e_T)\) be the "smallest" triangulation of the sphere, i.e., where \( T_0 \) is a triangle, the inner face is signed +1 and the other −1. Clearly this signing is the unique Heawood signing of \( T_0 \). Let \((K_0, e_K)\) be the signed triangulation of the sphere obtained from \((T_0, e_T)\) by applying once the move (II) (see Fig. 4).

![Prime triangulations](image_url)
The aim of this paper is to prove that \((K_0, e_K)\) is the prime signed triangulation for all the strict Heawood triangulations under moves (I) and (II), and that \((T_0, e_T)\) is the prime signed triangulation for all the Heawood triangulations under moves (I), (II) and (III) or (IV).

**Theorem 1.3.** Let \((G, e)\) be a Heawood triangulation of the sphere. Then \((K_0, e_K)\) is \(\alpha_0(G, e)\) if and only if \(c(G, e)\) is a strict 4-coloring of \(G\).

**Corollary 1.4.** Let \((G, e)\) be a Heawood triangulation of the sphere. Then \((T_0, e_T)\) is \(\sigma_1(G, e)\) if and only if \(c(G, e)\) is a 4-coloring of \(G\).

By the 4-colors theorem [1, 2], we obtain

**Corollary 1.5.** For all triangulations of the sphere \(G\), there is a Heawood triangulation \((G, e)\) which satisfies \((T_0, e_T)\) is \(\sigma_1(G, e)\).

Moreover Theorem 1.3 implies a result due to Penrose [7] and Vigneron [8]:

**Corollary 1.6.** Let \(s\) be a Heawood signing of a triangulation of the sphere. Then

\[
\sum_{F \in \mathcal{F}} s(F) \equiv 0 \mod (4).
\]

The proofs are given in the next section.

2. PROOFS

**Proof of Theorem 1.3.** Observe that the flip does not modify the coloring \(c(G, e)\), moreover if \(c(G, e)\) is a strict 4-coloring then adding a new vertex \(x\) to \(G\) provides still a strict 4-coloring of \(G + x\). Hence since \(c(K_0, e_K)\) is a strict 4-coloring of \(K_0\), if \((K_0, e_K)\) is \(\alpha_0(G, e)\) then \(c(G, e)\) is a strict 4-coloring of \(G\).

For the converse, assume that there is counter-example to Theorem 1.3. Let \((G, e)\) be a minimal such counter-example. By Proposition 1.1, there are at least 4 vertices of degree \(\leq 5\), since \(G\) has more than 4 vertices. Then at least one vertex \(x\) of degree \(\leq 5\) satisfies that \(G - x\) is still strictly 4-colored. Such a vertex will be called a removable vertex. Consider \(\mathcal{F}\) the set of Heawood triangulations of the sphere obtained from \((G, e)\) by applying move (I). Observe that all triangulations in \(\mathcal{F}\) are still minimal counter-examples of Theorem 1.3 with \(x\) removable. We restrict our attention...
to the subset $\mathcal{G}(x)$ of Heawood triangulations of $G$ where the degree of $x$ remains $\leq 5$. By hypothesis $(G, e) \in \mathcal{G}(x)$. Our aim is to prove that $\mathcal{G}(x) = \emptyset$, thus yielding a contradiction. First we partition $\mathcal{G}(x)$ in two disjoint subsets $\mathcal{G}_1$ and $\mathcal{G}_2$: a Heawood triangulation of $\mathcal{G}_1$ verifies that the neighborhood of $x$ is 3-colored with 3 colors and $\mathcal{G}_2 = \mathcal{G} \setminus \mathcal{G}_1$. Remark that in all triangulations of $\mathcal{G}_1$, the vertex $x$ has degree 3, 4, or 5 and in all triangulations of $\mathcal{G}_2$, the vertex $x$ has degree 2 or 4.

First we prove that $\mathcal{G}_1 = \emptyset$. Let $(G', e') \in \mathcal{G}_1$. We may assume that $x$ has color 4. If $x$ has degree 3 in $G'$ then remove it. By minimality of $G'$ and by applying (II), we have $(K_0, e_K) \circ (G' - x, e') \circ (G', e')$. By transitivity of $\alpha_0$, this yields a contradiction.

If $x$ has degree 5 in $G'$ then let $C = x_1, x_2, x_3, x_4, x_5$ be the cycle induced by the neighborhood of $x$. Every vertex $x_i$ has a color distinct from the one of $x$. Since there is a unique, up to rotation, 3-coloring of the cycle with 5 vertices, we may assume that $x_1$ is the only vertex colored 3. Hence, if we flip the edges $xx_2$ and $xx_5$ then we obtain by move (I) a new Heawood triangulation $(G'', e'')$ where $x$ has degree 3 and so we conclude like in the previous case.

If $x$ has degree 4 in $G'$ then let $C = x_1, x_2, x_3, x_4$ be the cycle induced by the neighborhood of $x$. Every vertex $x_i$ has a color distinct from the one of $x$. Since there is a unique, up to rotation, 3-coloring of the cycle with 4 vertices, we may assume that $x_1$ is the only vertex colored 3. Hence, if we flip the edge $xx_2$ then we obtain by move (I) a new Heawood triangulation $(G'', e'')$ where $x$ has degree 3 and so we conclude like in the previous case.

Now, we prove that $\mathcal{G}_2 = \emptyset$. For any triangulation in $\mathcal{G}_2$, we let $C$ the cycle induced by the neighborhood of $x$. Recall that $C$ is a four or two cycle. And we may assume that $x$ has color 3 and $C$ is 2-colored with colors 1 and 2. Among all elements of $\mathcal{G}_2$, choose one $(G', e')$ which minimizes the number of faces of a maximal polygon $P(G')$ induced by the vertices of colors 1, 2, and 3 and which contains $x \cup C$. This means that all edges of $P(G')$ are valuated 1 except the edges of the boundary of $P(G')$ which are valuated 0. Now, flip any edge $e$ of the boundary of $P(G')$ and so we obtain a new Heawood triangulation $(G'', e'')$ for which either $(G'', e'') \in \mathcal{G}_2$ and $G''$ has a maximal polygon $P(G'')$ with number of faces smaller than $P(G')$, or $x$ has degree 3 or 5 in $G''$ and so $(G'', e'') \in \mathcal{G}_1 = \emptyset$ (in this case $e$ is an edge of $C$). In both cases we obtain a contradiction. 

Proof of Corollary 1.4. First observe that if $c(G, e)$ is a 4-coloring then the Heawood coloring $c(G', e')$ obtained from $c(G, e)$ by applying (III) or (IV) is still a 4-coloring (not necessarily strict). For the converse, it is sufficient to observe that by Theorem 1.3 if $c(G, e)$ is a strict 4-coloring of $G$ then we are done. Else apply once (III) or (IV) to $(G, e)$ in order to obtain a new Heawood coloring $c(G', e')$ which is strictly 4-colored.
Proof of Corollary 1.6. By Corollary 1.4, it is sufficient to check that
\((T_0, e_0)\) satisfies the requirement and that moves (I), (II), (III), and (IV)
preserve this property.

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