Resolvability: a selective survey and some new results ☆

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Received 10 May 1995; revised 23 October 1995

Abstract

Following guidance from the Organizing Committee, the authors give a brief introduction to the theory of spaces which are *resolvable* in the sense introduced by Hewitt (1943).

The new results presented here are these. (A) A countably compact regular Hausdorff space without isolated points is ω-resolvable—that is, it admits an infinite family of pairwise disjoint dense subsets. (B) Among Tychonoff topologies without isolated points on a fixed set, no pseudocompact topology is maximal.

Keywords: Resolvable topological space; Resolvable topological group; κ-resolvable space; Maximal topology; Maximal regular topology; Maximal Tychonoff topology; S_ω-like space; Without isolated points; Dense-in-itself; Countably compact space; Pseudocompact space

AMS classification: Primary 54–02; 54A10; 54C05, Secondary 54G10; 22A05; 54D10; 54H11

Respectfully dedicated to Edwin Hewitt on the occasion of his 75th Birthday

1. Introductory remarks

It is an honor to address this distinguished audience.

My goal in this hour and with this manuscript is to bring to your attention some of the salient features, and some of the outstanding problems, associated with Edwin Hewitt's resolvable spaces [25]. We proceed according to the following plan. Section 2 contains basic generalities; Sections 3 and 4 show respectively the existence in profusion of spaces

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(including Tychonoff spaces) which are, and which are not, resolvable; in Section 5 we introduce and discuss a class of spaces (called S_ω-like spaces) derived from the familiar (countable, sequential, non-first-countable) space S_ω of Arhangel'skiĭ and Franklin [2], and in Sections 6 and 7 we use those spaces to establish the new results mentioned in our Abstract; in Section 8 we point the way down some of the paths followed by other workers, and from the list of many unsolved problems we cite those which seem to us most inviting.

2. Definitions and generalities

Following Hewitt [25], we say that a topological space $X = \langle X, \mathcal{T} \rangle$ is *resolvable* if there is a subset D of X such that both D and $X \backslash D$ are $\mathcal{T}$-dense in X. More generally, adopting terminology introduced subsequently by Ceder [7], we say for a cardinal number α that $X = \langle X, \mathcal{T} \rangle$ is α-*resolvable* if there is a family of α-many pairwise disjoint dense subsets of X. (According to this usage "resolvable" coincides with "2-resolvable"; every space, resolvable or not, is 1-resolvable; and the empty space $\emptyset$ is α-resolvable for every cardinal $\alpha \geqslant 1$.)

It is worth noting explicitly that every resolvable space X is dense-in-itself, i.e., no point of X is isolated in X.

Lemma 2.1. *Let $\alpha \geqslant 1$ and let X be a space. If Y is an α-resolvable subspace of X, then $\overline{Y}^X$ is α-resolvable.*

Proof. The relation "is dense-in" is transitive. □

It may be said informally that the union of α-resolvable spaces is α-resolvable. A more careful version of this useful statement, taken from [9] (the case $\alpha = 2$), reads as follows.

Theorem 2.2. *Let $\alpha \geqslant 1$ and let X be a space of the form $X = \bigcup_{i \in I} X_i$ with each X_i α-resolvable (in the subspace topology). Then X is α-resolvable.*

Proof. Let $\mathcal{Y}$ be a maximal family of pairwise disjoint α-resolvable subspaces of X. It is enough to show that $\bigcup \mathcal{Y}$ is dense in X—for in that case, choosing for $Y \in \mathcal{Y}$ a pairwise disjoint family $\{Y_\xi: \xi < \alpha\}$ of dense subsets of Y and defining $D_\xi = \bigcup_{Y \in \mathcal{Y}} Y_\xi$ for $\xi < \alpha$, one checks easily that $\{D_\xi: \xi < \alpha\}$ is a family of α-many pairwise disjoint dense subsets of X, as required.

If the open set $U := X \backslash \overline{\bigcup \mathcal{Y}}^X$ is nonempty then there is $i \in I$ such that $U \cap X_i \neq \emptyset$. Since an open subspace of an α-resolvable space is (clearly) α-resolvable, the family $\mathcal{Y} \cup \{U \cap X_i\}$ is pairwise disjoint with α-resolvable members, contrary to the maximality of $\mathcal{Y}$. □

Theorem 2.2 allows efficient proofs of a number of results first achieved in the literature by less direct arguments. The following are typical, and they are suggestive of paths of

investigation pursued over the years by many workers. Limitations of time and space prevent our pursuing these consequences in any detail here.

Corollary 2.3. *Let* $\alpha \geqslant 1$.

(a) *A homogeneous space with a nonempty α-resolvable subspace is α-resolvable.*

(b) *A (locally) arc-wise connected space is resolvable.*

(c) *A topological group with a nonclosed subgroup is resolvable.*

(d) *For every space X the subspace*

$$R_\alpha(X) := \bigcup \{Y: \ Y \subseteq X, \ Y \ is \ \alpha\text{-}resolvable\}$$

is α-resolvable and closed in X, and its complement $I_\alpha(X)$ is strongly α-irresolvable (in the sense that $I_\alpha(X)$ contains no nonempty α-resolvable subspace).

Proof. (a), (b) and (d) are immediate from 2.2 and 2.1. If H is a nonclosed subgroup of a topological group G then $\overline{H}^G$ is resolvable (since H and $\overline{H}^G \backslash H$ are dense in $\overline{H}^G$), so (c) follows from (a). $\square$

Remark 2.4. It is conventional to write a space X in the form $X = R \cup I$ with $R = R_2(X)$ and $I = I_2(X) = X \backslash R$. This decomposition of X into its *resolvable hull* R and the complementary (strongly irresolvable) open subspace I is due to Hewitt [25].

3. Some spaces are resolvable

Notation 3.1. Given a space $\langle X, \mathcal{T} \rangle$, the notation $\Delta(X)$ was introduced by Hewitt [25] to denote the so-called *dispersion character* of X; this is the cardinal number

$$\Delta(X) = \min \{|U|: \ \emptyset \neq U \in \mathcal{T}\}.$$

Subsequent workers have called a space X *maximally resolvable* if X is $\Delta(X)$-resolvable.

The concept of a π-network is useful in connection with the investigation of questions concerning maximal resolvability.

Definition 3.2. A family $\mathcal{S}$ of subsets of a space X is said to be a π-*network* if

(a) each $S \in \mathcal{S}$ is nonempty, and

(b) each nonempty open subset of X contains an element of $\mathcal{S}$.

The relevance of π-networks to the study of (maximal) resolvability is evident: if $\mathcal{S}$ is a π-network for X and if $D \subseteq X$ satisfies $D \cap S \neq \emptyset$ for each $S \in \mathcal{S}$, then D is dense in X.

While we are interested chiefly in (infinite) Hausdorff spaces without isolated points, it is worthwhile to notice *en passant* that every space of finite weight is maximally resolvable.

Theorem 3.3. *Every space* $X = \langle X, \mathcal{T} \rangle$ *such that* $w(X) < \omega$ *is maximally resolvable.*

Proof. Since $|\mathcal{T}| < \omega$ the family

$$\mathcal{S} := \{ S \in \mathcal{T} \setminus \{\emptyset\} \colon S \text{ contains properly no nonempty open set} \}$$

is a π-network for X. For $S \in \mathcal{S}$ let f_S be a one-to-one function from $\Delta(X)$ into S. For $\eta < \Delta(X)$ the set $D_\eta := \{f_S(\eta) \colon S \in \mathcal{S}\}$ is dense in X, and the sets D_η are pairwise disjoint since the elements of $\mathcal{S}$ are pairwise disjoint. $\square$

Our convention to the effect that every space X is 1-resolvable allows in 3.3 the banal case $\Delta(X) = 1$, i.e., X has an isolated point.

In order to show the existence of many infinite Hausdorff spaces without isolated points which are (not only resolvable but even) maximally resolvable, we appeal to the so-called disjoint refinement lemma and to a special case of one of its consequences noted by El'kin [17] (3.4 and 3.5 below, respectively); for a proof of 3.4 and appropriate citations to the literature, see [12] (7.5 and Notes for Section 7).

Lemma 3.4 (the disjoint refinement lemma). *Let* α *be an infinite cardinal and let* $\mathcal{S}$ *be a set of sets such that* $|\mathcal{S}| = \alpha$ *and each* $S \in \mathcal{S}$ *satisfies* $|S| = \alpha$. *Then there is a family* $\{T(S) \colon S \in \mathcal{S}\}$ *of pairwise disjoint sets such that*

(i) $T(S) \subseteq S$ *for all* $S \in \mathcal{S}$, *and*

(ii) $|T(S)| = \alpha$ *for all* $S \in \mathcal{S}$.

Lemma 3.5. *Let* X *be an infinite Hausdorff space with no isolated points. If* X *has a* π-network $\mathcal{S}$ *such that* $|\mathcal{S}| \leqslant \Delta(X)$ *and each* $S \in \mathcal{S}$ *satisfies* $|S| \geqslant \Delta(X)$, *then* X *is maximally resolvable.*

Proof. The disjoint refinement lemma with $\alpha = \Delta(X) \geqslant \omega$ gives a family $\{T(S) \colon S \in \mathcal{S}\}$ of pairwise disjoint sets such that $T(S) \subseteq S$ and $|T(S)| = \Delta(X)$ for each $S \in \mathcal{S}$. For $S \in \mathcal{S}$ let f_S be a one-to-one function from $\Delta(X)$ into $T(S)$, and for $\eta < \Delta(X)$ set $D_\eta = \{f_S(\eta) \colon S \in \mathcal{S}\}$. Then $\{D_\eta \colon \eta < \Delta(X)\}$ is a family of $\Delta(X)$-many pairwise disjoint dense subsets of X, as required. $\square$

Remark 3.6. (a) It follows in particular from Lemma 3.5 that a Hausdorff space without isolated points and with a countable pseudobase is maximally resolvable.

(b) Suppose that X has a π-network $\mathcal{S}$ such that $|\mathcal{S}| \leqslant \Delta(X)$, and also X has a π-network $\mathcal{S}'$ such that $S \in \mathcal{S}' \Rightarrow |S| \geqslant \Delta(X)$ (this latter condition is surely satisfied: one may take $\mathcal{S}' = \mathcal{T}$, or $\mathcal{S}'$ any base for $\mathcal{T}$). It does not then follow that X has a π-network such that simultaneously $|\mathcal{S}| \leqslant \Delta(X)$ and also $S \in \mathcal{S} \Rightarrow |S| \geqslant \Delta(X)$. For an example to this effect let $\langle X, \mathcal{T} \rangle$ be a countably infinite Hausdorff space without isolated points which for some $n < \omega$ is n-resolvable but not $(n + 1)$-resolvable (see Section 6 below for references in this connection). Since X is not maximally resolvable, we see from Lemma 3.5 that X admits no such π-network $\mathcal{S}$.

(c) We say that a space X is *card-homogeneous* if every nonempty open subset V of X satisfies $|V| = |X|$. It is immediate from 3.5 that every card-homogeneous space X satisfying $w(X) \leqslant |X|$ is maximally resolvable. Now in 3.7 we develop this thought a bit further, showing that many of the spaces familiar to mathematicians are maximally resolvable. The assertions of 3.7 are contained already in the original work of Hewitt [25]. See Pytke'ev [37] and El'kin [16] for generalizations. In its essentials our development has paralleled that of Ceder [7, Theorems 7 and 8].

Theorem 3.7. *Let X be a Hausdorff space without isolated points. If either*

(a) *X is locally compact, or*

(b) *X is metrizable,*

then X is maximally resolvable.

Proof. In (a) let $\mathcal{U}$ be the set of open card-homogeneous subsets U of X such that $\overline{U}^X$ is compact and $|\overline{U}^X| = |U|$; in (b) let $\mathcal{U}$ be the set of open card-homogeneous subsets of X. Since X is regular the set $\bigcup \mathcal{U}$ is dense in X, so it suffices by 2.2 and 2.1 to prove that each $U \in \mathcal{U}$ is $\Delta(X)$-resolvable; we show in fact that U is $\Delta(U)$-resolvable.

Let $\mathcal{S}$ be a base for U with $|\mathcal{S}| = w(U)$. Clearly each $S \in \mathcal{S}$ satisfies $|S| \geqslant \Delta(U)$. Every compact Hausdorff space Y satisfies $w(Y) \leqslant |Y|$ (cf. Engelking [20, 3.1.19–3.1.21]) and every metrizable space Y satisfies $w(Y) = d(Y)$ (cf. Engelking [20, 4.1.15]) so in (a) we have

$$|\mathcal{S}| = w(U) \leqslant w(\overline{U}^X) \leqslant |\overline{U}^X| = |U| = \Delta(U)$$

and in (b) we have

$$|\mathcal{S}| = w(U) = d(U) \leqslant |U| = \Delta(U).$$

That U is $\Delta(U)$-resolvable then follows from 3.5. □

4. Some spaces are irresolvable

While our search of the literature has not uncovered explicit, systematic use of the device given next, special cases of this idea appear in several of the papers cited in our list of references. We use Lemma 4.2 three times below: in 4.3, 7.4, and 7.6.

Here and in what follows, for a set X we use the notation $\mathcal{R}(X)$ (respectively $\mathcal{CR}(X)$) to denote the set of Hausdorff topologies on X which are regular (respectively completely regular).

Notation 4.1. For a space $X = \langle X, \mathcal{T} \rangle$ and $f \in {}^X\mathbb{R}$, the symbol $\mathcal{T}_f$ denotes the topology on X for which $\mathcal{T} \cup \{f^{-1}(V): V \text{ open in } \mathbb{R}\}$ is a subbase.

Lemma 4.2. *Let $\langle X, \mathcal{T} \rangle$ be a Hausdorff space and let $f \in {}^X\mathbb{R}$.*

(a) *If $\mathcal{T} \in \mathcal{R}(X)$ (respectively $\mathcal{T} \in \mathcal{CR}(X)$) then $\mathcal{T}_f \in \mathcal{R}(X)$ (respectively $\mathcal{T}_f \in \mathcal{CR}(X)$);*

(b) *If every $W \in \mathcal{T}$ and every open $V \subseteq \mathbb{R}$ satisfy either $W \cap f^{-1}(V) = \emptyset$ or $|W \cap f^{-1}(V)| \geqslant \omega$, then $\langle W, \mathcal{T}_f \rangle$ has no isolated points.*

Proof. (a) Clearly f is $\mathcal{T}_f$-continuous. Let $p \in N = W \cap f^{-1}(V)$ with $W \in \mathcal{T}$ and V open in $\mathbb{R}$.

(i) The case $\mathcal{T} \in \mathcal{R}(X)$. There are W_0 and V_0, open in $\langle X, \mathcal{T} \rangle$ and $\mathbb{R}$, respectively, such that $p \in W_0 \subseteq \overline{W_0}^{\mathcal{T}} \subseteq W$ and $f(p) \in V_0 \subseteq \overline{V_0}^{\mathbb{R}} \subseteq V$. The set $N_0 := W_0 \cap f^{-1}(V_0)$ then satisfies $N_0 \in \mathcal{T}_f$ and $p \in N_0 \subseteq \overline{N_0}^{\mathcal{T}_f} \subseteq N$.

(ii) The case $\mathcal{T} \in \mathcal{CR}(X)$. There is a $\mathcal{T}$-continuous function $g : X \to \mathbb{R}$ such that $g(p) = 0$ and $g \equiv 1$ on $X \backslash W$ and there is $\varepsilon > 0$ such that $(f(p) - \varepsilon, f(p) + \varepsilon) \subseteq V$. The function $h := (f - f(p))/\varepsilon$ is $\mathcal{T}_f$-continuous, so $k := g + h$ is $\mathcal{T}_f$-continuous (and satisfies $k(p) = 0$, $k \equiv 1$ on $X \backslash N$).

(b) This is obvious. $\square$

It is clear that if $\mathcal{C}$ is a chain of topologies on a set X such that $\mathcal{C} \subseteq \mathcal{R}(X)$ (respectively $\mathcal{C} \subseteq \mathcal{CR}(X)$), and if $\mathcal{U} = \cup \mathcal{C}$, then $\mathcal{U} \in \mathcal{R}(X)$ (respectively $\mathcal{U} \in \mathcal{CR}(X)$); if in addition each $x \in X$ is u-isolated for no $u \in \mathcal{C}$, then also $\langle X, \mathcal{U} \rangle$ has no isolated points. This remark validates the existence of a topology $\mathcal{T}$ as in the following theorem, which theorem in all its essentials is due to Hewitt [25].

Theorem 4.3. *Let $\langle X, t \rangle$ be a Hausdorff space.*

(i) *If $\langle X, t \rangle$ is irresolvable set $\mathcal{T} = t$.*

(ii) *If $\langle X, t \rangle$ is resolvable and $t \in \mathcal{R}(X)$ (respectively $t \in \mathcal{CR}(X)$) let $\mathcal{T}$ be maximal in $\mathcal{R}(X)$ (respectively in $\mathcal{CR}(X)$) with respect to these properties: $\mathcal{T} \supseteq t$ and no point of X is $\mathcal{T}$-isolated.*

Then $\langle X, \mathcal{T} \rangle$ is irresolvable.

Proof. Suppose there is $D \subseteq X$ such that D and $X \backslash D$ are both $\mathcal{T}$-dense, and define $f \equiv 0$ on D, $f \equiv 1$ on $X \backslash D$. Clearly $\mathcal{T}_f \supseteq \mathcal{T}$ and $\mathcal{T}_f \neq \mathcal{T}$, and from 4.2(a) follows $\mathcal{T}_f \in \mathcal{R}(X)$ (respectively $\mathcal{T}_f \in \mathcal{CR}(X)$). It is enough therefore by 4.2(b) to show that no $W \in \mathcal{T}$ and open $V \subseteq \mathbb{R}$ satisfy $0 < |W \cap f^{-1}(V)| < \omega$. If these inequalities hold for some W and V then since $\mathcal{T}$ is a Hausdorff topology with no isolated points we have $|W| \geqslant \omega$, so we may assume without loss of generality that, say, $0 \in V$ and $1 \notin V$. Then $W \backslash D = W \backslash f^{-1}(V)$ is a nonempty, $\mathcal{T}$-open set disjoint from the $\mathcal{T}$-dense set D, a contradiction. $\square$

Remark 4.4. (a) It should be noted explicitly, as pointed out to us in conversation by Jan van Mill (and perhaps noted by some of the authors cited in our list of references) that for every cardinal number α there are irresolvable Tychonoff spaces $\langle X, \mathcal{T} \rangle$ such that $\Delta(X) > \alpha$. That is, the irresolvable topologies of 4.3 are not "almost discrete", achieved by making all open sets "small". For specific examples let Y be an arbitrary Tychonoff space with $|Y| > 1$, let u be the product topology on the set $X := (Y)^{\alpha^+}$, and let t be the topology on X generated by the sets $\bigcap \mathcal{U}$ with $\mathcal{U} \subseteq u$ and $|\mathcal{U}| \leqslant \alpha$.

Fixing $p \in Y$ and defining $D = \{x \in X: |\{\xi < \alpha^+: x_\xi \neq p\}| \leqslant \alpha\}$, one sees easily that D and $X \backslash D$ are t-dense in X. The topology $\mathcal{T}$ defined on X from t as in 4.3(ii) is then irresolvable with no isolated points, and every nonempty $W \in \mathcal{T}$ satisfies $|W| > \alpha$: Otherwise, choosing $x \in W$, we find $\mathcal{U} \subseteq u$ such that $|\mathcal{U}| \leqslant \alpha$ and $W \cap (\cap \mathcal{U}) = \{x\}$; then from $\cap \mathcal{U} \in t \subseteq \mathcal{T}$ follows $\{x\} \in \mathcal{T}$, contrary to the definition of $\mathcal{T}$.

(b) Informally, Hewitt's theorem (4.3 above) is sometimes stated "Every topology without isolated points expands to an irresolvable topology without isolated points", or, alternatively, "Every maximal topology without isolated points is irresolvable". The use of Zorn's Lemma required here is hardly upsetting, but honesty compels us to admit at this point that we do not know of any infinite irresolvable Hausdorff space without isolated points which is explicitly defined, described, or constructed. In this connection see Problem 8.8 below.

5. Concerning S_ω-like spaces

With the following definition we introduce a tool which proves helpful in identifying resolvable spaces. We write $^{<\omega}\omega = \bigcup_{n<\omega} {}^n\omega$, and for $s \in {}^n\omega$ and $m < \omega$ we denote by $s^\frown m$ that function $t \in {}^{n+1}\omega$ such that $t|n = s$ and $t(n) = m$. For $s \in {}^n\omega$ we write $l(s) = n$; here $l(s)$ is called the *length* of s.

Notation 5.1. Let $X = \langle X, \mathcal{T} \rangle$ be a space.

(a) For $x \in X$ we write $\mathcal{N}(x) = \{U: x \in U \in \mathcal{T}\}$.

(b) For $A \subseteq X$, we denote by $\mathrm{acc}_X A$ (or by $\mathrm{acc}\, A$ if ambiguity is impossible) the set of points $x \in X$ for which every $U \in \mathcal{N}(x)$ satisfies $|U \cap A| \geqslant \omega$.

Definition 5.2. A space X is an S_ω-*like space* (with root $x \in X$) if X may be indexed by $^{<\omega}\omega$ so that

- $x = x_{\langle\rangle}$ with $\langle\rangle$ the empty sequence,
- $\{x_{s^\frown m}: m < \omega\}$ is discrete for each $s \in {}^{<\omega}\omega$, and
- $x_s \in \mathrm{acc}\{x_{s^\frown m}: m < \omega\}$ for each $s \in {}^{<\omega}\omega$.

(As indicated in our Introduction, this concept and terminology are derived from the familiar work of Arhangel'skiĭ and Franklin [2].)

Remarks 5.3. (a) If $x_s \in X$ with $s \in {}^n\omega$ and X an S_ω-like space, then $\bigcup_{k \geqslant n}\{x_t: t \in {}^k\omega, t|n = s\}$ is an S_ω-like subspace of X with root x_s.

(b) In order that a space X contain an S_ω-like space with root $x \in X$, it is necessary and sufficient that there be a countable space S with $x \in S \subseteq X$ such that every $p \in S$ satisfies $p \in \mathrm{acc}\, A$ for some discrete $A \subseteq S$.

Lemma 5.4. *Let* $X = \{x_s: s \in {}^{<\omega}\omega\}$ *be an* S_ω-*like space. Then* X *is* ω-*resolvable.*

Proof. If U is open in X and $x_s \in U$ with $l(s) = n < \omega$, then for each $k > n$ the set U contains points x_t with $l(t) = k$. It follows that if $\{A_m: m < \omega\}$ is a collection of

infinitely many pairwise disjoint infinite subsets of ω and if $D_m = \{x_s: l(s) \in A_m\}$, then the sets D_m $(m < \omega)$ are dense in X and are pairwise disjoint. $\square$

6. Concerning countably compact spaces

As usual, we say that a space X is *countably compact* if every infinite $A \subseteq X$ satisfies $\mathrm{acc}_X A \neq \emptyset$.

Our goal in this section is to prove that every regular, Hausdorff countably compact space without isolated points is ω-resolvable. Although the required infinite family of disjoint dense subsets can be enumerated explicitly using our techniques, we find it convenient and more efficient to proceed by way of the following pretty result of Illanes [26]. (For proofs of this result less complicated than the original, and for related new results, see Feng and Masaveu [21] and Bhaskara Rao [38]. This latter paper shows *inter alia* that if X is κ-resolvable for all $\kappa < \alpha$ with $\mathrm{cf}(\alpha) = \omega$, then X is α-resolvable.)

Theorem 6.1 (Illanes [26]). *If the space X is n-resolvable for all $n < \omega$, then X is ω-resolvable.*

Notation 6.2. For X a space and $Y \subseteq X$ we set

$$\widetilde{Y} = \bigcup \{\mathrm{acc}_X D: \ D \subseteq Y, \ |D| = \omega, \ D \text{ is discrete}\}.$$

Lemma 6.3. *If X is a regular, Hausdorff countably compact space and $Y \subseteq X$, then $Y \cup \widetilde{Y}$ is countably compact.*

Proof. We show that every infinite subset E of $Y \cup \widetilde{Y}$ has an accumulation point in $Y \cup \widetilde{Y}$. The separation hypotheses imply that each such E contains an infinite discrete subset, so we may assume without loss of generality that E itself is discrete and satisfies $|E| = \omega$. We assume also $E \subseteq Y$ or $E \subseteq \widetilde{Y}$. In the former case we have $\emptyset \neq \mathrm{acc}_X E \subseteq \widetilde{Y}$. In the latter case for each $e \in E$ we choose a countable, discrete subset $D(e)$ of Y such that $e \in \mathrm{acc}\, D(e)$ and we choose a pairwise disjoint family $\{U(e): \ e \in E\}$ of open subsets of X with $e \in U(e)$. The set $D = \bigcup_{e \in E} (U(e) \cap D(e))$ is discrete with $|D| = \omega$, and $\emptyset \neq \mathrm{acc}_X E \subseteq \mathrm{acc}_X D \subseteq \widetilde{Y}$. $\square$

Notation 6.4. For a space X we write $A(X) = \widetilde{X}$, $V(X) = \mathrm{int}_X A(X)$, and $I(X) = X \backslash A(X)$.

Remarks 6.5. (a) As a mnemonic, we think of $A(X)$ (respectively $I(X)$) as the set of points which are accessible (respectively inaccessible) through countable, discrete subsets of X.

(b) It is essential in what follows to recall that our definition of α-resolvable implies that the empty space $\emptyset$ is α-resolvable for all cardinals α. The proofs of 6.6–6.9 should be read with a view to the possibility that certain of the spaces $V(X)$, $I(X)$ and $F(X)$—even X itself, hence $A(X)$—may be empty.

Lemma 6.6. *Let X be a regular, Hausdorff countably compact space without isolated points. Then*

(a) $A(X)$ *is dense in X; and*

(b) $V(X)$ *is ω-resolvable.*

Proof. (a) is immediate from the hypothesized separation properties.

(b) Clearly every point of $V(X)$ is the base point of an S_ω-like space $S(x)$ such that $S(x) \subseteq V(X)$. Since $S(x)$ is ω-resolvable by 5.4, $V(X)$ is then ω-resolvable by 2.2. $\square$

A version of the following result, which is preliminary to Theorem 6.9, is given in the paper [11].

Theorem 6.7. *Let X be a regular, Hausdorff countably compact space without isolated points. Then X is resolvable.*

Proof. Since Lemmas 6.6(b) and 2.1 show that $\overline{V(X)}$ is resolvable, it is by 2.2 enough to show that $X \backslash \overline{V(X)}$ is resolvable. From 6.6(a) the set $A(X) \cap (X \backslash \overline{V(X)})$ is dense in $X \backslash \overline{V(X)}$; its relative complement $I(X) \cap (X \backslash \overline{V(X)})$ is also dense in $X \backslash \overline{V(X)}$, since if some nonempty open subset U of X satisfies $U \subseteq X \backslash \overline{V(X)}$ and $U \cap I(X) = \emptyset$, then $U \subseteq A(X)$ and hence $U \subseteq V(X) \subseteq \overline{V(X)}$. $\square$

Lemma 6.8. *Let X be a regular, Hausdorff countably compact space without isolated points. Let V_0 and V_1 be disjoint, dense subsets of $V(X)$, set $F = (A(X) \backslash \overline{V(X)}) \cup (A(X) \backslash \overline{V(X)})^\sim$ and define $D = V_0 \cup I(X)$ and $E = V_1 \cup F$. Then*

(1) *D and E are dense subsets of X;*

(2) *$D \cap E = \emptyset$; and*

(3) *F is a countably compact space without isolated points.*

Proof. (1) Let U be a nonempty open subset of X. If U meets $V(X)$ then U meets both V_0 and V_1, so we assume $U \cap V(X) = \emptyset$. If $U \cap I(X) = \emptyset$ we have the contradiction $U \subseteq V(X)$, so $\emptyset \neq U \cap I(X) \subseteq U \cap D$. From $A(X) \backslash \overline{V(X)} \subseteq E$ follows $U \cap E \neq \emptyset$.

(2) The four sets V_0, V_1, F and $I(X)$ are pairwise disjoint.

(3) That F is countably compact follows from Lemma 6.2. For every $W \in \mathcal{N}(x)$ with $x \in F$ the set $W \cap [A(X) \backslash \overline{V(X)}]$ is open, nonempty, and a subset of $W \cap F$, so $W \cap F$ is infinite. $\square$

Theorem 6.9. *Every regular, Hausdorff countably compact space without isolated points is ω-resolvable.*

Proof. According to 6.1, it is enough to show for $2 \leqslant n < \omega$ that X is n-resolvable. For $n = 2$ the statement is given by 6.7.

Suppose the statement true for $n = k$, let X be a space as hypothesized, and define F as in 6.8. Let $\{V_m : 0 \leqslant m \leqslant k + 1\}$ be disjoint, dense subsets of $V(X)$ as given by

6.6(b), let $\{F_m: 1 \leqslant m \leqslant k\}$ be disjoint, dense subsets of F as given by the inductive hypothesis, and define

$$D_0 = V_0 \cup I(X) \quad \text{and} \quad D_m = V_m \cup F_m \quad \text{for } 1 \leqslant m \leqslant k.$$

Then $\{D_m: 0 \leqslant m \leqslant k\}$ is a family of $(k+1)$-many pairwise disjoint, dense subsets of X, as required. $\square$

7. Concerning pseudocompact spaces

While the relation between countable compactness and resolvability is set forth adequately in Theorem 6.9, the situation with respect to pseudocompactness has not yet been fully determined. In particular, we do not know (in ZFC) whether every pseudocompact Tychonoff space without isolated points is resolvable. The best contributions to this question have been given, albeit peripherally since their principal interest is elsewhere, by Kunen, Szymański and Tall [28]. They established the following two results.

Theorem 7.1. *Assume $V = L$. Then every Baire space (in particular, every Tychonoff pseudocompact space) without isolated points is resolvable.*

Theorem 7.2. *If ZFC is consistent with the existence of a measurable cardinal, then ZFC is consistent with the existence of an irresolvable (zero-dimensional, Tychonoff) Baire space.*

(Strictly speaking, the authors of [28] proved 7.1 only for card-homogeneous Baire spaces of regular cardinality. We have been informed by Frank Tall, however, that the proof in [28] requires only a routine and straightforward modification to cover the unrestricted case.)

Since the spaces constructed in [28] are not pseudocompact, that paper leaves open the question whether in ZFC every pseudocompact space without isolated points is resolvable. In this section we establish a property of pseudocompact spaces without isolated points which according to Theorem 4.3 is weaker: no such topology is maximal in $\mathcal{R}(X)$ or in $\mathcal{CR}(X)$.

Definitions and Discussion 7.3. For a space X and $\mathcal{A} = \{A_i: i \in I\}$ an indexed family of subsets of X and $x \in X$, the family $\mathcal{A}$ is said to be *locally finite* at x if some $U \in \mathcal{N}(x)$ satisfies $|\{i \in I: U \cap A_i \neq \emptyset\}| < \omega$. In what follows, the set of points of X at which $\mathcal{A}$ is not locally finite is denoted $\mathrm{NLF}_X(\mathcal{A})$, or simply $\mathrm{NLF}(\mathcal{A})$ when ambiguity is impossible. In order that our arguments apply not only when $\mathcal{T} \in \mathcal{CR}(X)$ but also when $\mathcal{T} \in \mathcal{R}(X)$, we choose for our definition of pseudocompactness the following: a space $\langle X, \mathcal{T} \rangle$ is pseudocompact if every infinite $\mathcal{A} \subseteq \mathcal{T} \setminus \{\emptyset\}$ satisfies $\mathrm{NLF}(\mathcal{A}) \neq \emptyset$. Of course, a space $\langle X, \mathcal{T} \rangle$ with an unbounded continuous function $f: X \to \mathbb{R}$ is not pseudocompact (as is witnessed by the family $\mathcal{A} = \{f^{-1}(-n, +n): n \in \mathbb{N}\} \setminus \{\emptyset\}$); it is a well-known theorem that for completely regular spaces $\langle X, \mathcal{T} \rangle$ the converse holds: if some infinite $\mathcal{T}$-open

family is locally finite, then some $\mathcal{T}$-continuous function from X to $\mathbb{R}$ is unbounded (see Engelking [20, 3.10.22] for a proof and for citations to the relevant literature).

Lemma 7.4. *Let* $\kappa \geqslant \omega$ *and let* $\langle X, \mathcal{T} \rangle$ *be a regular, κ-resolvable space. Then there is* $f : X \to \mathbb{R}$ *such that* $\langle X, \mathcal{T}_f \rangle$ *is κ-resolvable and not pseudocompact.*

Proof. Let $\mathcal{D} = \{D_{\xi,n} \colon \xi < \kappa, \ n < \omega\}$ be a set of pairwise disjoint dense subsets of $\langle X, \mathcal{T} \rangle$ faithfully indexed by $\kappa \times \omega$, and let $\{q_n \colon n < \omega\}$ be an enumeration of $\mathbb{Q} \backslash \{0\}$. Define $f : X \to \mathbb{R}$ by the rule

$$f \equiv \begin{cases} q_n & \text{on } \bigcup_{\xi < \kappa} D_{\xi,n}, \\ 0 & \text{on } X \backslash \bigcup \mathcal{D}. \end{cases}$$

Since f is $\mathcal{T}_f$-continuous and $\mathbb{Q}$ is unbounded in $\mathbb{R}$, the space $\langle X, \mathcal{T}_f \rangle$ is not pseudo-compact.

For $\xi < \kappa$ let $E_\xi = \bigcup_{n < \omega} D_{\xi,n}$. Then for $\emptyset \neq W \in \mathcal{T}$ and nonempty open $V \subseteq \mathbb{R}$ and $\xi < \kappa$ we have

$$W \cap f^{-1}(V) \cap E_\xi \supseteq W \cap f^{-1}(\{q_n\}) \cap E_\xi = W \cap D_{\xi,n} \neq \emptyset$$

for each n such that $q_n \in V$. Thus the pairwise disjoint sets E_ξ are $\mathcal{T}_f$-dense in X, as required. $\square$

Using notation suggested by Frolík [22,23] or Bernstein [3], for X a space and $x \in X$ and $C_n \subseteq X$ ($n < \omega$) we write $x = \lim_n C_n$ if for every $U \in \mathcal{N}(x)$ there is $k < \omega$ such that $\bigcup_{k < n < \omega} C_n \subseteq U$. We use also the standard "refinement" notation $\prec$ defined as follows: If $\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{P}(X)$ then $\mathcal{A} \prec \mathcal{B}$ means that for each $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$.

Lemma 7.5. *Let* $\langle X, \mathcal{T} \rangle$ *be a regular pseudocompact Hausdorff space and let U be a nonempty $\mathcal{T}$-open set with this property:*

> *For every sequence* $C = \{C_n \colon n < \omega\}$ *of nonempty $\mathcal{T}$-open subsets of U* (∗)
> *and for every* $x \in X$*, the relation* $x = \lim_n C_n$ *is false.*

Then

(a) *Every sequence* $C = \{C_n \colon n < \omega\}$ *of nonempty $\mathcal{T}$-open subsets of U satisfies* $|\mathrm{NLF}_X(C)| \geqslant \omega$*; and*

(b) *If C is a sequence as in* (a) *and* $W \in \mathcal{T}$ *satisfies* $W \cap \mathrm{NLF}(C) \neq \emptyset$*, then* $|W \cap \mathrm{NLF}(C)| \geqslant \omega$*.*

Proof. (a) Let $C_0 = C$. Since X is pseudocompact there is $x_0 \in \mathrm{NLF}(C_0)$. Since the relation $\lim_n C_n = x_0$ fails and X is regular, there is $V_0 \in \mathcal{N}(x_0)$ such that the family C_1 of nonempty sets of the form $C_n \backslash \overline{V_0}$ with $C_n \in C$ is infinite; there is $x_1 \in \mathrm{NLF}(C_1)$. Now if $k < \omega$ and points x_i ($i < k$) and open families $C_0, C_1, \ldots, C_{k-1}$ have been chosen with $C_{i+1} \prec C_i$ for $0 \leqslant i < i + 1 < k$ and $x_i \in \mathrm{NLF}(C_i) \subseteq \mathrm{NLF}(C_0)$, let $\{C_{k-1,n} \colon n < \omega\}$ index C_{k-1} and find $V_{k-1} \in \mathcal{N}(x_{k-1})$ such that the family C_k of

nonempty sets of the form $C_{k-1,n} \backslash \overline{V_{k-1}}$ is infinite, and choose $x_k \in \mathrm{NLF}(C_k)$. The recursive definition is complete, and $\{x_k\colon k < \omega\}$ is a faithfuly indexed subset of $\mathrm{NLF}(\mathcal{C})$.

(b) Let $x \in W \cap \mathrm{NLF}(\mathcal{C})$, let $N \in \mathcal{N}(x)$ satisfy $N \subseteq \overline{N} \subseteq W$, and set $\mathcal{D} = \{C_n \cap N\colon C_n \cap N \neq \emptyset\}$. Since $\mathcal{D}$ is an infinite family of nonempty $\mathcal{T}$-open subsets of U we have $|\mathrm{NLF}(\mathcal{D})| \geqslant \omega$ by part (a), and

$$\mathrm{NLF}(\mathcal{D}) \subseteq \overline{N} \cap \mathrm{NLF}(\mathcal{C}) \subseteq W \cap \mathrm{NLF}(\mathcal{C}). \qquad \square$$

The following theorem and its corollary give the statement denoted (B) in our Abstract.

Theorem 7.6. *Let $\langle X, \mathcal{T} \rangle$ be a space with no isolated points such that $\mathcal{T} \in \mathcal{R}(X)$ (respectively $\mathcal{T} \in \mathcal{CR}(X)$). Then*

(a) *there is on X a topology $\mathcal{U} \in \mathcal{R}(X)$ (respectively $\mathcal{U} \in \mathcal{CR}(X)$) of the form $\mathcal{U} = \mathcal{T}_f$ such that $\langle X, \mathcal{U} \rangle$ has no isolated points, $\mathcal{U} \supseteq \mathcal{T}$, and $\langle X, \mathcal{U} \rangle$ is not pseudocompact; and*

(b) *if in addition $\langle X, \mathcal{T} \rangle$ is countably compact then $\mathcal{U} = \mathcal{T}_f$ may be chosen so that $\langle X, \mathcal{U} \rangle$ is ω-resolvable.*

Proof. Statement (b) is a combination of 6.9 and (the case $\kappa = \omega$ of) 7.4, included here for clarity and completeness. We prove (a). If $\mathcal{T}$ is not pseudocompact we may define $f \in {}^X\mathbb{R}$ by $f \equiv 0$ and set $\mathcal{U} = \mathcal{T}_f = \mathcal{T}$, so we assume that $\mathcal{T}$ is pseudocompact.

Case 1. For $\emptyset \neq U \in \mathcal{T}$ there is $x \in X$ and a sequence $\mathcal{C} = \{C_n\colon n < \omega\}$ of nonempty $\mathcal{T}$-open subsets of U such that $x = \lim_n C_n$. We will show in this case that every nonempty $U \in \mathcal{T}$ contains an S_ω-like space. First, given such U, let $V \in \mathcal{T}$ satisfy $\emptyset \neq V \subseteq \overline{V} \subseteq U$. Then set $C_{\langle\rangle} = V$ and find $x = x_{\langle\rangle} \in X$ and disjoint open subsets $C_n = C_{\langle\rangle\frown n} \subseteq V$ such that $x_{\langle\rangle} = \lim_n C_{\langle\rangle\frown n}$. If $k < \omega$ and x_s, $C_{s\frown n}$ have been defined for each $s \in {}^k\omega$ with $x_s = \lim_n C_{s\frown n}$ and $C_{s\frown n}$ disjoint open subsets of C_s, then for $t \in {}^{k+1}\omega$ (say $t = s \cup \{\langle k, m\rangle\} = s\frown m$) choose $x_t \in X$ and pairwise disjoint open subsets $C_{t\frown n}$ of C_t such that $x_t = \lim_n C_{t\frown n}$. Then $\{x_s\colon s \in {}^{<\omega}\omega\}$ is an S_ω-like subspace of $\overline{V} \subseteq U$. Indeed not only does each x_s satisfy $x_s \in \mathrm{acc}\{x_{s\frown n}\colon n < \omega\}$ but in fact each x_s is the limit of the sequence $x_{s\frown n}$.

It follows in this case from 5.4, 1.1 and 1.2 that $\langle X, \mathcal{T} \rangle$ is ω-resolvable, so from 7.4 (and 4.2(a)) there is $f \in {}^X\mathbb{R}$ such that $\langle X, \mathcal{U} \rangle$ with $\mathcal{U} = \mathcal{T}_f$ is as required.

Case 2. Case 1 fails. Then there is a nonempty open subset U of X satisfying condition (∗) of Lemma 7.5. Let $\mathcal{C} = \{C_n\colon n < \omega\}$ be a sequence of pairwise disjoint nonempty open subsets of U, for $n < \omega$ let $D_n \in \mathcal{T}$ satisfy $\emptyset \neq D_n \subseteq \overline{D_n} \subseteq C_n$, and set $\mathcal{D} = \{D_n\colon n < \omega\}$. For later use let us verify the relation

$$\mathrm{NLF}(\mathcal{D}) = \overline{\bigcup \mathcal{D}} \, \backslash \, \bigcup \{\overline{D_n}\colon n < \omega\}. \qquad (**)$$

($\subseteq$) The inclusion $\mathrm{NLF}(\mathcal{D}) \subseteq \overline{\bigcup\mathcal{D}}$ is clear, and if $x \in \overline{D_n}$ then from $C_n \in \mathcal{N}(x)$ and $C_n \cap D_k = \emptyset$ ($n \neq k$) follows $x \notin \mathrm{NLF}(\mathcal{D})$.

($\supseteq$) Let $x \in \overline{\bigcup \mathcal{D}} \backslash \bigcup \{\overline{D_n}: n < \omega\}$ and suppose for some $W \in \mathcal{N}(x)$ that the set $F = \{n < \omega: W \cap D_n \neq \emptyset\}$ is finite. Then $W \backslash \bigcup_{n \in F} \overline{D_n}$ is a neighborhood of x disjoint from $\bigcup \mathcal{D}$, a contradiction.

The verification of ($**$) is complete. Let $S = \{s_n: n < \omega\}$ be an unbounded subset of $\mathbb{R}$, and fix $s \in \mathbb{R} \backslash S$. Define $f: X \to \mathbb{R}$ by the rule

$$f \equiv \begin{cases} s_n & \text{on } \overline{D_n}, \\ s & \text{on } X \backslash \bigcup_n \overline{D_n}, \end{cases}$$

and set $\mathcal{U} = \mathcal{T}_f$. Then $\mathcal{U} \in \mathcal{R}(X)$ (respectively $\mathcal{U} \in \mathcal{CR}(X)$) by Lemma 4.2(a), and since f is $\mathcal{T}_f$-continuous and f is unbounded the space $\langle X, \mathcal{T}_f \rangle$ is not pseudocompact. To complete the proof it is enough (by Lemma 4.2(b)) to show that every nonempty set N of the form $N = W \cap f^{-1}(V)$ (with $W \in \mathcal{T}$, V open in $\mathbb{R}$) satisfies $|N| \geq \omega$.

If some $n < \omega$ satisfies $s_n \in V$ and $W \cap D_n \neq \emptyset$, or if $W \cap [X \backslash \overline{\bigcup \mathcal{D}}] \neq \emptyset$, then N contains a nonempty $\mathcal{T}$-open set (either $W \cap D_n$ or $W \backslash \overline{\bigcup \mathcal{D}}$) and hence $|N| \geq \omega$. We assume in what follows, therefore, writing $\mathcal{D}_0 = \{D_n \in \mathcal{D}: s_n \in V\}$ and $\mathcal{D}_1 = \mathcal{D} \backslash \mathcal{D}_0$ and $E_i = \bigcup \{\overline{D_n}: D_n \in \mathcal{D}_i\}$ for $i = 0, 1$, that $W \cap E_0 = \emptyset$ (hence, $W \cap \overline{E_0} = \emptyset$) and that $W \subseteq \overline{\bigcup \mathcal{D}}$. Now either of the two conditions (1) $s \notin V$ or (2) $W \subseteq E_0 \cup E_1$ would entail $W \subseteq E_1$ and hence $N = W \cap f^{-1}(V) = \emptyset$, so (1) and (2) are false and we have

$$\emptyset \neq W \cap \left(\overline{\bigcup \mathcal{D}} \backslash \bigcup \{\overline{D_n}: D_n \in \mathcal{D}\} \right) = W \cap f^{-1}(\{s\})$$

$$\subseteq W \cap f^{-1}(V) = N. \qquad (***)$$

Since $\overline{\bigcup \mathcal{D}} \backslash \bigcup \{\overline{D_n}: n < \omega\} = \mathrm{NLF}(\mathcal{D})$ by ($**$) and $W \cap \mathrm{NLF}(\mathcal{D}) \neq \emptyset$ yields $|W \cap \mathrm{NLF}(\mathcal{D})| \geq \omega$ by Lemma 7.5(b), from ($***$) we have $|N| \geq \omega$, as required. $\square$

Corollary 7.7. *Let $\langle X, \mathcal{T} \rangle$ be an infinite Hausdorff space and let $\mathcal{T} \in \mathcal{R}(X)$ (respectively $\mathcal{T} \in \mathcal{CR}(X)$) be maximal (among all topologies in $\mathcal{R}(X)$ (respectively in $\mathcal{CR}(X)$)) with respect to this property: no point of X is $\mathcal{T}$-isolated. Then $\langle X, \mathcal{T} \rangle$ is not pseudo-compact.*

Proof. From 7.6(a). $\square$

Remark 7.8 (added September, 1995). While reading a prepublication copy of this paper circulated by the authors to selected colleagues, Professor Ronnie Levy of George Mason University noticed a proof of Theorem 7.6(a) and Corollary 7.7 which is in the authors' opinion more efficient or pleasing than the one presented above. We are grateful to Professor Levy for permission to include his argument here. We continue the notation of 7.6 and 7.7. First, recall this familiar property of maximal topologies without isolated points: every such space $X = \langle X, \mathcal{T} \rangle$ is *extremally disconnected* in the sense that every $U \in \mathcal{T}$ satisfies $\overline{U}^X \in \mathcal{T}$. (The proof of this fact, already noted by Hewitt [25, Theorem 39], is not difficult: if some $U \in \mathcal{T}$ has $\overline{U}^X \notin \mathcal{T}$ then the topology $\mathcal{U}$ on X with subbase $\mathcal{T} \cup \{\overline{U}^X\}$ again has no isolated points and satisfies $\mathcal{U} \supseteq \mathcal{T}$ and $\mathcal{U} \neq \mathcal{T}$,

a contradiction.) Suppose now that $\langle X, \mathcal{T} \rangle$ is a counterexample to Corollary 7.7. From the remark just made it is clear that $\mathcal{T}$ has a base of open-and-closed sets so there is a sequence $\mathcal{D} = \{D_n: \ n < \omega\}$ of pairwise disjoint, nonempty open-and-closed subsets of X. Since X is pseudocompact the closed set $B := \mathrm{NLF}(\mathcal{D}) = \overline{\bigcup \mathcal{D}} \setminus (\bigcup \mathcal{D})$ is nonempty. It is clear that $B \notin \mathcal{T}$. We claim that no point of B is isolated in B. Indeed let $x \in B$ and $U \in \mathcal{N}(x)$ be given, choose $V \in \mathcal{N}(x)$ such that $V \subseteq \overline{V} \subseteq U$, and let I be the infinite set $I = \{n < \omega: \ V \cap D_n \neq \emptyset\}$. For $n \in I$ let $E_{n,i}$ ($i = 0, 1$) be disjoint nonempty open subsets of $V \cap D_n$ and set $\mathcal{E}_i = \{E_{n,i}: \ n \in I\}$. Since X is extremally disconnected the disjoint sets $\bigcup \mathcal{E}_i$ ($i = 0, 1$) have disjoint closures, so with $x_i \in \mathrm{NLF}(\mathcal{E}_i)$ ($i = 0, 1$) we have $x_0 \neq x_1$ and $x_i \in B \cap \overline{V} \subseteq B \cap U$. The claim is proved. Now let $\mathcal{U}$ be the topology on X for which $\mathcal{T} \cup \{B\}$ is a subbase. From $B \in \mathcal{U} \setminus \mathcal{T}$ it follows that $\mathcal{U} \neq \mathcal{T}$, and since each $U \in \mathcal{T}$ satisfies $U \cap B = \emptyset$ or $|U \cap B| \geqslant \omega$ the space $\langle X, \mathcal{U} \rangle$ has no isolated points. That $\mathcal{U} \in \mathcal{R}(X)$ (respectively $\mathcal{U} \in \mathcal{CR}(X)$) may be seen directly or by an appeal to Lemma 4.2(b): the set B is open-and-closed in $\mathcal{U}$, so in fact $\mathcal{U} = \mathcal{T}_f$ with $f : X \to \mathbb{R}$ the characteristic function of B.

8. Comments and questions

Concerning resolvability there is a rich and extensive literature not touched upon in earlier portions of this selective survey. Here we indicate briefly some directions which may interest the reader and we list a few questions which to our best knowledge remain unsolved.

8.1. For results on maximal resolvablility, especially in the context of product spaces, see Ceder [7], Ceder and Pearson [8], and El'kin [16,17]. It was proved by Velichko [39] that every Hausdorff k-space is ω-resolvable, and by Pytke'ev [37] that each such space is even maximally resolvable.

8.2. For examples of connected, Hausdorff irresolvable spaces, see Padmavally [34] and Douglas Anderson [1].

8.3. For results (not closely related to our 7.6 and 7.7) on properties of pseudocompactness type and their minimal or maximal status within various lattices of topologies, see Cameron [5,6]; see also Porter, Stephenson and Woods [35] and their list of references.

8.4. For fixed $n < \omega$ there are spaces which are n-resolvable but not $(n+1)$-resolvable; see El'kin [18], van Douwen [14, Section 5], Feng and Masaveu [21], and Eckertson [15] for a variety of constructions. Some of these spaces are Lindelöf (indeed, countable) and regular Hausdorff, hence normal and hence Tychonoff. The existence of such spaces may be compared with the theorem of Illanes [26] we cited in 6.1 above and with Question 8.14 below.

8.5. The extremally disconnected group topology constructed by Malykhin [30] on the countable Boolean group $B = \bigoplus_\omega \{0, 1\}$ in the system [ZFC + P(c)] is maximal among regular Hausdorff topologies without isolated points on B, hence is not resolvable. Supplementing Malykhin's result is a theorem of Comfort and van Mill [13] asserting in ZFC that every nondiscrete Hausdorff group topology on an Abelian group containing no iso-

morph of B is necessarily resolvable. For generalizations of this result see Masaveu [32] and Comfort, Masaveu and Zhou [11].

8.6. The above-cited works contain theorems asserting the resolvability of groups satisfying various boundedness conditions, as do Comfort, Gladdines and van Mill [10], Villegas-Silva [40] and Masaveu [32]. Many of these theorems can be simultaneously encompassed by using the following result of Protasov.

Theorem (Protasov [36]). *Let G be a nondiscrete topological group and let A be an infinite subset of G which is bounded (in the sense that for every $U \in \mathcal{N}(e)$ there is finite $F \subseteq G$ such that $A \subseteq FU \cup UF$). Then the set AA^{-1} contains an infinite, discrete, nonclosed subset.*

From that result it is immediate that a nondiscrete locally bounded group G contains a countably infinite, discrete, nonclosed subset, hence (in the notation of 6.4 above) satisfies $G = A(G) = \text{int}_G A(G) = V(G)$, hence is ω-resolvable by 6.6(a). It follows in particular that if a group G embeds isomorphically into some compact group (these are the groups G called *maximally almost periodic* in the sense of Von Neumann [33]) then there is a countably infinite family $\{D_n\colon n < \omega\}$ of pairwise disjoint subsets of G each of which is dense in every totally bounded group topology on G (for, such a group G admits a largest totally bounded group topology, and sets dense in that topology are evidently dense in every totally bounded group topology); for comments in this direction see [32, Section 5] and [11, 5.10]. Much more striking results have been announced recently by Malykhin and Protasov [31], as follows.

Theorem (Malykhin and Protasov [31]). *Every infinite group G admits a family of $|G|$-many pairwise disjoint subsets, each dense in each totally bounded group topology; if in addition $|G|$ is regular, the sets may be chosen dense in each group topology $\mathcal{T}$ on G with the property that for every $U \in \mathcal{T}\setminus\{\emptyset\}$ there is $F \subseteq G$ such that $|F| < |G|$ and $G = FU$.*

8.7. Every topological group of the form $G \times G$ (with G nondiscrete) or of the form $G_0 \times G_1$ with G_i Abelian and nondiscrete is resolvable [11]. (Thus in particular the irresolvable topological group B of Malykhin [30] described in 8.5 above, which is algebraically isomorphic to $B \times B$, has $B \times B$ resolvable in the product topology.) Every nondiscrete Abelian topological group which is a Baire space (in the sense that every intersection of countably many dense, open subsets is dense) is resolvable [11]. In recent work based in part on ideas from [28], Bešlagić and Levy [4] have shown that the consistency of the existence of Tychonoff spaces X and Y without isolated points such that $X \times Y$ is irresolvable is equivalent to the consistency of the existence of a measurable cardinal. (We note in passing that if such spaces X and Y exist then their "free disjoint union" Z is of course a Tychonoff space without isolated points such that $Z \times Z$ is not resolvable.)

8.8. To the authors' best knowledge, every known construction of an irresolvable space without isolated points depends on some form of the Axiom of Choice; consider in

this connection for example the maximal topologies of Hewitt [25], the maximal almost disjoint families of van Douwen [14], and the maximal θ-independent families of Kunen, Szymański and Tall [28]. This situation suggests the following question.

Problem. With or without the Axiom of Choice, give a concrete example of an irresolvable Hausdorff space without isolated points (the open sets being explicitly identified in concrete form).

The following unexpected results of Ganster [24] ((a) ⟺ (b)) and El'kin [19] ((a) ⟺ (c)) indicate natural constraints concerning a careful statement of 8.8 and its solution.

Theorem. *For a space* $X = \langle X, \mathcal{T} \rangle$, *the following conditions are equivalent:*

   (a) *X is irresolvable;*

   (b) *the family* $\{\mathrm{int}_X D: D$ *is dense in* $X\}$ *is a filterbase on* X;

   (c) *$\mathcal{T}$ contains a base for an ultrafilter on* X.

Question 8.9 (Malykhin). Let X be a Lindelöf Tychonoff space without isolated points such that $\Delta(X) > \omega$. Is X necessarily resolvable?

(In connection with 8.8 it should be noted, as has been pointed out to us by Malykhin, that although the countable Boolean group defined in [30] in the system [ZFC + P(c)] is irresolvable and Lindelöf, nevertheless every Hausdorff Lindelöf topological group G with $\Delta(G) > \omega$ is resolvable. To see this let $A \subseteq G$ with $|A| = \omega^+$ and let $H = \langle A \rangle \subseteq G$. For every nonempty relatively open subset U of H there is $F \subseteq H$ such that $|F| < \omega^+ = |H|$ and $H = FU$, so H is ω^+-resolvable by the theorem of Malykhin and Protasov cited in 8.6; then G is resolvable by 2.3(a). For a more direct argument as in [32], let us show for arbitrary $\alpha \geqslant \omega$ that every topological group G such that $|G| > \alpha$ and G is α-bounded (in the sense that for every $U \in \mathcal{N}(e)$ there is $F \subseteq G$ such that $|F| \leqslant \alpha$ and $G = FU$) is ω-resolvable. It is by 2.3(a) enough to produce an ω-resolvable subgroup H of G. Let $H_0 = \{0\}$, and suppose that $\xi < \alpha^+$ and that an increasing chain $\{H_\eta: \eta < \xi\}$ of subgroups of G has been defined, with each $|H_\eta| \leqslant \alpha$. If ξ is a limit ordinal set $H_\xi = \bigcup_{\eta<\xi} H_\eta$ and if $\xi = \zeta + 1$ choose $x \in G\backslash H_\zeta$ and set $H_\xi = \langle H_\zeta \cup \{x\} \rangle$. Let $H = \bigcup_{\xi<\alpha^+} H_\xi$ and for $0 \leqslant n < \omega$ set

$$D_n = \bigcup \{H_{\xi+1}\backslash H_\xi: \ \xi < \alpha^+, \ \xi = n \text{ or } \xi = \lambda + n \text{ for some limit ordinal } \lambda\}.$$

To check that the disjoint sets D_n are dense in H let U be nonempty and open in H and let $A \subseteq H$ satisfy $|A| \leqslant \alpha$ and $H = AU$. Given n there is $\xi < \alpha^+$ of the form $\xi = \lambda + n$ such that $A \subseteq H_\xi$, and then with $x \in H_{\xi+1}\backslash H_\xi \subseteq H$, say $x = au$ with $a \in A$, $u \in U$ one has $u \in U \cap D_n$, as required.)

The next two questions are suggested respectively by the work of Ceder and Pearson [8] and by Theorem 2.2 above. We suppose that the answer to Question 8.10(b) is "No"; its relation to Question 8.10(a) is given by Theorem 6.9 above.

Question 8.10. (a) Is there an ω-resolvable Tychonoff space which is not maximally resolvable?

(b) Is every countably compact Tychonoff space maximally resolvable?

Question 8.11. If a Tychonoff space X is the union of maximally resolvable subspaces, must X itself be maximally resolvable?

Concerning Question 8.10(a) it should be noted that El'kin [19] has shown the existence of an ω-resolvable T_1 space which is not maximally resolvable, and Malykhin [29] has shown the existence of such a T_1 space which is even hereditarily resolvable. More recently Eckertson [15], using the existence of a maximal κ-independent family of cardinal at least κ on an uncountable cardinal κ, constructed a Tychonoff space as in 8.10(a). It is known [27] that the existence of such a family is independent of the axioms of ZFC; we see then that (1) Eckertson [15] has given a consistent positive answer to 8.10(a) and (2) his argument cannot be easily restructured to give an absolute example.

Question 8.12. Is a pseudocompact Tychonoff space without isolated points necessarily resolvable?

Question 8.13. Is an infinite connected Tychonoff space necessarily resolvable?

Question 8.14. If α is a limit cardinal with $\mathrm{cf}(\alpha) > \omega$ and a space X is κ-resolvable for each $\kappa < \alpha$, must X be α-resolvable?

(If the inequality $\mathrm{cf}(\alpha) > \omega$ is replaced in the statement of Question 8.14 by the condition $\mathrm{cf}(\alpha) = \omega$, the resulting question is solved affirmatively by the theorem of Bhaskara Rao [38] cited prior to 6.1 above.)

Question 8.15 (Malykhin). Is there, in some model of ZFC, an irresolvable Hausdorff topological group G with $\Delta(G) > \omega$?

The last two questions, also concerning topological groups, should be viewed from the perspective of the results cited in 8.7 above.

Question 8.16. Is a product of nondiscrete Hausdorff topological groups necessarily resolvable?

Question 8.17. Is a nondiscrete Hausdorff group which is a Baire space necessarily resolvable?

Acknowledgements

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