On transient Bessel processes and planar Brownian motion reflected at their future infima

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Received: February 1995

Abstract

Let $R$ be a transient Bessel process and let $J(t) = \inf_{u \geq t} R(u)$ be its future infimum process. The main result of this paper is an integral test characterizing the upper functions of $R - J$, which turn out to be quite different from those of $R$. The test implies in particular an iterated logarithm law recently obtained by Khoshnevisan et al. (1994), and also solves the problem of characterizing the large gaps between the past supremum and future infimum of $R$. The corresponding local question for a planar Wiener process is studied.

AMS 1980 Subject Classification: 60J65, 60G17

Keywords: Future infimum; Bessel gap; Wiener process; Integral test

1. Introduction

Let $\{R(t); t \geq 0\}$ be a Bessel process of dimension $d \geq 2$, i.e. a linear diffusion with generator $\frac{1}{2} f''(x) + \frac{d-1}{2x} f'(x)$. When $d$ is an integer, $R$ can be realized as the Euclidean modulus of a $d$-dimensional Wiener process. See for example Revuz and Yor (1994, Chap. XI) for an account of general properties of Bessel processes. In case $d > 2$, the process $R$ is transient, i.e. $\lim_{t \to \infty} R(t) = \infty$ almost surely (Revuz and Yor, 1994, p. 423). We define the future infimum process

$$J(t) = \inf_{u \geq t} R(u), \quad t \geq 0.$$

The process $J$ has been investigated by Erdős and Taylor (1962) who were interested in the rate of escape of a random walk (Wiener process) in space, and by Pitman (1975) who related, through $J$, a three-dimensional Bessel process with a real-valued Wiener process.
process. Several recent papers are devoted to the study of $J$ in the above two directions. We cite for example Adelman and Shi (1995), Burdzy (1994), Chen and Shao (1993), Khoshnevisan (1995), Khoshnevisan et al. (1994, 1995) and Okoraofoar and Ugbebor (1991) for path properties and Bertoin (1992), Saisho and Tanemura (1990) and Yor (1995) for extensions of Pitman's theorem for general Bessel processes, diffusions and Lévy processes. The present work is concerned with the first aspect. Recall the following known laws of the iterated logarithm (LIL's):

\[
\limsup_{t \to \infty} \frac{R(t)}{(2t \log \log t)^{1/2}} = 1 \quad \text{a.s.} \quad (1.1)
\]

\[
\limsup_{t \to \infty} \frac{J(t)}{(2t \log \log t)^{1/2}} = 1 \quad \text{a.s.} \quad (1.2)
\]

\[
\limsup_{t \to \infty} \frac{R(t) - J(t)}{(2t \log \log t)^{1/2}} = 1 \quad \text{a.s.} \quad (1.3)
\]

The identity (1.1) is the classical LIL for Bessel processes (Revuz and Yor, 1994, Exercise XI.1.20). Okoroafoar and Ugbebor (1991) and Khoshnevisan et al. (1994) independently proved (1.2). See also Conjecture 1 in Chen and Shao (1993). The LIL (1.3) for $R - J$ is due to Khoshnevisan et al. (1994).

The processes $R$, $J$ and $R - J$ satisfying the same LIL's, a natural question is to distinguish their limiting behaviours by studying the upper functions. Those of $R$ are characterized by the Dvoretzky-Erdős-Orey-Pruitt (DEOP) test (stated in (1.4) below, usually referred to as the Orey-Pruitt test (Révész, 1990, Theorem 18.4), or as the Dvoretzky-Erdős test (Ito and McKean, 1965, p.163)), and those of $J$ recently by Khoshnevisan et al. (1995). Comparing these results, it is seen that $R$ and $J$ have different upper functions (though they are related to each other through a Ciesielski-Taylor-type relation, see Khoshnevisan et al. (1995). The aim of this paper is to investigate the corresponding problem for $R - J$. For example, one may ask: do $R$ and $R - J$ have the same upper functions?

The answer is negative. In Section 2, we show the following integral test:

**Theorem 1.** Let $d > 2$. For any non-decreasing function $f > 0$, we have

\[
\left[ R(t) - J(t) > t^{1/2} f(t), \ i.o. \ t \to \infty \right]
\]

\[= \begin{cases} 0 & \iff \int_1^\infty \frac{dt}{t} \frac{t^{4-d}(t) \exp \left( - \frac{f^2(t)}{2} \right)}{< \infty} \\ 1 & = \infty \end{cases}\]

Here and in the sequel, we adopt the usual notation "i.o." $(t \to \infty)$ meaning "infinitely often" (as $t$ tends to infinity).

**Remarks.** (i) Khoshnevisan (1995) studies $t \mapsto M(t) - J(t)$ (with $M(t) \equiv \sup_{0 \leq u \leq t} R(u)$), the gap between the past supremum and future infimum of $R$ (see also Adelman and Shi, 1995; Burdzy, 1994). Since $\sup_{0 \leq u \leq t} (M(u) - J(u)) = \sup_{0 \leq u \leq t} (R(u) - J(u))$, using an argument similar to that presented in Csörgő and Révész (1981, p.
28), it follows that the integral test in Theorem 1 also characterizes the upper functions of \( M - J \).

(ii) As usual, there is a "local" version of Theorem 1 for small times. The statement and the proof are omitted — they actually are very similar to those for large \( t \).

(iii) Recall the DEOP test:

\[
P \left( R(t) > t^{1/2} f(t), \ i.o. \ t \to \infty \right)
\]

\[
= \left\{ \begin{array}{ll}
0 & \iff \int_1^\infty \frac{dt}{t} f^d(t) \exp \left( -\frac{f^2(t)}{2} \right) < \infty \\
\infty & \iff \int_1^\infty \frac{dt}{t} f^d(t) \exp \left( -\frac{f^2(t)}{2} \right) = \infty
\end{array} \right.
\]

(1.4)

(which actually remains true for any dimension \( d \geq 1 \)). Since (1.4) differs from the test given in Theorem 1, \( R \) and \( R - J \) have different upper functions. Indeed, they are quite different, in the sense that \( R \) (stochastically) increases with \( d \), whereas according to Theorem 1, the bigger \( d \) is, the smaller \( R - J \) seems to become! This feature is nonetheless in agreement with the intuitive idea that when \( d \) is big, the \( d \)-dimensional Bessel process \( R \) has little chance to "sink" down to a small level once it achieves a big value.

(iv) It seems curious that \( R - J \) has the same upper functions as a Bessel process of dimension \( 4 - d \). Is this related to the Bessel time-reversal theorem (i.e. the \( d \)-dimensional Bessel process \( R \), killed when exiting from a given level \( a \) for the last time, is a time-reversed Bessel process of dimension \( 4 - d \), starting from \( a \), killed when hitting 0. See Revuz and Yor (1994, Exercise XI.1.23))?

(v) Taking \( d = 3 \) in Theorem 1, \( R - J \) being in this case a reflecting linear Wiener process according to Pitman’s theorem (Pitman, 1975; see also Revuz and Yor, 1994, Theorem VI.3.5); we recover the classical EFKP test (see for example Révész (1990, Theorem 6.2); also referred to as the Kolmogorov test (Ito and McKean, 1965, p.33)) for linear Wiener processes.

In dimension \( d = 2 \), the Bessel process \( R \) is the radial part of an \( \mathbb{R}^2 \)-valued Wiener process, which is no longer transient. The study of future infima is reduced to a local sense. Consider a two-dimensional Bessel process \( \{ R(u); 0 \leq u \leq 1 \} \) starting from 0, and define

\[
\tilde{J}(s) = \inf_{0 \leq u \leq 1} R(u), \quad 0 < s \leq 1,
\]

the local future infimum process (of course \( \tilde{J} \) can also be defined for transient Bessel processes, and it has exactly the same almost sure asymptotic behaviour as \( J \) when \( s \) tends to 0). In dimension 2, known results are as follows:

\[
\limsup_{s \to 0} \frac{R(s)}{(2s \log \log(1/s))^{1/2}} = 1 \quad \text{a.s.}
\]

(1.6)

\[
\limsup_{s \to 0} \frac{\tilde{J}(s)}{(2s \log \log(1/s))^{1/2}} = 1 \quad \text{a.s.}
\]

(1.7)

\[
\limsup_{s \to 0} \frac{R(s) - \tilde{J}(s)}{(2s \log \log(1/s))^{1/2}} = 1 \quad \text{a.s.}
\]

(1.8)
The equality (1.6) is again due to the usual LIL for Bessel processes (more generally, using time reversion for Bessel processes, the DEOP test (1.4) has the following local version:

\[
P\left( R(s) > s^{1/2}g(s), \text{i.o. } s \to 0 \right) = \left\{ \begin{array}{ll}
1 & \text{if } \int_{0^+} \frac{ds}{s} g^2(s) \exp \left( - \frac{g^2(s)}{2} \right) < \infty \\
0 & \text{if } \int_{0^+} \frac{ds}{s} g^2(s) \exp \left( - \frac{g^2(s)}{2} \right) = \infty 
\end{array} \right.
\]

for any non-increasing function \( g > 0 \). The somewhat intriguing identity (1.7) was proved by Okoroafor and Ugbebor (1991). We mention that a characterization of the upper functions of \( \bar{J} \) has recently been established in Hu and Shi (1995). Finally, (1.8) is a straightforward consequence of (1.6) and (1.7).

Roughly, when \( s \) is in the (positive) neighbourhood of 0, \( R(s) \) can reach as high a level of order as \( \left( \log \log \frac{1}{s} \right)^{1/2} \) (at least for a suitably chosen sequence), whereas \( \bar{J}(s) \) can at most go up to a level of order \( \left( \log \log \log \frac{1}{s} \right)^{1/2} \) only. A natural question is: in dimension \( d = 2 \), do \( R \) and \( R - \bar{J} \) have the same upper functions?

The answer is (almost) yes.

**Theorem 2.** Let \( d = 2 \) and let \( g > 0 \) be a non-increasing function such that \( s \mapsto s^{1/2}g(s) \) is non-decreasing. Then

\[
P\left( R(s) - \bar{J}(s) > s^{1/2}g(s), \text{i.o. } s \to 0 \right)
\]

equals 0 or 1 according as

\[
\int_{0^+} \frac{ds}{s} g^2(s) \exp \left( - \frac{g^2(s)}{2} \right)
\]

converges or diverges.

**Remark.** Comparing the test in Theorem 2 with the local DEOP test (see (1.9)), it is immediately seen that they are almost the same (in dimension 2, of course), except that for the upper functions of \( R - \bar{J} \), we have an additional condition that \( s \mapsto s^{1/2}g(s) \) be non-decreasing. Whether this condition can be removed remains open to the best of our knowledge.

Theorem 1 is proved in Section 2, and Theorem 2 in Section 3.

### 2. The proof of Theorem 1

In this section, \( R \) denotes a Bessel process of dimension \( d > 2 \), assumed to start from 0 without loss of generality, and \( J \) its future infimum process. Write

\[
\sigma(r) = \inf\{ t > 0 : R(t) = r \}, \quad r > 0,
\]

the first hitting time of \( R \) at level \( r \). By scaling, \( \sigma(r) \) has the same distribution as \( r^2 \sigma(1) \) for any given \( r > 0 \). From the characterizations of the upper and lower functions (the
DEOP and Dvoretzky-Erdős integral tests, see (1.4) and Révész (1990, Theorem 18.5), respectively, it is easily seen that with probability 1 for sufficiently large \( r \),

\[
    r \leq \sigma(r) \leq r^3. \tag{2.1}
\]

For notational convenience, we write in the sequel

\[
    \sigma \equiv \sigma(1), \quad J_\sigma \equiv J(\sigma(1)).
\]

The lower tail of the random variable \( \sigma \) is explicitly known. See for example Gruet and Shi (1995) who showed that

\[
\lim_{s \to 0} s^{d/2 - 1} e^{1/(2s)} \mathbb{P}(\sigma < s) = \frac{1}{2^{d/2 - 1} \Gamma(d/2)}. \tag{2.2}
\]

Consequently, there exists a finite constant \( K > 1 \) (depending only on the value of \( d \)) such that for any \( 0 < s \leq 2 \),

\[
    K^{-1} s^{1-d/2} \exp \left( -\frac{1}{2s} \right) \leq \mathbb{P}(\sigma < s) \leq K s^{1-d/2} \exp \left( -\frac{1}{2s} \right). \tag{2.3}
\]

In the rest of the paper, generic constants are denoted by \( K \), \( K_1 \), \( K_2 \) and \( K_3 \). Their value may vary from line to line (but not within the same line). Our next estimate concerns the distribution of a functional of \( \sigma \).

**Lemma 3.** We have, for any \( 0 < x < 1 \),

\[
    \mathbb{E} \left[ (1 - \sqrt{\sigma/t})^{d-2} \mathbb{1}_{\{\sigma < t\}} \right] \leq K x^{d/2 - 1} e^{1/(2x)}, \tag{2.4}
\]

where \( K \) is a constant depending only on \( d \).

**Proof of Lemma 3.** Actually \( x^{d/2 - 1} e^{1/(2x)} \) is also a lower bound for the term on the LHS of (2.4) (up to multiplication of a small constant), but we only prove the upper estimate, which is sufficient for our needs later. Using integration by parts, we have

\[
    \mathbb{E} \left[ (1 - \sqrt{\sigma/t})^{d-2} \mathbb{1}_{\{\sigma < t\}} \right] = \int_0^x (1 - \sqrt{t/x})^{d-2} d\mathbb{P}(\sigma < t)
    = \left( \frac{d}{2} - 1 \right) x^{-1/2} \int_0^x t^{-1/2} (1 - \sqrt{t/x})^{d-3} \mathbb{P}(\sigma < t) dt,
\]

which, according to (2.3), is

\[
    \leq K_1 x^{-1/2} \int_0^x t^{(1-d)/2} (1 - \sqrt{t/x})^{d-3} e^{-1/(2t)} dt
    = K_1 x^{1-d/2} \int_1^{\infty} \frac{dz}{z} (\sqrt{z} - 1)^{d-3} e^{-z/(2x)},
\]
by a change of variable $z = x/t$. Since $1/z(\sqrt{z} + 1)^{d-3} \leq 2$ for $z \geq 1$ and $d > 2$, we have
\[
\mathbb{E} \left[ (1 - \sqrt{x/z})^{d-2} \mathbb{1}_{\{\sigma \leq x \}} \right] \leq 2K_1 x^{1-d/2} \int_1^{\infty} dz (z - 1)^{d-3} e^{-z/(2x)}
= K_1 2^{d-1} \Gamma(d - 2)x^{d/2-1} e^{-1/(2x)},
\]
which yields (2.4). □

Proof of the convergent half of Theorem 1. Let $f > 0$ be non-decreasing such that
\[
\int_0^{\infty} \frac{df}{t} f^{4-d}(t) \exp \left( - \frac{f^2(t)}{2} \right) < \infty. \tag{2.5}
\]
Obviously $f(t) \uparrow \infty$ (otherwise the integral in (2.5) would diverge). Take a sufficiently large initial value $r_0$ and define the sequence $(r_k)_{k \geq 0}$ by $r_{k+1} = (1 + 1/f^2(r_k))r_k$ for $k \geq 0$. Thus $(r_k)$ increases to infinity. A standard argument (see for example Lemma 10 of Chung (1948)) shows that (2.5) is equivalent to
\[
\sum_k f^{2-d}(r_k) \exp \left( - \frac{f^2(r_k)}{2} \right) < \infty. \tag{2.6}
\]

The $d$-dimensional Bessel process \{ $R(\sigma + t); t \geq 0$ \} being a linear diffusion starting from 1, with scale function $-x^{2-d}$ (Revuz and Yor, 1994, p. 426), we have
\[
P(J_\sigma < x) = x^{d-2}, \tag{2.7}
\]
for any $0 < x < 1$. Since $\sigma$ and $J_\sigma$ are independent (this is easily seen from the Bessel strong Markov property), we have, by scaling, (2.7) and (2.3),
\[
P \left( J(\sigma(r_k)) + \sqrt{\sigma(r_k)} f(r_k) < r_{k+1} \right) = P \left( J_\sigma + \sqrt{\sigma} f(r_k) < \frac{r_{k+1}}{r_k} \right)
= \mathbb{E} \left[ \left( \frac{r_{k+1} - \sqrt{\sigma} f(r_k)}{r_k} \right)^{d-2} \mathbb{1}_{\{1/f^6(r_k) \leq \sigma (r_{k+1}/r_k f(r_k))^2 \}} \right] + P \left( \sigma < 1/f^6(r_k) \right)
= \left( 1 + \frac{1}{f^2(r_k)} \right)^{d-2} \mathbb{E} \left[ \left( 1 - \frac{\sqrt{\sigma}}{r_{k+1}/r_k f(r_k)} \right)^{d-2} \mathbb{1}_{\sigma (r_{k+1}/r_k f(r_k))^2} \right]
+ K_1 f^{3d-6}(r_k) \exp \left( - \frac{f^6(r_k)}{2} \right).
\]

Applying Lemma 3 to $x = (r_{k+1}/r_k f(r_k))^2$, it follows that the above expression is
\[
\leq K_2 f^{2-d}(r_k) \exp \left( - \frac{f^2(r_k)}{2 \left( 1 + 1/f^2(r_k)^2 \right)^2} \right) + K_1 f^{3d-6}(r_k) \exp \left( - \frac{f^6(r_k)}{2} \right)
\leq K f^{2-d}(r_k) \exp \left( - \frac{f^2(r_k)}{2} \right),
\]
which is summable (for $k$) according to (2.6). Using the Borel–Cantelli lemma, we have, (almost surely) for sufficiently large $k$, $J(\sigma(r_k)) + \sqrt{\sigma(r_k)} f(r_k) \geq r_{k+1}$. On the
other hand, by (2.1) we have \( \sigma(r_k) \geq r_k \) for large \( k \). Let \( t \in [\sigma(r_k), \sigma(r_{k+1})] \). Then

\[
R(t) - J(t) \leq r_{k+1} - J(\sigma(r_k)) \leq \sqrt{\sigma(r_k)} f(r_k) \leq t^{1/2} f(\sigma(r_k)) \leq t^{1/2} f(t),
\]

which implies the convergent half of Theorem 1. \( \square \)

Another preliminary result is needed in order to prove the divergent half of Theorem 1. Denote by \( P_x \) (with \( x \geq 0 \)) the probability under which \( R \) is a \( d \)-dimensional Bessel process starting from \( x \) (thus \( P_0 = P \)). The following is an estimate of the tail distribution of \( \sigma \) under \( P_x \), which generalizes the second part of (2.3).

**Lemma 4.** There exists a constant \( K \) depending only on \( d \) such that for any \( 0 < y < 1 \) and \( 0 < s < 1 \),

\[
P_y(\sigma < s) \leq K(s + y)^{1-d/2} \exp \left( -\frac{(1-y)^2}{2s} \right).
\]

**Proof of Lemma 4.** For any \( u > 0 \), we have, using the Bessel strong Markov property (recalling that \( \sigma = \sigma(1) \) by our notation),

\[
P_y(\sigma < s)P(\sigma(y) < u) = P(\sigma(y) < u, \sigma - \sigma(y) < s) \leq P(\sigma < s + u).
\]

Therefore by scaling, we have

\[
P_y(\sigma < s) \leq P(\sigma < s + u) / P(\sigma < u/y^2).
\]

In case \( s \geq y \), by taking \( u = y^2 \) in (2.9) and using (2.3), we obtain:

\[
P_y(\sigma < s) \leq K_1 P \left( \sigma < s + y^2 \right) \leq K_2 s^{1-d/2} \exp \left( -\frac{1}{2(s + y^2)} \right)
\]

\[
\leq K(s + y)^{1-d/2} \exp \left( -\frac{1}{2s} \right),
\]

which yields (2.8). If \( s \geq (1-y)^2 \), Lemma 4 is obviously true since the term on the RHS of (2.8) (without \( K \)) is bounded below by a positive constant and thus we can choose a large constant \( K \) such that (2.8) holds. Finally, if \( s < \min(y, (1-y)^2) \), we take \( u = sy/(1-y) \) in (2.9), giving

\[
P_y(\sigma < s) \leq P(\sigma < s/(1-y)) / P(\sigma < s/y(1-y)).
\]

Again, by means of (2.3), this is

\[
\leq K_1 y^{1-d/2} \exp \left( -\frac{1-y}{2s} + \frac{y(1-y)}{2s} \right) \leq K(s + y)^{1-d/2} \exp \left( -\frac{(1-y)^2}{2s} \right),
\]

as desired. \( \square \)

**Proof of the divergent half of Theorem 1.** Let \( f > 0 \) be non-decreasing such that the integral \( \int_0^\infty (dt/t) f^{4-d}(t) \exp \left( -f^2(t)/2 \right) \) diverges. Define \( h(t) = f(t^3) \) which is
again a non-decreasing function, with

$$\int_{t}^\infty \frac{dt}{t} h^{4-d}(t) \exp \left( - \frac{h^2(t)}{2} \right) = \infty. \quad (2.10)$$

In view of the form of our integral test in Theorem 1, we may assume without loss of generality (see for example Csáki (1989) for an elegant argument of justification) that $(\log \log t)^{1/2} \leq f(t) \leq 2(\log \log t)^{1/2}$. Thus for sufficiently large $t$,

$$ (\log \log t)^{1/2} \leq h(t) \leq 3(\log \log t)^{1/2}. \quad (2.11)$$

Pick a large initial value $k_0$ and let $r_k = \exp(k / \log k)$ for $k \geq k_0$. Write in the sequel $h_k \equiv h(r_k)$ for notational convenience. From (2.10), it is easily checked that

$$\sum_k h_k^{4-d} \exp \left( - \frac{h_k^2}{2} \right) = \infty. \quad (2.12)$$

A basic idea in the proof is that time $t$ is not a "nice" clock, but the first hitting time $\sigma(r)$ is. Consider the measurable events

$$E_k = \left\{ \left(1 - \frac{1}{h_k^2} \right)^2 \frac{r_{k-1}^2}{h_k^2} < \sigma(r_k) < \left(1 - \frac{1}{h_k^2} \right)^2 \frac{r_k^2}{h_k^2} \right\},$$

$$F_k = \{ J(\sigma(r_k)) < \frac{r_k}{h_k^2} \},$$

$$G_k = E_k \cap F_k,$$

for $k \geq k_0$. Obviously $E_k$ and $F_l$ are independent if $k \leq l$ (using the Bessel strong Markov property). By scaling, (2.2) and (2.7), we have

$$\mathbb{P}(E_k) = \mathbb{P} \left[ \left(1 - \frac{1}{h_k^2} \right)^2 \frac{r_{k-1}^2}{h_k^2} < \sigma < \left(1 - \frac{1}{h_k^2} \right)^2 \frac{r_k^2}{h_k^2} \right]$$

$$\geq K_1 h_k^{d-2} \left[ \exp \left( - \frac{h_k^2}{2 (1 - h_k^{-2})^2} \right) - \left( \frac{r_k}{r_{k-1}} \right)^{d-2} \exp \left( - \frac{r_k^2}{r_{k-1}^2} \frac{h_k^2}{2 (1 - h_k^{-2})^2} \right) \right]$$

$$\geq K h_k^{d-2} \exp \left( - \frac{h_k^2}{2} \right),$$

and

$$\mathbb{P}(F_k) = \mathbb{P} \left( J_\sigma < 1/h_k^2 \right) = h_k^{d-2d}. \quad (2.13)$$

Consequently,

$$\mathbb{P}(G_k) = \mathbb{P}(E_k) \mathbb{P}(F_k) \geq K h_k^{2-d} \exp \left( - \frac{h_k^2}{2} \right), \quad (2.14)$$

which, with the aid of (2.12), implies the divergence of $\sum_k \mathbb{P}(G_k)$. The next step is to apply the Borel–Cantelli lemma. Though the $G_k$'s are not independent, we shall show
that they are almost so (asymptotically). Indeed, consider $k_0 \leq k < l$. We have

$$P(G_k G_l) \leq P\left(E_k; \inf_{\sigma(r_k) \leq t \leq \sigma(r_l)} R(t) < \frac{r_k}{h_k^2} ; E_l; F_l\right) + P\left(E_k; E_l; J(\sigma(r_l)) < \frac{r_k}{h_k^2}\right)$$

$$= \Delta_1 + \Delta_2,$$

with obvious notation. We now evaluate $\Delta_1$ and $\Delta_2$. Since $(1 - h_k^{-2})^2 r_k^2 > r_k^2/2$ for $k \geq k_0$, it is easily seen that

$$\Delta_1 \leq P\left( \frac{r_k^2}{2h_k^2} < \sigma(r_k) < \frac{r_k^2}{h_k^2} ; \inf_{\sigma(r_k) \leq t \leq \sigma(r_l)} R(t) < r_k - h_k \sqrt{\sigma(r_k)} ; E_l; F_l\right). \tag{2.15}$$

Observe that by means of the strong Markov property, we have, for $0 < s < r_k^2/h_k^2$,

$$P\left( \inf_{\sigma(r_k) \leq t \leq \sigma(r_l)} R(t) < r_k - h_k \sqrt{\sigma(r_k)} ; E_l; F_l \mid \sigma(r_k) = s\right)$$

$$\leq P\left( \inf_{\sigma(r_k) \leq t \leq \sigma(r_l)} R(t) < r_k - h_k \sqrt{s} ; \sigma(r_l) - \sigma(r_k) < \frac{r_k^2}{h_k^2} \mid \sigma(r_k) = s\right) P(F_l)$$

$$= P_{r_k}\left( \sigma(r_k) - h_k \sqrt{s} < \sigma(r_l) ; \sigma(r_l) - \sigma(r_k) < \frac{r_k^2}{h_k^2} \right) P(F_l)$$

$$\leq P_{r_k}\left( \sigma(r_k) - h_k \sqrt{s} < \sigma(r_l) ; \sigma(r_l) - \sigma(r_k) < \frac{r_k^2}{h_k^2} \right) P(F_l)$$

$$= P_{r_k}\left( \sigma(r_k) - h_k \sqrt{s} < \sigma(r_l) \right) P_{r_k-h_k \sqrt{s}}\left( \sigma(r_l) - \frac{r_k^2}{h_k^2} \right) P(F_l).$$

Using Lemma 4, (2.13) and the trivial estimate $(a-1)/(b-1) \leq a/b$ (for $1 < a \leq b$), the above expression is

$$= \frac{(r_l/r_k)^{d-2} - 1}{(r_l/(r_k - h_k \sqrt{s}))^{d-2} - 1} P_{(r_k - h_k \sqrt{s})/r_l}\left( \sigma < \frac{1}{h_k^2} \right) P(F_l)$$

$$\leq K\left( 1 - \frac{h_k \sqrt{s}}{r_k} \right)^{d-2} h_k^{-d} \exp\left( - \frac{(1 - (r_k - h_k \sqrt{s})/r_l)^2}{2} \right).$$

Therefore by (2.15), we have

$$\Delta_1 \leq Kh_k^{2-d} \sum_{n} \left( 1 - \frac{h_k \sqrt{s}}{r_k} \right)^{d-2} \exp\left( - \frac{(1 - (r_k - h_k \sqrt{s})/r_l)^2}{2} \right) \mathbb{I}_A,$$

where $A \equiv \left\{ r_k^2/2h_k^2 < \sigma(r_k) < r_k^2/h_k^2 \right\}$. By scaling, this means

$$\Delta_1 \leq Kh_k^{2-d} \sum_{n} \left( 1 - h_k \sqrt{\sigma} \right)^{d-2} \exp\left( - \frac{(1 - (1 - h_k \sqrt{\sigma})r_k/r_l)^2}{2} \right) \mathbb{I}_{\{ \frac{1}{2} < h_k \sqrt{\sigma} < 1 \}}. \tag{2.16}$$
On the other hand, since \((1 - h_k^{-2})^2/h_k^2 \geq (1 - h_l^{-2})^2/h_l^2\), we have

\[
\Delta_2 \leq \P\left[\sigma(r_k) < \frac{r_k^2}{h_k^2}; \sigma(r_l) - \sigma(r_k) < \left(1 - \frac{1}{h_l^2}\right)^2 \frac{r_l^2}{h_l^2}, \right.
\]

\[
\left. - \left(1 - \frac{1}{h_k^2}\right)^2 \frac{r_k^2 - 1}{h_k^2}; J(\sigma(r_l)) < \frac{r_k}{h_k}\right] \leq \P\left(\sigma(r_k) < \frac{r_k^2}{h_k^2}; \sigma(r_l) - \sigma(r_k) < \frac{r_l^2 - r_k^2}{h_l^2}; J(\sigma(r_l)) < \frac{r_k}{h_k}\right)
\]

\[
= \P\left(\sigma(r_k) < \frac{r_k^2}{h_k^2}\right) \P\left(\sigma(r_l) < \frac{r_l^2 - r_k^2}{h_l^2}\right) \P\left(J(\sigma(r_l)) < \frac{r_k}{h_k}\right)
\]

\[
= \P\left(\sigma < \frac{1}{h_k^2}\right) \P_{r_k/r_l}\left(\sigma < \frac{1 - (r_k - 1/r_l)^2}{h_l^2}\right) \P\left(J_{\sigma} < \frac{r_k}{r_l} h_k^2\right),
\]

by scaling. Using (2.3) and (2.7), this yields

\[
\Delta_2 \leq K_h^{2-d} \left(\frac{r_k}{r_l}\right)^{d-2} \exp\left(-\frac{h_k^2}{2}\right) \P_{r_k/r_l}\left(\sigma < \frac{1 - (r_k - 1/r_l)^2}{h_l^2}\right). \tag{2.17}
\]

Now let us estimate \(P(G_k G_l)\) for \(l > k \geq k_0\). There are two possible situations. First, assume \(l \geq k + k^{1/133}\). In this case, \(l - l^{1/134} \geq k + k^{1/133} - (k + k^{1/133})^{1/134} \geq k\). Thus \(l - k \geq l^{1/134}\). It follows from the mean value theorem that

\[
\frac{l}{\log l} - \frac{k}{\log k} \geq \frac{l - k}{2 \log l} \geq \frac{l^{1/134}}{2 \log l} \geq 2 \log \log l,
\]

which yields

\[
r_k = \exp\left(\frac{\log k}{k} - \frac{\log l}{l}\right) \leq \frac{\log l}{(\log l)^2} \leq \frac{1}{h_l^2}, \tag{2.18}
\]

the last inequality being due to (2.11). Thus by (2.16), we obtain

\[
\Delta_1 \leq K_h^{2-d} E \left[(1 - h_k \sqrt{\sigma})^{d-2} \exp\left(-\frac{(1 - 1/h_l^2)^2}{2} h_l^2\right) 1_{[h_k^2 \sigma < 1]}\right].
\]

Applying Lemma 3 to \(x = 1/h_k^2\), we get

\[
\Delta_1 \leq K_1 h_l^{-d} \exp\left(-\frac{h_l^2}{2}\right) h_k^{2-d} \exp\left(-\frac{h_k^2}{2}\right) \leq K_2 P(G_k) P(G_l),
\]

using (2.14). Moreover, by (2.17), Lemma 4 and (2.18), we have

\[
\Delta_2 \leq K_2 h_k^{2-d} \left(\frac{r_k}{r_l}\right)^{d-2} \exp\left(-\frac{h_k^2}{2}\right) \P_{r_k/r_l}\left(\sigma < \frac{1}{h_l^2}\right)
\]

\[
\leq K_3 h_k^{2-d} \left(\frac{r_k}{r_l}\right)^{d-2} \exp\left(-\frac{h_k^2}{2}\right) \left(\frac{r_k}{r_l}\right)^{1-d/2} \exp\left(-\frac{(1 - r_k/r_l)^2}{2} h_l^2\right)
\]

\[
\leq K_2 h_k^{2-d} \left(\frac{h_k^2}{2}\right) h_k^{2-d} \exp\left(-\frac{h_k^2}{2}\right),
\]

by scaling.
which, according to (2.14), is again bounded above by $K_3 \mathbb{P}(G_k) \mathbb{P}(G_l)$. Since $\mathbb{P}(G_k G_l) \leq \Delta_1 + \Delta_2$, we obtain the desired estimate:

$$\sum \sum_{k_0 < k < l \leq n, l \geq k + 1/133} \mathbb{P}(G_k G_l) \leq K_1 \left( \sum_{k=k_0}^n \mathbb{P}(G_k) \right)^2.$$  \hspace{1cm} (2.19)

It remains to treat the case $k < l < k + 1/133$. Note that on the event \{ $h_k^2 \sigma > 1/2$ \}, we have

$$(1 - (h_k \sqrt{\sigma}) r_k / r_l)^2 \geq h_k^2 \sigma \geq 1/2.$$  

Hence, (2.16), Lemma 3, (2.14) and (2.11) together yield

$$\Delta_1 \leq K_1 h_k^2 - d e^{-h_k^2/4} E \left[ \left( 1 - h_k \sqrt{\sigma} \right)^{d-2} \mathbb{1}_{\{ h_k^2 \sigma < 1 \}} \right]$$
$$\leq K_1 h_k^2 - d e^{-h_k^2/4} h_k^{2-d} \exp \left( - \frac{h_k^2}{2} \right)$$
$$\leq K_2 \mathbb{P}(G_k) e^{-h_k^2/4}$$
$$\leq K_2 \mathbb{P}(G_k) k^{-1/5}.$$  \hspace{1cm} (2.20)

To estimate $\Delta_2$ (in case $k < l < k + 1/133$), observe that by the mean value theorem,

$$\frac{l}{\log l} - \frac{k}{\log k} \geq \frac{l - k}{2 \log k}, \quad \text{and} \quad \frac{l}{\log l} - \frac{k - 1}{\log(k - 1)} \leq \frac{l - k + 1}{\log(k - 1)} \leq \frac{3(l - k)}{\log k},$$

for $k \geq k_0$. Thus using elementary inequalities $1 - e^{-x} \leq x$ (for $x > 0$), $1 - e^{-y} \geq y/\sqrt{e}$ (for $0 < y \leq 1/2$) and $\min(1/4 e, (1 - e^{-1/2})^2) > 1/11$, we have

$$1 - \left( \frac{r_{k-1}}{r_l} \right)^2 \leq 6 \min \left( \frac{l - k}{\log k}, 1 \right),$$  \hspace{1cm} (2.21)

$$\left( 1 - \frac{r_k}{r_l} \right)^2 \geq \frac{1}{11} \min \left( \frac{(l - k)^2}{(\log k)^2}, 1 \right).$$  \hspace{1cm} (2.22)

Gradually by (2.17), (2.8), (2.14), (2.21), (2.22) and (2.11), we obtain

$$\Delta_2 \leq K_1 h_k^2 - d \left( \frac{r_k}{r_l} \right)^{d-2} \exp \left( - \frac{h_k^2}{2} \right) \left( \frac{r_k}{r_l} \right)^{1-d/2} \exp \left( - \frac{1}{2} \frac{(1 - r_k/r_l)^2}{(1 - (r_{k-1}/r_l)^2)} h_k^2 \right)$$
$$\leq K_2 \mathbb{P}(G_k) \left( \frac{r_k}{r_l} \right)^{d/2-1} \exp \left( - \frac{1}{133} \min \left( \frac{l - k}{\log k}, 1 \right) \log l \right)$$
$$\leq K_2 \mathbb{P}(G_k) \left( e^{-\log k}/133 + e^{-(l-k)/133} \right).$$

Since $\mathbb{P}(G_k G_l) \leq \Delta_1 + \Delta_2$, this together with (2.20) implies

$$\sum \sum_{k_0 < k < l \leq n, l < k + 1/133} \mathbb{P}(G_k G_l) \leq K_3 \sum_{k=k_0}^n \mathbb{P}(G_k) \left( \sum_{k<k<l<k+k^{1/133}} k^{1/133} + \sum_{l>k} e^{-(l-k)/133} \right)$$
$$\leq K_1 \sum_{k=k_0}^n \mathbb{P}(G_k).$$
Combining this estimate with (2.19), we obtain
\[
\liminf_{n \to \infty} \sum_{k=k_0}^{n} \sum_{l=k_0}^{n} \mathbb{P}(G_k G_l) / \left( \sum_{k=k_0}^{n} \mathbb{P}(G_k) \right)^2 \leq K.
\]
Since \(\sum_k \mathbb{P}(G_k)\) diverges, applying Kochen and Stone's Borel–Cantelli lemma (Kochen and Stone, 1964) yields
\[
\mathbb{P}\left( \left(1 - \frac{1}{h_k^2}\right) \frac{r_k^2}{h_k^2} < \sigma(r_k) < \left(1 - \frac{1}{h_k^2}\right) \frac{r_k^2}{h_k^2}, J(\sigma(r_k)) < \frac{r_k}{h_k^2}, \text{ i.o.} \right) \geq K^{-1}.
\]
A fortiori, we have
\[
\mathbb{P}\left( J(\sigma(r_k)) + h_k \sqrt{\sigma(r_k)} < r_k, \text{ i.o.} \right) \geq K^{-1}.
\]
According to (2.1), (almost surely) for sufficiently large \(k\), we have \(\sigma(r_k) \leq r_k^2\). Thus \(h_k = h(r_k) = f(r_k^3) \geq f(\sigma(r_k))\). Therefore (noticing that \(r_k = R(\sigma(r_k))\)) we obtain
\[
\mathbb{P}\left( R(\sigma(r_k)) - J(\sigma(r_k)) > \sqrt{\sigma(r_k)} f(\sigma(r_k)), \text{ i.o.} \right) \geq K^{-1},
\]
which in turn yields
\[
\mathbb{P}\left( R(t) - J(t) > \sqrt{t} f(t), \text{ i.o. } t \to \infty \right) \geq K^{-1}.
\]
By time inversion (i.e. \(\{t R(1/t); t > 0\}\) is again a \(d\)-dimensional Bessel process) and Blumenthal's 0–1 law, the above probability equals 1, which completes the proof of the divergent part of Theorem 1. □

3. The proof of Theorem 2

In this section, \(\{R(u); u \geq 0\}\) stands for a Bessel process of dimension 2 (i.e. the radial part of a planar Wiener process), starting from 0, and let \(\tilde{J}\) be its local future infimum process defined in (1.5).

As is often the case, time 1 is not a nice clock to work with. We introduce a modified process:

\[
I(s) = \begin{cases} 
\inf_{s \leq u \leq \sigma} R(u), & \text{if } s \leq \sigma, \\
1, & \text{otherwise},
\end{cases}
\]

where, as before, \(\sigma(r)\) stands for the first hitting time of \(R\) at level \(r > 0\), and \(\sigma \equiv \sigma(1)\). Obviously \(R(s) - I(s)\) and \(R(s) - \tilde{J}(s)\) have the same upper functions as \(s\) tends to \(0^+\). What we prove in this section is the following equivalent statement of Theorem 2.

**Theorem 2'**. If \(d = 2\) and if \(g > 0\) is a non-increasing function such that \(s \mapsto s^{1/2} g(s)\) is non-decreasing, then
\[
\mathbb{P}\left( R(s) - I(s) > s^{1/2} g(s), \text{ i.o. } s \to 0 \right)
\]
equals 0 or 1 according as
\[ \int_{0^+} \frac{ds}{s} g^2(s) \exp \left( - \frac{g^2(s)}{2} \right) \]
converges or diverges.

Only the divergent part needs to be shown, since the convergent part of Theorem 2 (or 2') trivially follows from the local DEOP test (1.9). Some of the estimates given in Section 2 still remain true in dimension 2, notably those concerning \( \sigma \). To begin with, (2.2) being valid for any \( d > 0 \) (Gruet and Shi, 1995, Theorem 6), we obtain:

\[ \lim_{s \to 0} e^{1/(2s)} \mathbb{P}(\sigma < s) = 2, \]  \( (3.1) \)

\[ K^{-1} \exp \left( - \frac{1}{2s} \right) \leq \mathbb{P}(\sigma < s) \leq K \exp \left( - \frac{1}{2s} \right), \quad 0 < s < 1, \]  \( (3.2) \)

where \( K > 1 \) is a universal constant. Likewise, Lemma 4, of which the proof is only based on (2.2), holds as well in dimension 2, and is restated as follows:

\[ \mathbb{P}_y(\sigma < s) \leq K \exp \left( - \frac{(1 - y)^2}{2s} \right), \]  \( (3.3) \)

for \( 0 < y < 1 \) and \( 0 < s < 1 \), with \( K > 0 \) an absolute constant. Concerning the modified future infimum process \( I \), we have

\[ \mathbb{P}(I(\sigma(x)) < y) = \frac{\log(1/x)}{\log(1/y)}, \]  \( (3.4) \)

for any \( y < x < 1 \). The proof of Theorem 2 is similar to (and slightly easier than) that of Theorem 1 presented in Section 2. Emphasizing on the differences between the two- and higher-dimensional cases, it is only sketched in order to avoid non-essential discussions.

**Sketch of the proof of the divergent part of Theorem 2'.** Let

\[ \int_{0^+} \frac{ds}{s} g^2(s) \exp \left( - \frac{g^2(s)}{2} \right) = \infty. \]  \( (3.5) \)

Choose a sufficiently large \( k_0 \) and let \( z_k = \exp(-k/\log k) \) for \( k \geq k_0 \). Write \( g_k \equiv g(z_k) \) for notational simplification. As usual, we only have to treat the case

\[ \log \log(1/s) \leq g(s) \leq 2 \log \log(1/s). \]  \( (3.6) \)

Consider

\[ B_k = \left\{ \left( 1 - \frac{1}{g_k^2} \right)^2 \frac{z_{k+1}^2}{g_k^2} < \sigma(z_k) < \left( 1 - \frac{1}{g_k^2} \right)^2 \frac{z_k^2}{g_k^2}, \quad I(\sigma(z_k)) < \frac{\sigma_k}{g_k^2} \right\}. \]
Since $I(\sigma(z_k))$ is independent of $\sigma(z_k)$, several lines of elementary calculation using (3.1) and (3.4) yield the desired estimate:

$$\mathbb{P}(B_k) \geq K \exp \left( - \frac{a_k^2}{2} \right),$$

(3.7)

which, in light of (3.5), confirms the divergence of $\sum_k \mathbb{P}(B_k)$. Let $k_0 \leq k < l$. Observe that

$$\mathbb{P}(B_k B_l) \leq \mathbb{P} \left[ \sigma(z_l) < \frac{z_l^2}{g_l^2}, \sigma(z_k) - \sigma(z_l) < \left( 1 - \frac{1}{g_k^2} \right)^2 \frac{z_k^2}{g_k^2} - \left( 1 - \frac{1}{g_l^2} \right)^2 \frac{z_l^2}{g_l^2} \right]$$

$$\leq \mathbb{P} \left( \sigma(z_l) < \frac{z_l^2}{g_l^2} \right) \mathbb{P}_{z_l/z_k} \left( \sigma(z_k) < \frac{z_k^2}{g_k^2} - \frac{z_l^2}{g_l^2} \right)$$

$$\leq K \mathbb{P}(B_l) \mathbb{P}_{z_l/z_k} \left( \sigma < \frac{1}{g_k^2} - \frac{z_l^2}{z_k^2 g_l^2} \right).$$

(3.8)

As in the higher-dimensional situation, we distinguish two possible cases. First, assume $l \geq k + k^{1/265}$. Then $z_l/z_k \leq 1/g_k^2$ as in (2.18). From (3.8), (3.3) and (3.7), it follows that

$$\mathbb{P}(B_k B_l) \leq K_1 \mathbb{P}(B_l) \mathbb{P}_{z_l/z_k} \left( \sigma < 1/g_k^2 \right)$$

$$\leq K_2 \mathbb{P}(B_l) \exp \left( - \frac{(1 - z_l/z_k)^2 g_k^2}{2} \right)$$

$$\leq K \mathbb{P}(B_l) \mathbb{P}(B_k),$$

which yields

$$\sum_{k_0 \leq k \leq l < n, l \geq k + k^{1/265}} \mathbb{P}(B_k B_l) \leq K_1 \left( \sum_{k=k_0}^{n} \mathbb{P}(B_k) \right)^2.$$  

(3.9)

Now let $k < l < k + k^{1/265}$. The function $s \mapsto s^{1/2} g(s)$ being non-decreasing, we have

$$\frac{1}{g_k^2} - \frac{z_{l+1}^2}{z_k^2 g_l^2} \leq \frac{1}{g_k^2} \left( 1 - \frac{z_l z_{l+1}}{z_k^3} \right) \leq \frac{2}{g_k^2} \left( 1 - \frac{z_{l+1}^2}{z_k^2} \right).$$

With the aid of (3.8) and (3.3), we obtain

$$\mathbb{P}(B_k B_l) \leq K_1 \mathbb{P}(B_l) \exp \left( - \frac{(1 - z_l/z_k)^2 g_k^2}{4(1 - z_{l+1}^2/z_k^2) g_k^2} \right).$$

Since $z_k = 1/r_k$, using (2.21), (2.22) and (3.6), we have

$$\mathbb{P}(B_k B_l) \leq K_1 \mathbb{P}(B_l) \exp \left( - \frac{1}{265} \min \left( \frac{l - k}{\log k}, 1 \right) \log k \right)$$

$$\leq K_1 \mathbb{P}(B_l) \left( e^{- \log k/265} + e^{- (l - k)/265} \right),$$
from which it immediately follows that
\[ \sum_{k_0 \leq k < l \leq n} \sum_{l < k + k^{1/265}} P(B_k B_l) \leq K_2 \sum_{k = k_0}^n P(B_k). \]

This together with (3.9) allows us, by means of Kochen and Stone’s Borel–Cantelli lemma (Kochen and Stone, 1964), to establish the following estimate:
\[ P(\sup_{k_0 < k < n} |\sigma(z_k)| + g_\epsilon \sqrt{\sigma(z_k)} < z_k, \ i.o.) \geq K^{-1}. \]

Since \( \sigma(z_k) \leq z_k \) for sufficiently large \( k \), this implies
\[ P\left( R(s) - I(s) > \sqrt{s} g(s), \ i.o. \ s \rightarrow 0^+ \right) \geq K^{-1}. \]

The divergent part of Theorem 2′ is proved using Blumenthal’s 0–1 law. \( \square \)

**Acknowledgement**

The problem originates from a lecture at Université Paris VI by Jean-Claude Gruet, based on the paper of Khoshnevisan et al. (1994).

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