Journal of Combinatorial Theory, Series A **97**, 203–224 (2002)
doi:10.1006/jcta.2001.3207, available online at http://www.idealibrary.com on IDEAL®

Inverse Relations and Schauder Bases

I-Chiau Huang[1]

Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan, Republic of China
E-mail: ichuang@math.sinica.edu.tw

Communicated by George Andrews

Received September 5, 2000; published online November 5, 2001

The concept of inter-changes of Schauder bases is used to interpret inverse relations for sequences. For a given power series, the interplay between different representations by Schauder bases can result in combinatorial identities, new or known. Local cohomology residues and local duality are used for computations. The viewpoint of Riordan arrays is examined using Schauder bases. © 2001 Elsevier Science

Key Words: combinatorial identity; inverse relation; Lagrange inversion formula; local duality; power series ring; residue; Riordan array; Riordan group; Schauder basis.

1. INTRODUCTION

Let κ be a field. An inverse relation is a pair of identities of the form

$$\begin{cases} b_n = \sum_{k=0}^{n} c_{nk} a_k \\ a_n = \sum_{k=0}^{n} d_{nk} b_k, \end{cases}$$

where $a_i, b_i, c_{ji}, d_{ji} \in \kappa$ $(0 \leqslant i \leqslant j)$. For convenience, we let $c_{ji} = d_{ji} = 0$, for $i > j$. Usually, we assume that a_k's (resp. b_k's) are linearly independent over a certain subfield of κ which contains c_{nk} and d_{nk}. This independence implies the following orthogonal property

$$\sum_{k=0}^{\infty} c_{mk} d_{kn} = \delta_{mn}, \tag{1}$$

where δ_{mn} is the Kronecker delta function.

[1] Supported in part by NSC Grant 88-2115-M-001-011.

0097-3165/01 $35.00

Inverse relations are frequently encountered in combinatorial problems and have been extensively studied by Riordan [22]. Systematic and unified methods were developed by Egorychev [6] using contour integrals and by Krattenthaler [12] using operator methods. See also [28] for Möbius inversion over partially ordered sets and [19] for computational aspects. More recent work on one-dimensional inverse relations appears in [13, 21].

In this article, we propose a new approach to inverse relations. This approach, based on inter-change of Schauder bases, is conceptually elementary. It stems from the observation that formal power series rings can be variable-free. Traditionally, a formal power series ring is defined in terms of variables. However, it is truly a local ring characterized by certain additional algebraic properties free from the notion of variables. To illuminate this variable-free phenomenon, it is helpful to first look at vector spaces: An n-dimensional vector space over a field κ is an abelian group with certain additional algebraic structures. Its elements, once a basis $u_1, ..., u_n$ is chosen, can be represented uniquely as

$$\sum a_i u_i,$$

where $a_i \in \kappa$. One knows that there is no canonical choice of basis. In a formal power series ring, variables (i.e., regular system of parameters) play the role as bases in vector spaces. Once variables $X_1, ..., X_n$ are chosen for a (Krull) n-dimensional formal power series ring over a field κ, every element in this ring can be represented uniquely as

$$\sum a_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n},$$

where $a_{i_1 \cdots i_n} \in \kappa$. As the case of vector spaces, a formal power series ring, in its natural guise, has no canonical choice of variables. See the beginning of Section 2 for the one-dimensional case.

Interplay of different sets of variables has combinatorial significance. Let $X_1, ..., X_n$ and $Y_1, ..., Y_n$ be two sets of variables of an n-dimensional formal power series ring over a field κ with the following relations:

$$Y_j = \sum c^{(j)}_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n} \quad (c^{(j)}_{i_1 \cdots i_n} \in \kappa)$$

$$X_j = \sum d^{(j)}_{i_1 \cdots i_n} Y_1^{i_1} \cdots Y_n^{i_n} \quad (d^{(j)}_{i_1 \cdots i_n} \in \kappa)$$

It is the theme of Lagrange inversion formulae to find explicit formulae of $d^{(j)}_{i_1 \cdots i_n}$ in terms of $c^{(j)}_{i_1 \cdots i_n}$. Such variable-free viewpoint is supported by local cohomology residues, which provide a convenient framework to compute the coefficients $d^{(j)}_{i_1 \cdots i_n}$, see [10].

In this article, we focus on a one-dimensional formal power series ring R over a field. We introduce Schauder bases, which represent elements of R more flexible than variables do. Such generalization enables us to produce inverse relations. Our main result in Theorem 2.1 below provides a new characterization of inverse relations in terms of Schauder bases. The first part of our result says: *Representations of an element in R by two strictly monotone Schauder bases give rise to an inverse relation with the orthogonal property.*

Compared with the approaches to inverse relations by the umbral calculus [24, Theorem 3.1] or by the Riordan group [26, Section 4], our approach goes further by characterizing inverse relations with the orthogonal property. The converse of the above statement holds, that is, *every inverse relation with the orthogonal property comes from representations of an element in R by two strictly monotone Schauder bases.*

A strictly monotone Schauder basis $\{g_i\}$ and an infinite lower triangular matrix (c_{ij}) with $c_{ii} \neq 0$ give rise to a new strictly monotone Schauder basis $\{f_i\}$. One may ask whether another infinite lower triangular matrix (d_{ij}) with $d_{ii} \neq 0$, together with $\{f_i\}$, gives rise to $\{g_i\}$. This is equivalent to the orthogonal property (1), or in terms of matrices, equivalent to that (d_{ij}) is the inverse of (c_{ij}). Solving (d_{ij}) from an explicitly given (c_{ij}) may lead to interesting inverse relations. See [13] for an inverse relation obtained by an operator method and [19, Theorem 1.7] for a characterization of (d_{ij}).

For Schauder bases to be of significant value, it is necessary to have some technical tools for computing their coefficients. In this article, we recall some definitions and properties of local cohomology residues, which serve as our computing tool. Local duality is used to translate the method in [13] to our language. Besides inverse relations, our applications of Schauder bases include an Abel identity, a Gould identity, their analogues and a generalization. The Abel and Gould identities were also generalized by Sprugnoli [27] using Riordan arrays. We will examine the concept of Riordan arrays using Schauder bases. In this article, terminology of commutative algebra is used. The reader is referred to [15, Chap. IV, Sect. 9] for a quick introduction and to [17] for further details.

2. SCHAUDER BASES

A one-dimensional formal power series ring R over a field κ is a complete local ring containing κ, whose maximal ideal $\mathfrak{m}$ is generated by one element, and satisfying the condition that the canonical map from κ to the residue field of R is an isomorphism. The maximal ideal $\mathfrak{m}$ of R gives the notion of convergence of a sequence (g_i) in R: a sequence (g_i) is said to be converging to $g \in R$ if and only if, for any power $\mathfrak{m}^\nu$, there exists an integer N such that for all $n \geqslant N$ we have

$$g_n - g \in \mathfrak{m}^\nu.$$

If X is a generator for the maximal ideal, every element g in R can be written uniquely as a power series over κ with X acting as a variable

$$g = \sum_{i=0}^{\infty} a_i X^i \qquad (a_i \in \kappa)$$

(i.e., $g = \lim_{n\to\infty} \sum_{i=0}^{n} a_i X^i$). In such case, we write $R = \kappa[[X]]$. This notation not only indicates that R is a one-dimensional formal power series ring over κ but also specify a variable X.

Borrowing the terminology from Banach spaces, we define *Schauder bases*.

DEFINITION 2.1. Let R be a one-dimensional formal power series ring over a field κ. A sequence (f_i) of R is called a Schauder basis for R if for every $g \in R$ there exists a unique sequence $(a_i) \subset \kappa$ such that

$$g = \sum_{i=0}^{\infty} a_i f_i.$$

We call a_i the coefficient of g at f_i with respect to the Schauder basis (f_i).

The coefficients of representations by a Schauder basis (f_i) are κ-linear in the following sense: If $a \in \kappa$ and

$$g_1 = \sum_{i=0}^{\infty} a_i^{(1)} f_i \qquad (a_i^{(1)} \in \kappa),$$

$$g_2 = \sum_{i=0}^{\infty} a_i^{(2)} f_i \qquad (a_i^{(2)} \in \kappa),$$

then

$$g_1 + g_2 = \sum_{i=0}^{\infty} (a_i^{(1)} + a_i^{(2)})\, f_i,$$

$$a g_1 = \sum_{i=0}^{\infty} (a a_i^{(1)})\, f_i.$$

We look at some examples of Schauder bases. Let $R = \kappa[[X]] = \kappa[[Y]]$.

EXAMPLE 2.1. The sequence (X^i) is a Schauder basis, which we name an ordinary Schauder basis, or an *ordinary basis* for short.

EXAMPLE 2.2. Given $p \in \mathbb{Z}$, the sequence $(Y^i(1+X)^p)$ is a Schauder basis, which we name a Gould–Schauder basis, or a *Gould basis* for short. See Identity 4.1 for the choice of such name.

Assume furthermore that κ is of characteristic zero.

EXAMPLE 2.3. The sequence $(X^i/i!)$ is a Schauder basis, which we name an exponential Schauder basis, or an *exponential basis* for short.

EXAMPLE 2.4. Given $p \in \kappa$, the sequence $(Y^i e^{pX})$ is a Schauder basis, which we name an Abel–Schauder basis, or an *Abel basis* for short. See Identity 4.1 for the choice of such name.

EXAMPLE 2.5. Given $p \in \mathbb{Z}$, the sequence $(Y^i(X/(e^X-1))^p)$ is a Schauder basis, which we name a Bernoulli–Schauder basis, or a *Bernoulli basis* for short.

The above examples of Schauder bases are of the type $(Y^i\varphi_i)$, where $\varphi_i \in R = \kappa[[Y]]$ is invertible. They are encountered in concrete problems and deserve a name.

DEFINITION 2.2. Let R be a one-dimensional formal power series ring over a field κ. Let $\mathfrak{m}$ be the maximal ideal of R. A Schauder basis (f_i) is called strictly monotone if $f_i \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$.

If $R = \kappa[[X]]$, it is easy to see that a Schauder basis (f_i) is strictly monotone if and only if it is of the form $(X^i\varphi_i)$, where $\varphi_i \in R$ is invertible.

The method of generating functions is a powerful tool in combinatorics. To transform a sequence $a_0, a_1, a_2, \ldots$ in a combinatorial problem to a power series for analytic or algebraic machineries, one usually uses the ordinary generating function

$$a_0 + a_1 X + \cdots + a_n X^n + \cdots$$

or the exponential generating function

$$a_0 + a_1 X + \cdots + a_n \frac{X^n}{n!} + \cdots.$$

These two generating functions correspond to representations by the ordinary basis (X^i) and the exponential basis $(X^i/i!)$ respectively. The concept of Schauder bases suggests that generating functions may be put in a more general context. The following theorem underlies the importance of Schauder bases.

THEOREM 2.1. *Let* (f_i) *and* (g_i) *be two strictly monotone Schauder bases for R with the following inter-relations:*

$$f_i = c_{0i}\, g_0 + c_{1i}\, g_1 + c_{2i}\, g_2 + c_{3i}\, g_3 + \cdots \tag{2}$$

$$g_i = d_{0i}\, f_0 + d_{1i}\, f_1 + d_{2i}\, f_2 + d_{3i}\, f_3 + \cdots \tag{3}$$

$(c_{ji}, d_{ji} \in \kappa)$. *Then the following orthogonal relation holds.*

$$\sum_{k=0}^{\infty} c_{mk} d_{kn} = \delta_{mn}. \tag{4}$$

Given $h \in R$ *represented as*

$$h = a_0 f_0 + a_1 f_1 + a_2 f_2 + \cdots = b_0 g_0 + b_1 g_1 + b_2 g_2 + \cdots \tag{5}$$

$(a_j, b_j \in \kappa)$, *the following pair of identities holds.*

$$\begin{cases} b_n = \sum_{k=0}^{n} c_{nk} a_k \\ a_n = \sum_{k=0}^{n} d_{nk} b_k. \end{cases} \tag{6}$$

Conversely, given any pair of identities (6) *with the orthogonal property* (4) *and the convention* $c_{ji} = d_{ji} = 0$ *for* $i > j$, *there exist strictly monotone Schauder bases* (f_i) *and* (g_i) *for R and an element* $h \in R$ *satisfying* (2), (3), *and* (5).

Proof. Let (f_i) and (g_i) be strictly monotone Schauder bases for R with the relations (2) and (3). Then $c_{ji} = d_{ji} = 0$ for $i > j$. So $\sum_{k=0}^{\infty} d_{kn} c_{mk}$ is a finite sum. We compare g_n and $\sum_{m=0}^{\infty} (\sum_{k=0}^{\infty} d_{kn} c_{mk})\, g_m$: For any $i > 0$, their images in $R/\mathfrak{m}^{i+1}$ are the same as the images of

$$\sum_{k=0}^{i} d_{kn} f_k = \sum_{k=0}^{i} d_{kn} \left(\sum_{m=0}^{\infty} c_{mk}\, g_m \right) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{i} d_{kn} c_{mk} \right) g_m$$

and

$$\sum_{m=0}^{i}\left(\sum_{k=0}^{i} d_{kn}c_{mk}\right)g_m = \sum_{m=0}^{i}\left(\sum_{k=0}^{\infty} d_{kn}c_{mk}\right)g_m.$$

Therefore

$$g_n - \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty} d_{kn}c_{mk}\right)g_m \in \bigcap_{i=0}^{\infty} \mathfrak{m}^i = (0).$$

Equating coefficients, we get

$$\sum_{k=0}^{\infty} c_{mk}d_{kn} = \delta_{mn}.$$

Now we compare $\sum_{n=0}^{\infty} b_n g_n$ and $\sum_{n=0}^{\infty} (\sum_{k=0}^{\infty} a_k c_{nk})\, g_n$: For any $i > 0$, their images in $R/\mathfrak{m}^{i+1}$ are the same as the images of

$$\sum_{k=0}^{i} a_k f_k = \sum_{k=0}^{i} a_k\left(\sum_{n=0}^{\infty} c_{nk}g_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{i} a_k c_{nk}\right)g_n$$

and

$$\sum_{n=0}^{i}\left(\sum_{k=0}^{i} a_k c_{nk}\right)g_n = \sum_{n=0}^{i}\left(\sum_{k=0}^{\infty} a_k c_{nk}\right)g_n.$$

Therefore

$$\sum_{n=0}^{\infty} b_n g_n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} a_k c_{nk}\right)g_n.$$

Equating coefficients, we get

$$b_n = \sum_{k=0}^{\infty} a_k c_{nk} = \sum_{k=0}^{n} c_{nk}a_k.$$

Similarly, we have

$$a_n = \sum_{k=0}^{n} d_{nk}b_k.$$

Conversely, given the inverse relation (6) with the orthogonal property (4) and the convention $c_{ji} = d_{ji} = 0$ for $i > j$, we consider the lower triangular matrices $C=(c_{ji})_{n\times n}$ and $D=(d_{ji})_{n\times n}$ for a fixed $n\in\mathbb{N}$. As the product of C and D is the $n\times n$ unit matrix, we know $c_{nn}\neq 0$ for any n. Let

$$f_i := c_{0i}+c_{1i}X+c_{2i}X^2+c_{3i}X^3+\cdots.$$

Then (f_i) is a Schauder basis. Let

$$g_i = X^i$$
$$h = b_0+b_1X+b_2X^2+b_3X^3+\cdots.$$

As in the previous paragraph, we may switch summations:

$$\sum_{n=0}^{\infty} d_{ni} f_n = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} d_{ni}c_{\ell n}X^{\ell} = \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} d_{ni}c_{\ell n}X^{\ell} = g_i$$

$$\sum_{n=0}^{\infty} a_n f_n = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} a_n c_{\ell n}X^{\ell} = \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} a_n c_{\ell n}X^{\ell} = \sum_{\ell=0}^{\infty} b_{\ell}X^{\ell} = h. \quad \blacksquare$$

3. COMPUTING TECHNIQUES

Our main techniques of computation is based on local cohomology residues. We recall some definitions and properties. The reader is referred to [9] for details.

Let R be a one-dimensional formal power series ring over a field κ with maximal ideal $\mathfrak{m}$. The universal finite differential module of R over κ is a finite R-module $\tilde{\Omega}_{R/\kappa}$ together with a κ-derivation $d\colon R\to\tilde{\Omega}_{R/\kappa}$ (that is, a κ-linear map satisfying $d(fg)=fd(g)+gd(f)$) which is universal among all such finite κ-derivations. If $R=\kappa[[X]]$, then elements of $\tilde{\Omega}_{R/\kappa}$ can be written uniquely as $f\,dX$ for some $f\in R$.

Let $\mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})$ be the first local cohomology of $\tilde{\Omega}_{R/\kappa}$ supported at $\mathfrak{m}$. If $g\in R$ is not invertible and not zero, then there is a canonical exact sequence

$$\tilde{\Omega}_{R/\kappa}\to(\tilde{\Omega}_{R/\kappa})_g\to\mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})\to 0,$$

where $(\tilde{\Omega}_{R/\kappa})_g$ is the localization of $\tilde{\Omega}_{R/\kappa}$ at the multiplicatively closed set $\{1, g, g^2, \ldots\}$. If $R=\kappa[[X]]$, elements of $(\tilde{\Omega}_{R/\kappa})_g$ are of the form $f\,dX/g^n$, where $f\in R$ and $n\in\mathbb{N}$. We denote the image of $f\,dX/g^n$ in $\mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})$ by

$$\begin{bmatrix} f\,dX \\ g^n \end{bmatrix}$$

and call it a generalized fraction. Generalized fractions satisfy the following properties.

PROPOSITION 3.1 (Linearity Law). *For* $\omega_1, \omega_2 \in \tilde{\Omega}_{R/\kappa}$, *and* $k_1, k_2 \in R$,

$$\begin{bmatrix} k_1\omega_1 + k_2\omega_2 \\ g^n \end{bmatrix} = k_1 \begin{bmatrix} \omega_1 \\ g^n \end{bmatrix} + k_2 \begin{bmatrix} \omega_2 \\ g^n \end{bmatrix}.$$

PROPOSITION 3.2 (Vanishing law).

$$\begin{bmatrix} f\,dX \\ g^n \end{bmatrix} = 0$$

if and only if $f \in Rg^n$.

PROPOSITION 3.3 (Transformation Law). *For a non-zero element* $h \in R$,

$$\begin{bmatrix} f\,dX \\ g^n \end{bmatrix} = \begin{bmatrix} hf\,dX \\ hg^n \end{bmatrix}.$$

If $R = \kappa[[X]]$, the above properties imply that elements in $\mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})$ can be written uniquely as a finite sum

$$\sum_{n>0} \begin{bmatrix} a_n dX \\ X^n \end{bmatrix} \qquad (a_n \in \kappa).$$

So we can make the following definition.

DEFINITION 3.1. The residue map

$$\mathrm{res}_X : \mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa}) \to \kappa$$

is defined to be the κ-linear map satisfying

$$\mathrm{res}_X \begin{bmatrix} dX \\ X^n \end{bmatrix} = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

What makes residue map interesting is that it is independent of the choice of a variable. Then next theorem also explains why universal finite differential modules are needed in our definition of residue map.

THEOREM 3.1. *If* $R=\kappa[[X]]=\kappa[[Y]]$. *Then* $\mathrm{res}_X=\mathrm{res}_Y$.

EXAMPLE 3.1. Assume that $R=\kappa[[X]]$ and $Y=X/(1-X)$. Consider the elements

$$\omega_1=\begin{bmatrix} dX \\ X \end{bmatrix}=\begin{bmatrix} dY \\ Y \end{bmatrix},$$

$$\omega_n=\begin{bmatrix} dX \\ X^n \end{bmatrix}=\begin{bmatrix} (1+Y)^{n-2}\,dY \\ Y^n \end{bmatrix},$$

where $n\geqslant 2$. We see directly from the definition that $\mathrm{res}_X\,\omega_1=\mathrm{res}_Y\,\omega_1=1$ and $\mathrm{res}_X\,\omega_n=\mathrm{res}_Y\,\omega_n=0$ for $n\geqslant 2$.

We will write res_X simply as res. Generalized fractions and residues can be extended to formal power series rings of several variables over a complete local ring; see [8]. Inverse relations also have multidimensional generalizations. See [2, 14, 18, 20, 25] for some recent developments. Viewpoints and methods in this article are expected to extend naturally to the multidimensional case.

Given an element $\omega\in \mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})$, there is a κ-linear map Φ_ω from R to κ defined by $f\mapsto \mathrm{res}(f\omega)$. The kernel of Φ_ω contains a power of $\mathfrak{m}$. Therefore Φ_ω is continuous for the $\mathfrak{m}$-adic topology of R and the discrete topology of κ. We denote the R-module of these continuous homomorphisms by $\mathrm{Hom}^c_\kappa(R,\kappa)$.

THEOREM 3.2 (Local Duality). *The map*

$$H^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})\to \mathrm{Hom}^c_\kappa(R,\kappa)$$

given by $\omega\mapsto\Phi_\omega$ *is an isomorphism.*

Now we illustrate how local cohomology residues are used for computing coefficients of a representation by a Schauder basis.

EXAMPLE 3.2. Let $(f_i)=(X^i/(1-X)^{p+i+1})$ be a Schauder basis of $\kappa[[X]]$. We want to find a representation

$$X^k=d_{0k}f_0+d_{1k}f_1+d_{2k}f_2+d_{3k}f_3+\cdots \qquad (d_{nk}\in\kappa).$$

Let $Y=X/(1-X)$. Then

$$d_{nk}=\operatorname{res}\left[\frac{X^k(1-X)^{p+1}\,dY}{Y^{n+1}}\right]$$

$$=\operatorname{res}\left[\frac{\dfrac{Y^k}{(1+Y)^{p+k+1}}\,dY}{Y^{n+1}}\right]$$

$$=(-1)^{n+k}\binom{p+n}{p+k}.$$

EXAMPLE 3.3. Let κ be a field with characteristic zero and $(f_i)=(X^i e^{(p+i)X})$ be an Abel basis of $\kappa[[X]]$. The coefficient d_{nk} of the representation

$$X^k=d_{0k}f_0+d_{1k}f_1+d_{2k}f_2+d_{3k}f_3+\cdots$$

is given by

$$d_{nk}=\operatorname{res}\left[\frac{X^k e^{-pX}d(Xe^X)}{(Xe^X)^{n+1}}\right]$$

$$=\operatorname{res}\left[\frac{(1+X)\,e^{(-p-n)\,X}dX}{X^{n-k+1}}\right]$$

$$=\frac{(-p-n)^{n-k}}{(n-k)!}+\frac{(-p-n)^{n-k-1}}{(n-k-1)!}$$

$$=\frac{p+k}{p+n}\frac{(-p-n)^{n-k}}{(n-k)!}.$$

The next example involves Bernoulli numbers $B_i^{(n)}$ of order $n\in\mathbb{Z}$, which are rational numbers defined by

$$\left(\frac{X}{e^X-1}\right)^n=\sum_{i=0}^{\infty}\frac{B_i^{(n)}}{i!}X^i;$$

see [16].

EXAMPLE 3.4. Let κ be a field with characteristic zero and $(f_i)=(X^i(X/(e^X-1))^{p+i})$ be a Bernoulli basis of $\kappa[[X]]$. The coefficient d_{nk} of the representation

$$X^k=d_{0k}f_0+d_{1k}f_1+d_{2k}f_2+d_{3k}f_3+\cdots$$

is given straightforwardly by

$$d_{nk} = \operatorname{res}\left[\frac{X^k\left(\frac{X}{e^X-1}\right)^{-p} d\left(X\frac{X}{e^X-1}\right)}{\left(X\frac{X}{e^X-1}\right)^{n+1}}\right]$$

$$= \operatorname{res}\left[\frac{2\left(\frac{X}{e^X-1}\right)^{-p-n} dX}{X^{n-k+1}}\right] - \operatorname{res}\left[\frac{\left(\frac{X}{e^X-1}\right)^{-p-n} dX}{X^{n-k}}\right]$$

$$- \operatorname{res}\left[\frac{\left(\frac{X}{e^X-1}\right)^{-p-n+1} dX}{X^{n-k+1}}\right]$$

$$= 2\frac{B_{n-k}^{(-p-n)}}{(n-k)!} - \frac{B_{n-k-1}^{(-p-n)}}{(n-k-1)!} - \frac{B_{n-k}^{(-p-n+1)}}{(n-k)!}.$$

Using the following relation

$$nB_i^{(n+1)} = (n-i)\, B_i^{(n)} - niB_{i-1}^{(n)},$$

see [16, Eq. (15)], we may simplify the above expression for d_{nk} and get

$$d_{nk} = \frac{p+k}{p+n} \frac{B_{n-k}^{(-n-p)}}{(n-k)!}.$$

4. INVERSE RELATIONS AND DUAL BASES

This section consists of examples of inverse relations and combinatorial identities. Assume that $R = \kappa[[X]]$. Apply Theorem 2.1 with the ordinary basis (X^i) and the Schauder basis

$$(f_i) = \left(\frac{X^i}{(1-X)^{p+i+1}}\right),$$

where $p \geqslant 0$, we get the following inverse relation:

Inverse Relations 4.1 [22, Table 2.1, class 4].

$$\begin{cases} b_n = \sum_{k=0}^{n} \binom{p+n}{p+k} a_k \\ a_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{p+n}{p+k} b_k. \end{cases}$$

Assume that κ has characteristic zero. Apply Theorem 2.1 with the ordinary basis (X^i) and the Abel basis $(f_i) = (X^i e^{(p+i)X})$, where $p \in \mathbb{N}$, we get the following inverse relation:

Inverse Relations 4.2

$$\begin{cases} b_n = \sum_{k=0}^{n} \dfrac{(k+p)^{n-k}}{(n-k)!} a_k \\ a_n = \sum_{k=0}^{n} \dfrac{p+k}{p+n} \dfrac{(-n-p)^{n-k}}{(n-k)!} b_k \end{cases}$$

Apply Theorem 2.1 with the ordinary basis (X^i) and the Bernoulli basis $(f_i) = (X^i(X/(e^X - 1))^{p+i})$, where $p \in \mathbb{N}$, we get the following inverse relation:

Inverse Relations 4.3

$$\begin{cases} b_n = \sum_{k=0}^{n} \dfrac{B_{n-k}^{(p+k)}}{(n-k)!} a_k \\ a_n = \sum_{k=0}^{n} \dfrac{p+k}{p+n} \dfrac{B_{n-k}^{(-n-p)}}{(n-k)!} b_k \end{cases}$$

Now we translate the operator method in [13] to our language. Let (c_{ij}) be an infinite lower triangular matrix with $c_{ii} \neq 0$. To find its inverse, let $R = \kappa[[X]]$ and

$$f_j = c_{0j} + c_{1j} X + c_{2j} X^2 + \cdots.$$

An infinite lower triangular matrix (d_{ij}) is the inverse of (c_{ij}) if and only if the elements

$$\omega_i := \sum_{n=0}^{i} \left[\frac{d_{in} dX}{X^{n+1}} \right] \in \mathrm{H}^1_{\mathrm{m}}(\tilde{\Omega}_{R/\kappa})$$

satisfy $\operatorname{res}(f_j \omega_i) = \delta_{ij}$. If (d_{ij}) is the inverse of (c_{ij}), the elements ω_i form a κ-vector basis for $\mathrm{H}^1_{\mathrm{m}}(\tilde{\Omega}_{R/\kappa})$. In fact,

$$\omega = \sum_{i=0}^{\infty} \operatorname{res}(f_i \omega)\, \omega_i$$

for any $\omega \in \mathrm{H}^1_{\mathrm{m}}(\tilde{\Omega}_{R/\kappa})$. In such case, we call (ω_i) the dual basis of (f_i). The dual basis defined above depends only on a strictly monotone Schauder basis for R. It is independent of the representation $R = \kappa[[X]]$.

Let U be a continuous linear operator on R (that is, a κ-linear map $R \to R$ continuous for the $\mathfrak{m}$-adic topology of R). Each $\omega \in \mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})$ determines a continuous κ-linear map $R \to \kappa$ by $f \mapsto \operatorname{res}((Uf)\,\omega)$. By local duality, there exists a unique $U^*\omega \in \mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})$ such that

$$\operatorname{res}((Uf)\,\omega) = \operatorname{res}(f(U^*\omega)).$$

U^* is a linear operator on $\mathrm{H}^1_{\mathfrak{m}}(\tilde{\Omega}_{R/\kappa})$, that is, the map $\omega \mapsto U^*\omega$ is κ-linear.

LEMMA 4.1 (cf. [13]). *Let (c_{ij}) and f_j be as above and let ω_i be the dual basis of (f_i). Suppose that*

$$Uf_j = e_j V f_j$$

holds for $j \geqslant 0$, where $e_j \in \kappa$ satisfying $e_i \neq e_j$ for $i \neq j$ and U, V are continuous linear operators on R. If ξ_k is a solution of

$$U^*\xi_k = e_k V^*\xi_k$$

for $k \geqslant 0$, then

$$\operatorname{res}(f_k V^*\xi_k)\,\omega_k = V^*\xi_k.$$

Proof. If $i \neq k$, from the computation

$$\begin{aligned} e_k \operatorname{res}(f_i V^*\xi_k) &= \operatorname{res}(f_i U^*\xi_k) = \operatorname{res}((Uf_i)\,\xi_k) = e_i \operatorname{res}((Vf_i)\,\xi_k) \\ &= e_i \operatorname{res}(f_i V^*\xi_k), \end{aligned}$$

we get $\operatorname{res}(f_i V^*\xi_k) = 0$. Therefore

$$V^*\xi_k = \sum_{i=0}^{\infty} \operatorname{res}(f_i V^*\xi_k)\,\omega_i = \operatorname{res}(f_k V^*\xi_k)\,\omega_k. \quad \blacksquare$$

Let (α_i), (β_i), and (e_i) be sequences of elements of κ such that $e_i \neq e_j$ if $i \neq j$. Let A, B, E be continuous linear operators defined by $AX^k = \alpha_k X^k$, $BX^k = \beta_k X^k$ and $EX^k = e_k X^k$. Let

$$c_{nk} = \frac{\prod_{j=k}^{n-1}(\alpha_j + e_k\beta_j)}{\prod_{j=k+1}^{n}(e_j - e_k)}.$$

(By convenience, products of the form $\prod_{j=n}^{n-1}$ are defined to be equal to 1.) and $f_k = \sum_{n=0}^{\infty} c_{nk} X^n$. As observed in [13],

$$(E - XA)\,f_k = e_k(1 + XB)\,f_k$$

and

$$\xi_k = \sum_{\ell=0}^{k} \frac{\prod_{j=\ell}^{k-1}(\alpha_j + e_k \beta_j)}{\prod_{j=\ell}^{k-1}(e_j - e_k)} \left[\frac{dX}{X^{\ell+1}} \right]$$

satisfies the equation

$$(E^* - A^* X)\, \xi_k = e_k (1 + B^* X)\, \xi_k.$$

By Lemma 4.1, we can compute the dual basis of (f_i) and obtain an inverse relation.

Inverse Relations 4.4 [13]. Assume that $\alpha_n + e_n \beta_n \neq 0$ for all n.

$$\begin{cases} b_n = \sum_{k=0}^{n} \frac{\prod_{j=k}^{n-1}(\alpha_j + e_k \beta_j)}{\prod_{j=k+1}^{n}(e_j - e_k)} a_k \\ a_n = \sum_{k=0}^{n} \frac{\alpha_k + e_k \beta_k}{\alpha_n + e_n \beta_n} \frac{\prod_{j=k+1}^{n}(\alpha_j + e_n \beta_j)}{\prod_{j=k}^{n-1}(e_j - e_n)} b_k. \end{cases}$$

This inverse relation is quite general and useful. The case $e_k = k$ is an inverse relation obtained by Gould and Hsu [7]. The case $e_k = q^k$ is equivalent to a q-analogue of [7] obtained by Carlitz [5]. Andrews [1] showed that the Bailey transform is equivalent to the case $\alpha_j = 1$, $\beta_j = -bq^j$, $e_k = q^k$. The case $\alpha_j = aq^{-j-1} + b^2 q^{j-1}$, $\beta_j = -b/q$, $e_k = q^{-k} + aq^k$ is equivalent to a transform used by Bressoud in finite forms of Rogers–Ramanujan identities [3, 4].

Every inverse relation consists of two identities. Some of them are interesting in their own right.

IDENTITY 4.1. *Let p, q, r be non-negative integers and $\varphi \in \kappa[[X]]$ be an invertible formal power series. Assume that q does not divide r. Write*

$$\varphi^n = \sum_{i=0}^{\infty} b_i^{(n)} X^i \quad (b_i^{(n)} \in \kappa).$$

Then

$$\sum_{k=0}^{n} \frac{r}{r - qk} b_k^{(r-qk)} b_{n-k}^{(p+qk)} = b_n^{(p+r)}.$$

Proof. Let (f_i) be the Schauder basis $(X^i \varphi^{p+iq})$, (g_i) be the ordinary Schauder basis (X^i) and $h = \varphi^{p+r}$. We use the notation as in Theorem 2.1.

It is easy to check that $b_n = b_n^{(p+r)}$ and $c_{nk} = b_{n-k}^{(p+qk)}$. The identity follows from the first equation of (6) and the computation

$$\begin{aligned}
a_k &= \operatorname{res}\left[\frac{\varphi^r d(X\varphi^q)}{(X\varphi^q)^{k+1}}\right] \\
&= \operatorname{res}\left[\frac{\varphi^{r-qk-q}\left(\varphi^q + qX\varphi^{q-1}\dfrac{d\varphi}{dX}\right)dX}{X^{k+1}}\right] \\
&= \operatorname{res}\left[\frac{\varphi^{r-qk}dX}{X^{k+1}}\right] + \frac{q}{r-qk}\operatorname{res}\left[\frac{Xd\varphi^{r-qk}}{X^{k+1}}\right] \\
&= b_k^{(r-qk)} + \frac{qk}{r-qk}b_k^{(r-qk)} \\
&= \frac{r}{r-qk}b_k^{(r-qk)}. \quad \blacksquare
\end{aligned}$$

This identity generalizes some classical identities. For instance, with $\varphi = e^X$, we recover the Abel identity. Note that (f_i) in this case is an Abel basis.

IDENTITY 4.2 (Abel). *Let p, q, r be elements in a field κ of characteristic zero. Assume that $r - qk \neq 0$ for all $k \geqslant 0$. Then*

$$\sum_{k=0}^{n} \frac{r}{r-qk} \frac{(r-qk)^k}{k!} \frac{(p+qk)^{n-k}}{(n-k)!} = \frac{(p+r)^n}{n!}.$$

With $\varphi = 1 + X$, we recover the Gould identity. Note that (f_i) in this case is a Gould basis.

IDENTITY 4.3 (Gould). *Let $p, q, r \in \mathbb{N}$. Assume that q does not divide r. Then*

$$\sum_{k=0}^{n} \frac{r}{r-qk}\binom{r-qk}{k}\binom{p+qk}{n-k} = \binom{p+r}{n}.$$

If $\varphi = 1/(1-X)$, we get the following identity.

IDENTITY 4.4. *Let p, q, r be non-negative integers. Assume that q does not divide r. Then*

$$\sum_{k=0}^{n} \frac{r}{r-qk} \binom{r-qk+k-1}{k} \binom{p+qk+n-k-1}{n-k} = \binom{p+r+n-1}{n}.$$

If $\varphi = X/(e^X - 1)$, we get the following identity.

IDENTITY 4.5. *Let $p, q, r \in \mathbb{Z}$. Assume that q does not divide r. Then*

$$\sum_{k=0}^{n} \frac{r}{r-qk} \frac{B_k^{(r-qk)}}{k!} \frac{B_{n-k}^{(p+qk)}}{(n-k)!} = \frac{B_n^{(p+r)}}{n!}.$$

5. RIORDAN ARRAYS

The Abel and Gould identities were also generalized in [27, Theorem 3.1] using Riordan arrays. In this section, we examine the viewpoint of Riordan arrays and compare it with that of Schauder bases. In short, a variable in the theory of Riordan arrays is a "dummy variable"; while variables from the viewpoint of Schauder bases are elements in a formal power series ring with relations.

Let κ be a field. Given $g \in \kappa[[X]]$ and f in the maximal ideal $\mathfrak{m}$ of R, write

$$g(X) = a_0 + a_1 X + a_2 X^2 + \cdots \qquad (a_i \in \kappa),$$
$$f(X) = b_1 X + b_2 X^2 + \cdots \qquad (b_i \in \kappa).$$

We define the value of g at f with respect to X by

$$g(f(X)) := a_0 + a_1(b_1 X + b_2 X^2 + \cdots) + a_2(b_1 X + b_2 X^2 + \cdots)^2 + \cdots.$$

Let R_0 be the subset of R consisting of all invertible elements. With respect to X, for $f, g \in R_0$, we define

$$f * g = f(X)\, g(Xf(X)).$$

It is easy to check

$$f * 1_\kappa = 1_\kappa * f = f$$

$$f * (g * h) = (f * g) * h$$

for all $f, g, h \in R_0$. Given $f \in R_0$, let $Y = Xf$, we represent $1/f$ as power series in Y:

$$\frac{1}{f} = a_0 + a_1 Y + a_2 Y^2 + \cdots + \qquad (a_i \in \kappa).$$

In the theory of the Lagrange groups and the Riordan groups, X and Y are treated as dummy variables. Replacing Y by X on the left hand side of the above equation, we define (with respect to X)

$$\bar{f} = a_0 + a_1 X + a_2 X^2 + \cdots + .$$

It was shown that

$$f * \bar{f} = \bar{f} * f = 1_\kappa.$$

Hence $(R_0, \, *)$ is a group, which we call the Lagrange group.

Given $f \in R_0$ and $g \in R$, let $Y = Xf$, we represent g as power series in Y:

$$g = b_0 + b_1 Y + b_2 Y^2 + \cdots + \qquad (b_i \in \kappa).$$

Replacing Y by X, we define

$$\hat{g}_{(f)} = b_0 + b_1 X + b_2 X^2 + \cdots + .$$

As one of the main theorems in [27], it was proved that

$$\hat{g}_{(f)}(Xf(X)) = g(X) \tag{7}$$

and

$$g(X\bar{f}(X)) = \hat{g}_{(f)}(X). \tag{8}$$

In our language, where X and Y are not dummy variables but satisfy the relation $Y = Xf(X)$, identities (7) and (8) are conceptually trivial: Identity (7) simply states that the same elements $\hat{g}_{(f)}(Y)$ and $g(X)$ are represented in different ways. Identity (8) should be stated in the equivalent form

$$g(Y\bar{f}(Y)) = \hat{g}_{(f)}(Y). \tag{9}$$

Note that

$$X = Xf(X)\,\bar{f}(Y) = Y\,\bar{f}(Y).$$

Hence identity (9) also states that the same elements $\hat{g}_{(f)}(Y)$ and $g(X)$ are represented in different ways.

A Riordan array is a pair (g, h), where $g, h \in R_0$. Given two Riordan arrays (g_1, h_1) and (g_2, h_2), we define

$$(g_1, h_1) * (g_2, h_2) = (g_1(X)\, g_2(Xh_1), h_1 * h_2). \tag{10}$$

It is straightforward to check that

$$(g, h) * (1_\kappa, 1_\kappa) = (1_\kappa, 1_\kappa) * (g, h) = (g, h),$$

$$((g_1, h_1) * (g_2, h_2)) * (g_3, h_3) = (g_1, h_1) * ((g_2, h_2) * (g_3, h_3)),$$

$$(g, h) * (\hat{g}^{-1}_{(\bar{h})}, \bar{h}) = (\hat{g}^{-1}_{(\bar{h})}, \bar{h}) * (g, h) = (1_\kappa, 1_\kappa),$$

for all Riordan arrays (g, h), (g_1, h_1), (g_2, h_2) and (g_3, h_3). Hence the set of Riordan arrays together with the operation (10) forms a group, which we call the Riordan group. We remark that our definition of the Riordan group, following [27], is essentially the same as the original definition in [26], although in slightly different guise. We remark also that the group structure of Riordan arrays was found in [23, p. 43] in terms of Sheffer operators using the umbral calculus.

A Riordan array (g, h) determines an infinite lower triangular matrix (c_{nk}), where $c_{nk} \in \kappa$, by the relation

$$g(Xh)^k = \sum_{n=0}^{\infty} c_{nk} X^n.$$

For instance, the Riordan array $(1/(1-X), 1/(1-X))$ determines the Pascal triangle

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The product in the Riordan group is in fact coming from product of matrices: Given two infinite lower triangular matrices $(c_{n_1k_1})$ and $(c_{n_2k_2})$ determined by Riordan arrays (g_1, h_1) and (g_2, h_2), the product $(c_{n_1k_1})(c_{n_2k_2})$ of matrices is determined by the product $(g_1, h_1) * (g_2, h_2)$ of Riordan arrays.

Given an infinite lower triangular matrix (c_{nk}) defined by the Riordan array (g, h) and a power series

$$f = a_0 + a_1 X + a_2 X^2 + \cdots \qquad (a_i \in \kappa),$$

the generating functions of the column of

$$\begin{pmatrix} c_{00} & 0 & 0 & 0 & \cdots \\ c_{10} & c_{11} & 0 & 0 & \cdots \\ c_{20} & c_{21} & c_{22} & 0 & \cdots \\ c_{30} & c_{31} & c_{32} & c_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}$$

is $g(X)\, f(Xh(X))$. From the viewpoint of the Riordan group, this fact with a suitable choice of f gives rise to the following identity.

IDENTITY 5.6 [27, Theorem 3.1]. *Assume that* $R = \kappa[[X]] = \kappa[[Y]]$ *and* $g, f \in R$ *are invertible. Let*

$$g = b_0 + b_1 X + b_2 X^2 + b_3 X^3 + \cdots,$$

$$g = a_0 f^{-1} g + a_1 f^{-1} Y g + a_2 f^{-1} Y^2 g + a_3 f^{-1} Y^3 g + \cdots, \tag{11}$$

$$f^{-1} Y^i g = c_{0i} + c_{1i} X + c_{2i} X^2 + c_{3i} X^3 + \cdots, \tag{12}$$

where $a_j, b_j, c_{ji} \in \kappa$. *Then*

$$b_n = \sum_{k=0}^{n} a_k c_{nk}.$$

We interpret the above generalization in terms of Schauder bases. Equation (11) describes the element g in terms of $f^{-1} Y^j g$. Equation (12) gives the relation between the ordinary basis (X^i) and the Schauder basis $(f^{-1} Y^j g)$. So we are able to find the representation of g by the ordinary basis (X^j):

$$b_n = \operatorname{res}\left[\frac{g\,dX}{X^{n+1}}\right] = \operatorname{res}\left[\frac{\sum_{k=0}^{\infty} a_k f^{-1} Y^k g\,dX}{X^{n+1}}\right] = \sum_{k=0}^{n} \operatorname{res}\left[\frac{a_k f^{-1} Y^k g\,dX}{X^{n+1}}\right]$$

$$= \sum_{k=0}^{n} a_k c_{nk}.$$

ACKNOWLEDGMENT

The author thanks the referee for comments and for suggesting related references, especially [13].

REFERENCES

1. G. E. Andrews, Connection coefficient problems and partitions, *in* "Relations between Combinatorics and Other Parts of Mathematics, Ohio State Univ., Columbus, Ohio, 1978," Proc. Sympos. Pure Math., pp. 1–24. Amer. Math. Soc., Providence, 1979.
2. G. Bhatnagar and S. C. Milne, Generalized bibasic hypergeometric series and their u(n) extensions, *Adv. Math.* **131** (1997), 188–252.
3. D. M. Bressoud, Some identities for terminating q-series, *Math. Proc. Cambridge Philos. Soc.* **89** (1981), 211–223.
4. D. M. Bressoud, A matrix inverse, *Proc. Amer. Math. Soc.* **88** (1983), 446–448.
5. L. Carlitz, Some inverse relations, *Duke Math. J.* **40** (1973), 893–901.
6. G. P. Egorychev, "Integral Representation and the Computation of Combinatorial Sums," Translation of Mathematical Monographs, Vol. 59, Amer. Math. Soc., Providence, 1984.
7. H. W. Gould and L. C. Hsu, Some new inverse series relations, *Duke Math. J.* **40** (1973), 885–891.
8. I-C. Huang, Pseudofunctors on modules with zero dimensional support, *Mem. Amer. Math. Soc.* **114** (1995), xii–53.
9. I-C. Huang, Applications of residues to combinatorial identities, *Proc. Amer. Math. Soc.* **125** (1997), 1011–1017.
10. I-C. Huang, Reversion of power series by residues, *Comm. Algebra* **26** (1998), 803–812.
11. I-C. Huang and S.-Y. Huang, Bernoulli numbers and polynomials via residues, *J. Number Theory* **76** (1999), 178–193.
12. C. Krattenthaler, Operator methods and Lagrange inversion: a unified approach to Lagrange formulas, *Trans. Amer. Math. Soc.* **305** (1988), 431–465.
13. C. Krattenthaler, A new matrix inverse, *Proc. Amer. Math. Soc.* **124** (1996), 47–59.
14. C. Krattenthaler and M. Schlosser, A new multidimensional matrix inverse with applications to multiple q-series, *Discrete Math.* **204** (1999), 249–279.
15. S. Lang, *Algebra*, 3rd ed., Addison–Wesley, Reading, MA, 1993.
16. M. É. Lucas, Sur les développements en séries, *Bull. Soc. Math. France* **6** (1877), 57–68.
17. H. Matsumura, "Commutative Ring Theory," Cambridge Univ. Press, Cambridge, UK, 1986.
18. S. C. Milne, Balanced ${}_3\phi_2$ summation theorems for u(n) basic hypergeometric series, *Adv. Math.* **131** (1997), 93–187.
19. S. C. Milne and G. Bhatnagar, A characterization of inverse relations, *Discrete Math.* **193** (1998), 235–245.
20. S. C. Milne and G. M. Lilly, The A_l and C_l Bailey transform and lemma, *Bull. Amer. Math. Soc. (N.S.)* **26** (1992), 258–263.
21. E. Neuwirth, Recursively defined combinatorial functions: Extending Galton's board, *Discrete Math.*, in press.
22. J. Riordan, "Combinatorial Identities," Wiley, New York, 1968.
23. S. Roman, "The Umbral Calculus," Pure and Applied Mathematics, Vol. 111, Academic Press, New York, 1984.

24. S. Roman, P. N. De Land, R. C. Shiflett, and H. S. Shultz, Inverse relations and the umbral calculus, *J. Combin. Inform. System Sci.* **8** (1983), 185–198.
25. M. Schlosser, A new multidimensional matrix inversion in A_r, *in* "q-series from a Contemporary Perspective, South Hadley, MA, 1998," pp. 413–432, Amer. Math. Soc., Providence, 2000.
26. L. W. Shapiro, S. Getu, W. J. Woan, and L. C. Woodson, The Riordan group, *Discrete Appl. Math.* **34** (1991), 229–239.
27. R. Sprugnoli, Riordan arrays and the Abel–Gould identity, *Discrete Math.* **142** (1995), 213–233.
28. R. P. Stanley, "Enumerative Combinatorics, Vol. 1," Cambridge Studies in Advanced Mathematics, Vol. 49, Cambridge Univ. Press, Cambridge, UK, 1997.