ELSEVIER

Discrete Mathematics 132 (1994) 377–381

DISCRETE MATHEMATICS

Note

An application of linear species

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Received 15 September 1992

Abstract

Combinatorial operations on linear species are used in order to obtain, in a simple manner, the identity

$$\frac{1}{1-\alpha x}=1+\sum_{k\geqslant 1} S(\alpha;k)\frac{x^k}{1-x^k},$$

where $S(\alpha;k)$ denotes the number of aperiodic words of length k over an alphabet with α elements. The cyclotomic identity follows as an immediate corollary.

1. Introduction

In this note we use combinatorial operations on linear species in order to obtain some fundamental results on generating functions. Our main result is the identity

$$\frac{1}{1-\alpha x}=1+\sum_{k\geqslant 1} S(\alpha;k)\frac{x^k}{1-x^k}, \tag{1.1}$$

where $S(\alpha;k)$ denotes the number of aperiodic words of length k over an alphabet with α elements. As an immediate corollary of 1.1 we obtain the cyclotomic identity

$$\frac{1}{1-\alpha x}=\prod_{j\geqslant 1}\left(\frac{1}{1-x^j}\right)^{M(\alpha;j)},$$

where $M(\alpha;j)$ denotes the number of aperiodic necklaces of length j over an alphabet having α letters (see [2]).

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SSDI 0012-365X(92)00574-V

All species we consider here are linear species, i.e., functors from the category $\mathbb{L}$ of finite linearly ordered sets and order preserving bijections to the category $\mathbb{E}$ of finite sets and functions (see [1] for details). Notice that any object in $\mathbb{L}$ is isomorphic in a unique manner to some ordered set $[n]=(1, 2, \ldots, n)$, $n \geqslant 0$. Hence $\mathbb{L}$ is equivalent to the discrete category of non-negative integers, having only identity morphisms. Defining a linear species ($\mathbb{L}$-species, for short) F is thus equivalent to giving a sequence of finite sets $(F[n])_{n \geqslant 0}$. For psychological reasons, however, we prefer to think of F as defining a set $F[E]$ of *F-structures* on any given linearly ordered set E.

The *cardinality* of an $\mathbb{L}$-species F is the formal series

$$f(F, x)= \sum_{n \geqslant 0} |F[n]| x^n,$$

where $F[n]$ denotes the set of F-structures on $[n]$, and $|F[n]|$ denotes the cardinality of $F[n]$. We will write $F=G$, whenever F and G are isomorphic $\mathbb{L}$-species. It is clear that $f(F, x)=f(G, x)$ whenever F and G are isomorphic. The converse is true for $\mathbb{L}$-species, but not for ordinary species (see [1]).

All sets considered in this paper will be finite. By an *ordered* set, we shall always mean a linearly ordered set.

2. Operations on linear species

We review here the operations on $\mathbb{L}$-species introduced in [1].

If F and G are $\mathbb{L}$-species, the *sum* $F+G$ is the $\mathbb{L}$-species defined by letting $(F+G)[E]$ be the disjoint union $F[E]+G[E]$, for any ordered set E. The *product* FG is the $\mathbb{L}$-species defined by

$$(FG)[E]= \sum_{E_1+E_2=E} F[E_1] \times G[E_2],$$

where E_1+E_2 is the disjoint union of the ordered sets E_1 and E_2, ordered by concatenation. The above sum is thus over all expressions for E as a concatenation of two disjoint (possibly empty) ordered sets.

If the $\mathbb{L}$-species G satisfies $G[\emptyset]=\emptyset$, then the *composition* $F \circ G$ is the $\mathbb{L}$-species defined by

$$F \circ G[E]= \sum_{k \geqslant 0} \sum_{E_1+\cdots+E_k=E} F[(E_1, \ldots, E_k)] \times G[E_1] \times \cdots \times G[E_k],$$

where $(E_1, \ldots, E_k)$ denotes the set $\{E_1, \ldots, E_k\}$ with the natural ordering.

It is straightforward to verify that cardinalities behave just as one would like with respect to these operations, i.e.,

$$f(F+G, x)=f(F, x)+f(G, x),$$

$$f(FG,x)=f(F,x)f(G,x),$$

for all $\mathbb{L}$-species F and G; if $G[\emptyset]=\emptyset$, then

$$f(F\circ G,x)=f(F,f(G,x)). \tag{2.1}$$

A sequence $F_1, F_2, \ldots$ of $\mathbb{L}$-species is *summable* if, for any ordered set E, there is some $k\geqslant 1$ (which depends only on $|E|$) such that $F_n[E]=\emptyset$, for all $n\geqslant k$. In this case, the sum $\sum_{n\geqslant 1}F_n$ is a well-defined $\mathbb{L}$-species, and $f(\sum F_n, x)=\sum f(F_n, x)$.

As an example, consider the *singleton* species $X:\mathbb{L}\to\mathbb{E}$ defined by

$$X[E]=\begin{cases}\{E\} & \text{if } |E|=1,\\ \emptyset & \text{otherwise.}\end{cases}$$

It is easy to see that $f(X,x)=x$, and so $f(X^n,x)=f(X,x)^n=x^n$, for all $n\geqslant 1$. The $\mathbb{L}$-species $\sum_{k\geqslant 1}X^n$ is isomorphic to the $\mathbb{L}$-species of non-empty sets U_0, given by

$$U_0[E]=\begin{cases}\{E\} & \text{if } E\neq\emptyset,\\ \emptyset & \text{otherwise.}\end{cases}$$

Hence, $f(U_0,x)=\sum_{n\geqslant 1}f(X^n,x)=\sum_{n\geqslant 1}x^n=x/(1-x)$.

3. The cyclotomic identity

Let W_α and S_α be the $\mathbb{L}$-species of non-empty words and aperiodic words, respectively, over an alphabet of cardinality α. Hence, for all $n\geqslant 0$,

$$W_\alpha[n]=\begin{cases}\{f:[n]\to[\alpha]\} & \text{if } n\neq 0,\\ \emptyset & \text{if } n=0\end{cases}$$

and

$$S_\alpha[n]=\{f:[n]\to[\alpha]\text{: } (f(1),\ \ldots\ ,f(n)) \text{ is an aperiodic sequence}\}.$$

By convention, $S_\alpha[\emptyset]=\emptyset$. Therefore

$$f(S_\alpha,x)=\sum_{n\geqslant 1}S(\alpha,n)\,x^n \tag{3.1}$$

and

$$f(W_\alpha,x)=\sum_{n\geqslant 1}\alpha^n x^n=\frac{1}{1-\alpha x}-1. \tag{3.2}$$

Our basic combinatorial result is the following:

Proposition 3.1. *W_α and $K_\alpha=\sum_{n\geqslant 1}(S_\alpha\circ X^n)$ are isomorphic $\mathbb{L}$-species.*

Proof. Let E be an ordered set. Then

$$K_\alpha[E] = \sum_{n\geq 1} S_\alpha \circ X^n[E]$$

$$= \sum_{n\geq 1} \sum_{k\geq 1} \sum_{E_1+\cdots+E_k=E} S_\alpha[(E_1, \ldots, E_k)] \times X^n[E_1] \times \cdots \times X^n[E_k]$$

$$= \sum_{n\geq 1} \sum_{k\geq 1} \sum_{\substack{E_1+\cdots+E_k=E \\ |E_i|=n}} S_\alpha[(E_1, \ldots, E_k)]$$

$$= \sum_{n\geq 1} \sum_{nk=|E|} S_\alpha[k]$$

$$= W_\alpha[E].$$

The equal signs above do not denote equality of sets, but designate natural bijections. The last of these bijections comes from the fact that any element w of $W_\alpha[n]$ can be written uniquely as $w = v^{n/k}$, for some $v \in S_\alpha[k]$, where k divides n. □

We can now prove equation (1.1).

Theorem 3.2. *For all* $\alpha \geq 1$,

$$\frac{1}{1-\alpha x} = 1 + \sum_{k\geq 1} S(\alpha; k) \frac{x^k}{1-x^k}.$$

Proof. By equations (3.1) and (2.1), we have

$$f(S_\alpha \circ X^n, x) = \sum_{k\geq 1} S(\alpha; k) x^{nk}.$$

Hence

$$f(K_\alpha, x) = \sum_{n\geq 1} \sum_{k\geq 1} S(\alpha; k) x^{nk}$$

$$= \sum_{k\geq 1} S(\alpha; k) \sum_{n\geq 1} (x^k)^n$$

$$= \sum_{k\geq 1} S(\alpha; k) \frac{x^k}{1-x^k}.$$

Therefore, by Proposition (3.1) and equation (3.2),

$$1 + \sum_{k\geq 1} S(\alpha; k) \frac{x^k}{1-x^k} = 1 + f(K_\alpha, x)$$

$$=1+f(W_\alpha, x)$$
$$=\frac{1}{1-\alpha x}. \qquad \square$$

Corollary 3.3 (The cyclotomic identity). *For all* $\alpha \geqslant 1$,

$$\frac{1}{1-\alpha x}=\prod_{j\geqslant 1}\left(\frac{1}{1-x^j}\right)^{M(\alpha;j)}.$$

Proof. From Theorem (3.2) and the fact that $kM(\alpha;k)=S(\alpha;k)$, we have

$$\frac{\alpha}{1-\alpha x}=\sum_{k\geqslant 1} M(\alpha;k)\frac{kx^{k-1}}{1-x^k}.$$

Integrating both sides with respect to x, we obtain

$$\ln\left(\frac{1}{1-\alpha x}\right)=C+\sum_{k\geqslant 1} M(\alpha;k)\ln\left(\frac{1}{1-x^k}\right),$$

for some constant C. Hence, by exponentiating,

$$\frac{1}{1-\alpha x}=e^C\prod_{k\geqslant 1}\left(\frac{1}{1-x^k}\right)^{M(\alpha;k)}.$$

By setting $x=0$ we see that $C=0$. Thus the identity follows. $\square$

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[2] N. Metropolis and G.-C. Rota, Witt vectors and the algebra of necklaces, Adv. in Math. 50 (1983) 95–125.