Acyclic 5-choosability of planar graphs without 4-cycles

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Abstract

A proper vertex coloring of a graph \( G = (V, E) \) is acyclic if \( G \) contains no bicolored cycle. A graph \( G \) is acyclically \( L \)-list colorable if for a given list assignment \( L = \{L(v) : v \in V\} \), there exists a proper acyclic coloring \( \pi \) of \( G \) such that \( \pi(v) \in L(v) \) for all \( v \in V \). If \( G \) is acyclically \( L \)-list colorable for any list assignment with \( |L(v)| \geq k \) for all \( v \in V \), then \( G \) is acyclically \( k \)-choosable. In this paper we prove that every planar graph without 4-cycles and without two 3-cycles at distance less than 3 is acyclically 5-choosable. This improves a result in [M. Montassier, P. Ochem, A. Raspaud, On the acyclic choosability of graphs, J. Graph Theory 51 (2006) 281–300], which says that planar graphs of girth at least 5 are acyclically 5-choosable.

Keywords: Planar graphs; Acyclic coloring; Choosable; Cycle

1. Introduction

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). A proper vertex coloring of \( G \) is an assignment \( \pi \) of integers (or labels) to the vertices of \( G \) such that \( \pi(u) \neq \pi(v) \) if two vertices \( u \) and \( v \) are adjacent in \( G \). A \( k \)-coloring is a proper vertex coloring using \( k \) colors. A proper vertex coloring of a graph is acyclic if there is no bicolored cycle in \( G \). The \textit{acyclic chromatic number}, denoted by \( \chi_a(G) \), of a graph \( G \) is the smallest integer \( k \) such that \( G \) has an acyclic \( k \)-coloring.

The acyclic coloring of graphs were introduced by Grünbaum in [5] and studied by Mitchem [8], Albertson and Berman [1] and Kostochka [6]. In 1979, Borodin [2] proved Grünbaum’s conjecture that every planar graph is acyclically 5-colorable. This bound is the best possible. In 1973, Grünbaum [5] gave an example of 4-regular planar graph which is not acyclically 4-colorable. Furthermore, bipartite planar graphs which are not acyclically 4-colorable were constructed in [7]. Borodin, Kostochka and Woodall [4] proved that every planar graph of girth at least 7 is acyclically 3-colorable and every planar graph of girth at least 5 is acyclically 4-colorable. We recall that the girth of a graph \( G \) is the length of its shortest cycle.

A graph \( G \) is \textit{acyclically \( L \)-list colorable} if for a given list assignment \( L = \{L(v) : v \in V\} \), there is an acyclic coloring \( \pi \) of the vertices such that \( \pi(v) \in L(v) \). We say that \( \pi \) is an \( L \)-coloring of \( G \). If \( G \) is acyclically \( L \)-list

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colorable for any list assignment $L$ with $|L(v)| \geq k$ for all $v \in V$, then $G$ is acyclically $k$-choosable. The acyclic list chromatic number of $G$, $\chi_a^l(G)$, is the smallest integer $k$ such that $G$ is acyclically $k$-choosable.

Borodin et al. [3] first investigated the acyclically list coloring of planar graphs to show that every planar graph is acyclically 7-choosable. They also put forward the following challenging conjecture:

Conjecture 1. Every planar graph is acyclically 5-choosable.

If Conjecture 1 were true, then it would strengthen the Borodin’s acyclically 5-colorable theorem and the Thomassen’s 5-choosable theorem [12] about planar graphs.

By investigating the maximum average degree of graphs, Montassier, Ochem, and Raspaud [9] showed that if $G$ is a planar graph with girth $g$ then $\chi_a^l(G) \leq 3$ if $g \geq 8$, $\chi_a^l(G) \leq 4$ if $g \geq 6$, and $\chi_a^l(G) \leq 5$ if $g \geq 5$. Some sufficient conditions for a planar graph to be acyclically 4-choosable were established in [10]. Recently, Montassier, Raspaud and Wang [11] proved that every planar graph $G$ without 4-cycles and 5-cycles, or without 4-cycles and 6-cycles is acyclically 5-choosable.

To attack Conjecture 1, we would like to put forward the following weak version about this conjecture:

Conjecture 2. Every planar graph without 4-cycles is acyclically 5-choosable.

Let us consider the acyclic 5-choosability of planar graphs $G$ having neither 4-cycles nor 3-cycles at distance $d$. Obviously, the case $d = 0$ corresponds to Conjecture 2. The case $d = \infty$ means that $G$ is a planar graph with girth at least 5, which is shown to be acyclically 5-choosable [9]. In this paper, we handle the case $d = 3$. More precisely, we will prove the following result:

Theorem 1. Every planar graph without 4-cycles and without triangles at distance less than 3 is acyclically 5-choosable.

Our result partially confirms Conjecture 1 and gives an improvement to a result in [9].

2. Notation

Only simple graphs are considered in this paper. A plane graph is a particular drawing of a planar graph in the Euclidean plane. For a plane graph $G$, we denote its face set by $F(G)$. $k$-vertex, $k^+$-vertex and $k^-$-vertex are vertices of degree $k$, at least $k$ and at most $k$, respectively. Similarly, we can define $k$-face, $k^+$-face, $k^-$-face, etc. We say that two cycles (or faces) are adjacent if they share at least one common edge. A triangle is synonymous with a 3-cycle. Usually, a face $f \in F(G)$ is written as $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of $f$ in a cyclic order. For a vertex $v \in V(G)$ and an integer $i \geq 1$, let $n_i(v)$ denote the number of $i$-vertices adjacent to $v$. For a face $f \in F(G)$ and an integer $j \geq 2$, let $n_j(f)$ denote the number of $j$-vertices incident to $f$. For $x \in V(G) \cup F(G)$, let $t(x)$ denote the number of 3-faces adjacent or incident to $x$. Let $N(v)$ denote the set of neighbors of a vertex $v$.

A 3-face $f = [v_1v_2v_3]$ is called an $(a_1, a_2, a_3)$-face if the degree of the vertex $v_i$ is $a_i$ for $i = 1, 2, 3$. An edge $uv$ is a $(b_1, b_2)$-edge if $d(u) = b_1$ and $d(v) = b_2$. A 3-vertex $v$ is light if it is incident to a 3-face. If a vertex $v$ is adjacent to a 3-vertex $u$ such that the edge $uv$ is not incident to any 3-face, then we say $u$ a pendant 3-vertex of $v$. A pendant light 3-vertex is a light and pendant 3-vertex. If $v$ is a pendant light 3-vertex which is incident to an $(a_1, a_2, a_3)$-face, then we call $v$ a pendant light $(a_1, a_2, a_3)$-vertex. Let $p_3(v)$ denote the number of pendant light 3-vertices of a vertex $v$. For a pendant light 3-vertex $u$ of $v$, if $d(v) = 4$ and $u$ is a pendant light $(3, 5^+, 5^+)$-vertex, then we call $u$ a bad pendant light 3-vertex of $v$.

Suppose that $f = [uvwxyz \cdots]$ is a face of degree at least 5 such that $d(w) = 2$, $d(v) \geq 6$ and $d(x) = 3$. We say that $f$ is a heavy face of the edge $uv$ if one of the following conditions holds:

1. $d(y) = 3$;
2. $d(y) = 4$, $d(z) \geq 5$, and $yz$ lies on a 3-face that is adjacent to $f$.

3. Structural properties

Suppose that $G$ is a counterexample to Theorem 1 with the least vertices. Then the following Lemma 1 holds, whose proof was provided in [11]:
Lemma 1. (C1) There are no 1-vertices.
(C2) No 2-vertex is adjacent to a 4-vertex.
(C3) Let \( v \) be a 3-vertex. Then
   - (C3.1) If \( v \) is adjacent to a 3-vertex, then \( v \) is not adjacent to other 4-vertex;
   - (C3.2) \( v \) is not adjacent to any pendant light 3-vertex.
(C4) Let \( v \) be a 5-vertex. Then
   - (C4.1) \( v \) is adjacent to at most one 2-vertex;
   - (C4.2) If \( n_2(v) = 1 \), then \( v \) is not adjacent to any pendant light 3-vertex.
(C5) Let \( v \) be a 6-vertex. Then
   - (C5.1) \( v \) is adjacent to at most four 2-vertices;
   - (C5.2) If \( n_2(v) = 4 \), then \( v \) is not adjacent to any 3-vertex.
(C6) Each 7-vertex is adjacent to at most five 2-vertices.
(C7) No 3-face \([xyz]\) with \( d(x) \leq d(y) \leq d(z) \) satisfies one of the following:
   - (C7.1) \( d(x) = 2 \);
   - (C7.2) \( d(x) = d(y) = 3 \) and \( d(z) \leq 5 \);
   - (C7.3) \( d(x) = 3 \) and \( d(y) = d(z) = 4 \).
(C8) There is no 5-face \([x_1x_2 \cdots x_5]\) with \( d(x_1) = 2 \), \( d(x_2) = 5 \) and \( d(x_3) = 3 \).

In what follows, let \( L \) be a list assignment of \( G \) with \( |L(v)| = 5 \) for all \( v \in V(G) \).

Lemma 2. Suppose that \( v \) is a pendant light 3-vertex of \( v_3 \), i.e., \( f = [vuyv] \) is a 3-face. Then
   - (A1) \( d(v_3) \geq 4 \);
   - (A2) If \( d(v_3) = 4 \), then \( d(v_1), d(v_2) \geq 5 \).

Proof. (A1) Suppose to the contrary that \( d(v_3) \leq 3 \). Let \( u_1, \ldots, u_k \) be the neighbors of \( v_3 \) different from \( v \). Then \( k \leq 2 \). By the minimality of \( G \), \( G - v \) admits an acyclic \( L \)-coloring \( \pi \). Clearly, \( \pi(v_1) \neq \pi(v_2) \), since \( v_1 \) is adjacent to \( v_2 \) in \( G - v \). If \( v_1, v_2 \) and \( v_3 \) have mutually distinct colors, then we color \( v \) with a color different from the colors of its neighbors (i.e., a proper coloring). Otherwise, by the symmetry, we may suppose \( \pi(v_1) = \pi(v_2) \). Color \( v \) with a color in \( L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(u_1), \ldots, \pi(u_k)\} \). Since \( k \leq 2 \), the resulting coloring is an acyclic \( L \)-coloring of \( G \). This contradicts the choice of \( G \).

(A2) Without loss of generality, assume that \( d(v_3) \leq 4 \). Let \( w_1, w_2 \) and \( w_3 \) be the neighbors of \( v_3 \) different from \( v \), and \( u_1, \ldots, u_m \) be the neighbors of \( v_1 \) different from \( v \) and \( v_2 \). Clearly, \( m \leq 2 \). By the minimality of \( G \), \( G - v \) admits an acyclic \( L \)-coloring \( \pi \). If \( v_1, v_2 \) and \( v_3 \) have mutually distinct colors, it is enough to color \( v \) properly. If \( \pi(v_1) = \pi(v_3) \), we color \( v \) with a color \( c \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(u_1), \ldots, \pi(u_m)\} \).

Now, we assume that \( \pi(v_2) = \pi(v_3) \) and \( L(v) = \{1, 2, \ldots, 5\} \). If there exists a color \( c \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(u_1), \pi(u_2), \pi(w_2), \pi(w_3)\} \), then we color \( v \) with \( c \). Otherwise, we may suppose that \( \pi(u_1) = 1, \pi(v_2) = \pi(v_3) = 2, \pi(u_1) = 3, \pi(u_2) = 4 \) and \( \pi(w_3) = 5 \). If \( L(v_3) \neq L(v) \), we recolor \( v_3 \) with a color in \( L(v_3) \setminus L(v) \) and reduce to the previous case. Otherwise, we recolor \( v_3 \) with 1 and again reduce to the previous case.

Lemma 3. Suppose that \( v \) is a 5-vertex with \( n_2(v) = 1 \). If \( v \) is incident to a 3-face \( f \), then \( n_3(f) = 0 \).

Proof. Let \( v_1, v_2, \ldots, v_5 \) be the neighbors of \( v \) with \( d(v_1) = 2 \) and \( N(v_1) = \{v, u_1\} \). Assume that \( v \) is incident to a 3-face \( f = [vuvv] \) such that \( n_3(f) \geq 1 \). By (C7.2), we derive that \( n_3(f) = 1 \), say \( d(v_2) = 3 \). Let \( x_2 \) be the neighbor of \( v_2 \) different from \( v \) and \( v_3 \). By the minimality of \( G \), \( G - v \) has an acyclic \( L \)-coloring \( \pi \). Suppose that \( L(v_1) = \{1, 2, 3, 4, 5\} \). If \( \pi(u_1) \neq \pi(v) \), we color properly \( v_1 \). Otherwise, if \( v_1 \) cannot be acyclically colored, we may assume that \( \pi(v) = \pi(u_1) = \pi(v_2) = 1, \pi(v) = i \) for \( i = 2, 3, 4, 5 \). If \( L(v) \neq L(v_1) \), we recolor \( v \) with a color in \( L(v) \setminus L(v_1) \) and then give \( v_1 \) a proper coloring. If \( L(v) = L(v_1) \), we recolor \( v \) with 2 and color \( v_1 \) with 3, then recolor \( v_2 \) with a color different from 1, 2 and 3. □

In the following proofs of Lemmas 4 and 5, we let \( v_1, v_2, \ldots, v_d(v) \) be the neighbors of the vertex \( v \) considered. If \( v_1 \) is a 2-vertex, we use \( u_i \) to denote the neighbor of \( v_j \) different from \( v \). If \( v_j \) is a 3-vertex, we use \( x_j \) and \( y_j \) to denote the neighbors of \( v_j \) different from \( v \).
Lemma 4. Suppose that $v$ is a 6-vertex. Then the following hold:

(B1) If $n_2(v) = 2$ and $v$ is incident to a (3, 3, 6)-face, then $n_3(v) \leq 2$;

(B2) If $n_2(v) = 3$, then $n_3(v) \leq 1$;

(B3) If $n_2(v) = 4$, then $t(v) = 0$.

Proof. (B1) Suppose that $v_1, v_2$ are 2-vertices, $v_3, v_4, v_5$ are 3-vertices such that $[v_3v_4]$ is a 3-face. Let $N(v_3) = \{v_1, v_4, x_3\}$ and $N(v_4) = \{v_3, x_4\}$. By the minimality of $G$, $G - \{v_1, v_2, v_3, v_4\}$ admits an acyclic $L$-coloring $\pi$.

If $\pi(v_5) \neq \pi(v_6)$, there exists a color $c \in L(v) \setminus \{\pi(v_5), \pi(v_6)\}$ which appears at most once on the set $\{u_1, u_2, x_3, x_4\}$. So we color $v$ with $c$. If $\pi(u_1) = c$, we further color $v_1$ with a color different from $c$, $\pi(v_5)$, $\pi(v_6)$ and then give a proper coloring for $v_2, v_3, v_4$. If $\pi(x_3) = c$, we color $v_3$ with a color different from $\pi(v_5)$, $\pi(v_6)$, $\pi(v)$, then color $v_4$ with a color in $L(v) \setminus \{c, \pi(x_4), \pi(v_3)\}$, and finally give a proper coloring for $v_1$ and $v_2$.

Now, we suppose that $\pi(v_5) = \pi(v_6)$. If $\pi(x_3) \neq \pi(x_5)$, we recolor $v_5$ with a color in $L(v) \setminus \{\pi(x_5), \pi(x_3), \pi(v_3)\}$ and reduce the argument to the previous case. Otherwise, since $|L(v) \setminus \{\pi(v_5), \pi(x_3)\}| \geq 3$, the proof can also be given with a similar argument to the previous case.

(B2) Assume to the contrary that $v_1, v_2, v_3$ are 2-vertices and $v_4, v_5$ are 3-vertices. Let $\pi$ be an acyclic $L$-coloring of $G - \{v_1, v_2, v_3\}$. Let $\alpha = |\{\pi(v_4), \pi(v_5), \pi(v_6)\}|$. We only need to handle the following three cases:

(a) $\alpha = 3$. There is a color $c \in L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6)\}$ appearing at most once on $\{u_1, u_2, u_3\}$, e.g., $\varphi(u_1) = c$.

We color $v$ with $c$, $v_1$ with a color in $L(v_1) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6), c\}$ and then give a proper coloring for $v_2$ and $v_3$.

(b) $\alpha = 2$. It suffices to discuss the following two situations.

(b1) $\pi(u_4) = \pi(v_5)$. If $\pi(x_4) \neq \pi(x_5)$, we recolor $v_4$ with a color different from $\{\pi(x_4), \pi(x_5), \pi(v_4), \pi(v_6)\}$ and go back to the former case. If $\pi(x_4) \neq \pi(x_5)$, we have a similar argument. Now assume that $\pi(x_4) = \pi(x_5) = \pi(v_5)$.

There exists a color $c \in L(v) \setminus \{\pi(v_4), \pi(v_6), \pi(x_4)\}$ appearing at most once on $\{u_1, u_2, u_3\}$, say $\pi(u_1) = c$. We color $v$ with $c$, $v_1$ with a color different from that of $u_1, v_4, u_6, x_4$, and give a proper coloring for $v_2$ and $v_3$.

(b2) $\pi(v_5) = \pi(v_6)$. If $\pi(x_3) \neq \pi(x_5)$, we do a similar recoloring for $v_5$ and then reduce to the previous case. Otherwise, since $|L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(x_5)\}| \geq 2$, we also have a similar discussion as above.

(c) $\alpha = 1$. This means that $\pi(v_4) = \pi(v_5) = \pi(v_6)$. Similarly, we may assume that $\pi(x_4) = \pi(x_5)$ and $\pi(x_5) = \pi(y_5)$. Now, since $|L(v) \setminus \{\pi(v_4), \pi(x_4), \pi(x_5)\}| \geq 2$, we can reduce the proof to the previous case.

(B3) Assume to the contrary that $v_1, v_2, v_3, v_4$ are 2-vertices and $[v_1v_2v_3]$ is a 3-face. Let $\pi$ be an acyclic $L$-coloring of $G - \{v_1, v_2, v_3\}$. Obviously, $\pi(v_5) \neq \pi(v_6)$. There exists a color $c \in L(v) \setminus \{\pi(v_5), \pi(v_6)\}$ appearing at most once on $\{u_1, u_2, u_3, u_4\}$. The remaining argument is similar to the previous case.

Lemma 5. Let $v$ be a 7-vertex. Then

(F1) If $n_2(v) = 4$, then $n_3(v) \leq 2$;

(F2) If $n_2(v) = 5$, then $n_3(v) = 0$ and $t(v) = 0$.

Proof. (F1) Assume to the contrary that $v_1, v_2, v_3, v_4$ are 2-vertices and $v_5, v_6, v_7$ are 3-vertices. By the minimality of $G$, $G - \{v_1, v_2, v_3, v_4\}$ has an acyclic $L$-coloring $\varphi$. Suppose that $L(v) = \{1, 2, \ldots, 5\}$. Let $\beta = |\{\varphi(v_5), \varphi(v_6), \varphi(v_7)\}|$. We consider the following possibilities:

(a) $\beta = 3$. This means that $v_5, v_6, v_7$ are colored with mutually distinct colors. If there exists a color $c \in L(v) \setminus \{\varphi(v_5), \varphi(v_6), \varphi(v_7)\}$ appearing at most once on $\{u_1, u_2, u_3, u_4\}$, we have a similar argument to the previous case. Otherwise, we may suppose that $\varphi(v_5) = 1, \varphi(v_6) = 2, \varphi(v_7) = 3, \varphi(u_1) = \varphi(u_2) = 4$ and $\varphi(u_3) = \varphi(u_4) = 5$.

If $4 \notin \{\varphi(x_j), \varphi(y_j)\}$ for some fixed $j \in \{5, 6, 7\}$, say $j = 5$, then we color $v$ with 4, $v_1$ with a color different from 2, 3, 4, and $v_2$ with a color different from 2, 3, 4, $\varphi(u_1)$, and give a proper coloring for $v_3$ and $v_4$. Suppose that $4 \in \{\varphi(x_j), \varphi(y_j)\}$ for all $j \in \{5, 6, 7\}$, and similarly $5 \in \{\varphi(x_j), \varphi(y_j)\}$ all $j \in \{5, 6, 7\}$. This shows that $\varphi(x_j), \varphi(y_j) = \{4, 5\}$ for all $j = 5, 6, 7$. In this case, we color $v$ with 1, recolor $v_5$ with a color different from 1, 4, 5 and then give a proper coloring for $v_1, v_2, v_3, v_4$.

(b) $\beta = 2$. Without loss of generality, we assume that $\varphi(v_5) = \varphi(v_6) = 1$ and $\varphi(v_7) = 2$. If $\varphi(x_3) \neq \varphi(y_3)$ or $\varphi(x_6) \neq \varphi(y_6)$, we can recolor $v_5$ or $v_6$ to reduce to the previous case (a). Thus, suppose $\varphi(x_3) = \varphi(y_3)$ and $\varphi(x_6) = \varphi(y_6)$. There exists a color $c \in L(v) \setminus \{1, 2, \varphi(x_3)\}$ appearing at most twice on $\{u_1, u_2, u_3, u_4\}$, say $\varphi(u_1) = \varphi(u_2) = c$. We color $v$ with $c$, $v_1$ with a color different from 1, 2, $c$, $v_2$ with a color different from 1, 2, $c$, $\varphi(u_1)$, and give a proper coloring for $v_3$ and $v_4$. 


(c) $\beta = 1$. This means that $\pi(v_5) = \pi(v_6) = \pi(v_7)$. If there exists $j \in \{5, 6, 7\}$ such that $\pi(x_j) \neq \pi(y_j)$, then we recolor $v_j$ to reduce to the former case. Otherwise, we have that $\pi(x_j) = \pi(y_j)$ for all $j \in \{5, 6, 7\}$. There exists a color $c \in L(v) \setminus \{\pi(v_5), \pi(x_5), \pi(x_6)\}$ appearing at most twice on $\{u_1, u_2, u_3, u_4\}$, say $\pi(u_1) = \pi(u_2) = c$. We color $v$ with $c$, $v_1$ with a color in $L(v_1) \setminus \{\pi(v_5), c\}$, $v_2$ with a color different from $\{\pi(v_5), c, \pi(v_1)\}$, then properly color $v_3$ and $v_4$.

(F2) The proof is analogous to that of (C5.2) and (B3). \qed

**Lemma 6.** Every 8-vertex is adjacent to at most six 2-vertices.

**Proof.** The proof is similar to (C6) in Lemma 1. \qed

4. Discharging process

In order to complete the proof, we suppose that $G$ is a counterexample to Theorem 1 with the least vertices. Let $L$ be a list assignment such that $|L(v)| = 5$ for all $v \in V(G)$. Thus, $G$ satisfies Lemma 1 to 6.

Using Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the relation $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$, we can derive the following identity:

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8. \quad (1)$$

We define a weight function $w$ by $w(x) = d(x) - 4$ for all $x \in V(G) \cup F(G)$. It follows from identity $(1)$ that the total sum of weights is equal to $-8$. We design appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function $w'$ is produced. However, the total sum of weights is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, the new weight function $w'(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction,

$$0 \leq \sum_{x \in V(G) \cup F(G)} w'(x) = \sum_{x \in V(G) \cup F(G)} w(x) = -8. \quad (2)$$

For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the amount of weights transferred from $x$ to $y$. Suppose that $f = [v_1v_2v_3]$ is a 3-face with $d(v_1) \leq d(v_2) \leq d(v_3)$. We use $(d(v_1), d(v_2), d(v_3)) \to (c_1, c_2, c_3)$ to denote that the vertex $v_i$ gives $f$ the amount of weight $c_i$ for $i = 1, 2, 3$.

Our discharging rules are as follows:

(R1) Let $f = [v_1v_2v_3]$ be a 3-face with $d(v_1) \leq d(v_2) \leq d(v_3)$. We set

$$(3, 3, 6^+) \to \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right);$$

$$(3, 4, 5^+) \to \left(\frac{1}{3}, 0, \frac{1}{3}\right);$$

$$(3, 5^+, 5^+) \to \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right);$$

$$(4, 4, 5^+) \to \left(0, 0, \frac{1}{3}\right);$$

$$(4, 5^+, 5^+) \to \left(0, \frac{1}{6}, \frac{1}{6}\right);$$

$$(5^+, 5^+, 5^+) \to \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

(R2) Let $v$ be a 2-vertex adjacent to a vertex $u$.

If $d(u) = 5$, then $\tau(u \to v) = \frac{1}{2}$.
If \( d(u) \geq 6 \), we set

\[
\tau(u \rightarrow v) = \begin{cases} 
\frac{5}{6} & \text{if } vu \text{ is incident to two heavy faces;} \\
\frac{2}{3} & \text{if } vu \text{ is incident to exactly one heavy face;} \\
\frac{1}{2} & \text{if } vu \text{ is not incident to any heavy faces.}
\end{cases}
\]

(R3) Each \( 5^+ \)-vertex \( v \) gives \( \frac{1}{2} \) to each adjacent pendant light \( (3, 4, 5^+) \)-vertex, \( \frac{1}{3} \) to each other pendant light \( 3 \)-vertex, and \( \frac{1}{6} \) to each adjacent \( 3 \)-vertex \( u \) such that the edge \( uv \) is incident to a \( 3 \)-face.

(R4) Let \( f \) be a \( 5^+ \)-face. Then

(R4.1) \( f \) gives \( \frac{1}{2} \) to each adjacent \( 3 \)-face through a common \( (4, 4^+) \)-edge, \( \frac{1}{2} \) to each incident bad pendant light \( 3 \)-vertex, \( \frac{1}{2} \) to each other incident \( 3 \)-vertex.

(R4.2) If \( f \) is incident to a \( 2 \)-vertex \( v \), then \( \tau(f \rightarrow v) = \frac{1}{2} \) if \( f \) is a heavy face of \( vu \), where \( u \) is a neighbor of \( v \) in the boundary of \( f \); Otherwise, \( \tau(f \rightarrow v) = \frac{1}{3} \).

Let \( f \in F(G) \). Since \( G \) contains no \( 4 \)-cycles, \( d(f) \neq 4 \). The proof is divided into the following cases.

1. \( d(f) = 3 \). Then \( w(f) = -1 \). Let \( f = [xyz] \) such that \( d(x) \leq d(y) \leq d(z) \). Since \( G \) has no \( 4 \)-cycles, \( f \) is not adjacent to any \( 3 \)-face. Thus, each of the faces adjacent to \( f \) is of degree at least 5. By (C7.1), we derive that \( d(x) \geq 3 \).

(1.1) Assume that \( d(x) = 3 \). If \( d(y) = 3 \), then by (C7.2), \( d(z) \geq 6 \), and hence by (R1), \( w'(f) \geq -1 + \frac{1}{6} \times 2 + \frac{2}{3} = 0 \).

If \( d(y) = 4 \), then \( z \) is a \( 5^+ \)-vertex by (C7.3), and hence \( w'(f) \geq -1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 0 \) by (R1) and (R4.1). If \( d(y) \geq 5 \), then \( w'(f) \geq -1 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 0 \) by (R1).

(1.2) Assume that \( d(x) \geq 4 \). If \( d(x) \geq 5 \), then \( f \) is a \( (5^+, 5^+, 5^+) \)-face and therefore \( w'(f) \geq -1 + \frac{1}{3} \times 3 = 0 \) by (R1). So suppose that \( d(x) = 4 \). If at least one of \( y \) and \( z \) is a \( 4 \)-vertex, then by (R1) and (R4), \( w'(f) \geq -1 + \frac{1}{4} \times 2 = 0 \). Otherwise, it follows that \( d(z) \geq d(y) \geq 5 \), and \( w'(f) \geq -1 + \frac{1}{3} \times 2 = 0 \) by (R1) and (R4).

2. \( d(f) = 5 \). Then \( w(f) = 1 \). Let \( f = [x_1x_2 \cdots x_5] \). By (C2) and (C3.1), we have \( n_2(f) \leq 2 \), and \( n_2(f) + n_3(f) \leq 3 \).

Since the distance between any two triangles is at least 3, \( t(f) \leq 1 \).

(2.1) Assume that \( t(f) = 0 \). If \( n_2(f) + n_3(f) \leq 2 \), we have \( w'(f) \geq 1 - \frac{1}{3} \times 2 = 0 \) by (R4). So suppose that \( n_2(f) + n_3(f) = 3 \). When \( n_2(f) = 2 \), it is easy to see that \( n_3(f) = 0 \) by (C2). Thus, suppose that \( n_2(f) \leq 1 \).

If \( n_2(f) = 0 \), then \( n_3(f) = 3 \). By (C3.1), \( f \) is not incident to any bad pendant light \( 3 \)-vertex. Thus, \( w'(f) \geq 1 - \frac{1}{3} \times 3 = 0 \) by (R4.1).

If \( n_2(f) = 1 \), then \( n_3(f) = 2 \). Without loss of generality, we assume that \( d(x_1) = 2 \). It follows from (C2) and (C8) that \( d(x_2), d(x_3) \geq 6 \) and \( d(x_4) = d(x_5) = 3 \), implying that \( f \) is a heavy face of the edge \( x_1x_5 \). By (R4.2), \( \tau(f \rightarrow x_1) \leq \frac{1}{2} \). Noting that neither \( x_3 \) nor \( x_4 \) is a bad pendant light \( 3 \)-vertex, we have \( w'(f) \geq 1 - \frac{1}{3} \times 3 = 0 \) by (R4.1).

(2.2) Assume that \( t(f) = 1 \) and \( f' = [x_1x_2x^*] \) is a \( 3 \)-face. Then none of \( x_3, x_4, x_5 \) is a bad pendant light \( 3 \)-vertex, since the distance between any two triangles is at least 3.

(2.2.1) Suppose that \( n_2(f) = 0 \). We consider three subcases as follows:

If \( d(x_1), d(x_2) \geq 4 \), at most two of \( x_3, x_4, x_5 \) are \( 3 \)-vertices, and thus \( w'(f) \geq 1 - \frac{1}{3} \times 3 = 0 \) by (R4.1).

If exactly one of \( x_1 \) and \( x_2 \) is a \( 3 \)-vertex, say \( d(x_1) = 3 \), then \( d(x_3) \geq 4 \) by (A1). If \( d(x_3) = d(x_4) = 3 \), then \( x_1 \) is not a bad pendant light \( 3 \)-vertex by (C3.1). Thus, \( w'(f) \geq 1 - \frac{1}{3} \times 3 = 0 \) by (R4.1). Otherwise, at most one of \( x_3 \) and \( x_4 \) is a \( 3 \)-vertex, we have \( w'(f) \geq 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3} \) by (R4.1).

If \( d(x_1) = d(x_2) = 3 \), then neither \( x_1 \) nor \( x_2 \) is a bad pendant light \( 3 \)-vertex. Both \( x_3 \) and \( x_5 \) are \( 5^+ \)-vertices. Thus, \( w'(f) \geq 1 - \frac{1}{3} \times 3 = 0 \) by (R4.1).

(2.2.2) Suppose that \( n_2(f) = 1 \). Then exactly one of \( x_3, x_4, x_5 \) is a \( 2 \)-vertex. By symmetry, we consider the following subcases:

Assume that \( d(x_1), d(x_2) \geq 4 \). We consider, without loss of generality, two cases: (a) \( d(x_4) = 2 \). It is easy to see that both \( x_3 \) and \( x_5 \) are \( 5^+ \)-vertices by (C2). Thus, \( n_3(f) = 0 \) and \( w'(f) \geq 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{6} \).
by (R4); (b) \(d(x_3) = 2\). Then \(d(x_1), d(x_4) \geq 5\) by (C2). If \(d(x_3) \geq 4\), then \(w'(f) \geq 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{6}\) by (R4). Otherwise, \(d(x_3) = 3\) and \(d(x_4) \geq 6\) by (C8). If \(d(x_2) \geq 5\), then \(f\) gives nothing to \(f'\) and hence 
\[w'(f) \geq 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}\] 
by (R4). If \(d(x_2) = 4\), then \(f\) is a heavy face of the edge \(x_4x_5\). By (R4.2), 
\(\tau(f \rightarrow x_5) \leq \frac{1}{3}\) and consequently \(w'(f) \geq 1 - \frac{1}{3} - \frac{1}{3} = 0\) by (R4).

Assume that \(d(x_1) = 3\) and \(d(x_2) \geq 4\). It follows that \(d(x_3) \geq 4\) by (A1) and \(\tau(f \rightarrow f') = 0\) by our rules. Thus, \(w'(f) \geq 1 - \frac{1}{3} - \frac{1}{3} = 0\) by (R4).

Assume that \(d(x_1) = d(x_2) = 3\). This implies that \(d(x_4) = 2\) and \(d(x_3), d(x_5) \geq 6\). Moreover, \(f\) is a heavy face of \(x_3x_4\) and \(\tau(f \rightarrow x_i) \leq \frac{1}{3}\) for \(i = 1, 2\). Thus, \(w'(f) \geq 1 - \frac{1}{3} \times 3 = 0\) by (R4). 

(2.2.3) Suppose that \(n_2(f) = 2\). It is immediate to see that \(d(x_3) = d(x_5) = 2\) and \(d(x_1), d(x_2), d(x_4) \geq 5\) by (C2). Thus, \(\tau(f \rightarrow f') = 0\) and \(w'(f) \geq 1 - \frac{1}{2} \times 2 = 0\) by (R4).

3. \(d(f) \geq 6\). Let \(f = [v_1v_2 \cdots v_3]\). We use \(t^*(f)\) to denote the number of 3-faces each of which is adjacent to \(f\) and gets \(\frac{1}{3}\) from \(f\), and \(n^*(f)\) the number of 3-vertices each of which is incident to \(f\) and gets \(\frac{1}{2}\) from \(f\). For simplicity, we write \(t^*\) for \(t^*(f)\), \(n^*_2\) for \(n^*_2(f)\), \(n^*_3\) for \(n^*_3(f)\), etc.

Claim 1. \(2t^* + 2n^*_2 + 3n^*_3 \leq d(f)\).

It is easy to see that if \(2 \leq d(v_i) \leq 3\) then \(f\) gives nothing to adjacent faces through both edges \(v_{i-1}v_i\) and \(v_iv_{i+1}\).

In particular, when \(d(v_i) = 2\), we have \(d(v_{i-1}), d(v_{i+1}) \geq 5\) by (C2). Thus, the number of \((4, 4^+)-\)edges incident to \(f\) is at most \(d(f) - n_3 - 2n_2\). Since there are no triangles at distance less than 3, it follows that \(t^*\) is at most \((d(f) - n_3 - 2n_2)/2\), which shows Claim 1.

The following Claim 2 follows immediately from the fact that there do not exist two triangles at distance less than 3:

Claim 2. \(n^*_3 = \lfloor \frac{2}{3} d(f) \rfloor\).

Using (R4), we derive

\[
w'(f) \geq d(f) - 4 - \frac{2}{3} t^* - \frac{1}{2} n^*_2 - \frac{1}{2} n^*_3 - \frac{1}{3} (n_3 - n^*_3)
\]

\[
= d(f) - 4 - \frac{2}{3} t^* - \frac{1}{2} n^*_2 - \frac{1}{6} n^*_3 - \frac{1}{3} n_3
\]

\[
\geq d(f) - 4 - \frac{2}{3} t^* - \frac{1}{2} n^*_2 - \frac{1}{6} n^*_3 - \frac{1}{3} d(f) - 2n_2 - 2t^*
\]

\[
= \frac{2}{3} d(f) - 4 + \frac{2}{3} t^* + \frac{1}{6} n^*_2 - \frac{1}{6} n^*_3
\]

\[
\geq \frac{2}{3} d(f) - 4 + \frac{2}{3} t^* + 1\n\]

\[
\geq \frac{2}{3} d(f) - 4 + \frac{2}{3} t^* + 1.
\]

If \(d(f) \geq 7\), then \(w'(f) \geq \frac{3}{2} d(f) - 4 \geq \frac{3}{2} \times 7 - 4 = \frac{1}{2}\).

If \(d(f) = 6\), then \(w'(f) = 2\), \(\tau(f) \leq 1\) and so \(n^*_3 \leq 1\). If \(n^*_3 = 0\), the above expression shows that 
\(w'(f) \geq \frac{2}{3} d(f) - 4 + \frac{2}{3} t^* + \frac{1}{6} n^*_2 \geq \frac{2}{3} \times 6 - 4 = 0\). Otherwise, we assume that \(v_1\) is a bad pendant light 3-vertex, say \(d(v_1) = 3\), \(d(v_2) \geq 5\) and \(d(v_3) = 4\). Noting that \(v_5\) cannot be a 2-vertex by (C2), we have that \(n_2(v) \leq 1\) and \(w'(f) \geq 2 - \frac{1}{2} \times 2 - \frac{1}{3} \times 2 = \frac{1}{3}\).

Let \(v \in V(G)\). Let \(v_1, v_2, \ldots, v_d(v)\) denote the neighbors of \(v\) in a cyclic order. Let \(f_i\) denote the incident face of \(v\) with \(v_1v_i\) and \(v_iv_{i+1}\) as two boundary edges for \(i = 1, 2, \ldots, d(v)\), where indices are taken modulo \(d(v)\). We see that \(d(v) \geq 2\) by (C1). Since \(G\) contains no triangles at distance less than 3, \(\tau(v) \leq 1\), \(p_3(v) \leq 1\) and \(p_3(v) + \tau(v) \leq 1\).

The proof is divided into some subcases according to the value of \(d(v)\).

1. \(d(v) = 2\). Then \(w(v) = -2\), \(d(v_1), d(v_2) \geq 5\) by (C2), and \(d(f_1), d(f_2) \geq 5\) by (C7.1). By (R2), \(\tau(v_1) \rightarrow v \geq \frac{1}{2}\).

By symmetry, we need to consider the following three possibilities:

(1.1) Assume that \(d(v_1) = d(v_2) = 5\). Neither \(f_1\) nor \(f_2\) is a heavy face of \(vv_1\) for each \(i = 1, 2\). Thus, 
\(\tau(f_i \rightarrow v) = \tau(v_1 \rightarrow v) = \frac{1}{2}\) for \(i = 1, 2\) by (R2) and (R4). Thus, \(w'(f) \geq -2 + \frac{1}{2} \times 4 = 0\).
Assume that $d(v_1) = 5$ and $d(v_2) \geq 6$. If neither $f_1$ nor $f_2$ is a heavy face of $vv_2$, then $\tau(v_2 \to v) = \frac{1}{2}$ and $w'(v) \geq -1 + \frac{1}{2} \times 4 = 0$ by (R2) and (R4). If exactly one of $f_1$ and $f_2$ is a heavy face of $vv_2$, say $f_1$ is but $f_2$ is not, then $\tau(v_2 \to v) = \frac{3}{2}$, $\tau(f_1 \to v) = \frac{1}{2}$, $w'(v) \geq -2 + \frac{1}{3} + \frac{2}{3} + \frac{1}{2} + \frac{1}{4} = 0$ by (R2) and (R4). If both $f_1$ and $f_2$ are heavy faces of $vv_2$, then $\tau(v_2 \to v) = \frac{5}{6}$, $\tau(f_1 \to v) = \frac{1}{4}$ for $i = 1, 2$, $w'(v) \geq 2 + \frac{5}{6} + \frac{1}{2} + \frac{1}{4} \times 2 = 0$ by (R2) and (R4).

Assume that $d(v_1), d(v_2) \geq 6$. If neither $f_1$ nor $f_2$ is a heavy face of $vv_i$ for any $i \in \{1, 2\}$, then $\tau(f_i \to v) = \frac{1}{2}$ for $i = 1, 2$, and $w'(v) \geq -2 + \frac{5}{6} + \frac{1}{2} + \frac{1}{4} \times 2 = 0$ by (R2) and (R4).

If neither $f_1$ nor $f_2$ is a heavy face of $vv_i$ for any $i \in \{1, 2\}$, then $\tau(v_2 \to v) = \frac{3}{2}$, $\tau(f_1 \to v) = \frac{1}{2}$, $w'(v) \geq -2 + \frac{1}{3} + \frac{2}{3} + \frac{1}{2} + \frac{1}{4} = 0$ by (R2) and (R4). If both $f_1$ and $f_2$ are heavy faces of $vv_i$ for some $i \in \{1, 2\}$, then $\tau(v_2 \to v) = \frac{5}{6}$, $\tau(f_1 \to v) = \frac{1}{4}$ for $i = 1, 2$, and $w'(v) \geq 2 + \frac{5}{6} + \frac{1}{2} + \frac{1}{4} \times 2 = 0$ by (R2) and (R4).

2. $d(v) = 3$. Then $w(v) = -1$, $t(v) \leq 1$, $p_3(v) = 0$ by (C3.2). If $t(v) = 0$, then $w'(v) \geq -1 + \frac{1}{3} \times 3 = 0$ by (R4). Assume that $t(v) = 1$. Let $f_1 = [vv_1v_2]$ be a 3-face with $d(v) \leq d(v_1) \leq d(v_2)$.

If $d(v_1) = 3$, then by (C7.2) and (C3.1), $d(v_2) \geq 6$, $d(v_3) \geq 5$, and $w'(v) \geq -1 - \frac{1}{6} + \frac{1}{3} \times 3 = 0$ by (R1), (R3) and (R4).

If $d(v_1) = 4$, then $d(v_2) \geq 5$ by (C7.3), $d(v_3) \geq 5$ by (A2), and $w'(v) \geq -1 - \frac{1}{6} + \frac{1}{3} \times 2 + \frac{1}{6} + \frac{1}{2} = 0$ by (R1), (R3) and (R4).

If $d(v_1) \geq 5$, then $\tau(f_3 \to v) \geq \frac{1}{2}$, $\tau(v_1 \to v) \geq \frac{1}{6}$ and $\tau(v_2 \to v) \geq \frac{1}{6}$ by (R3) and (R4). When $d(v_3) = 4$, $v$ is a pendant light 3-vertex, so that $\tau(f_2 \to v) = \tau(f_3 \to v) = \frac{1}{2}$ and $w'(v) \geq -1 - \frac{1}{3} + \frac{1}{3} \times 2 + \frac{1}{6} \times 2 = 0$.

When $d(v_3) \geq 5$, we have $\tau(v \to f_1) = \frac{1}{3}$, $\tau(v_3 \to v) = \frac{1}{3}$, and $w'(v) \geq -1 - \frac{1}{3} + \frac{1}{3} \times 2 + \frac{1}{6} \times 2 = 0$ by (R1) and (R3).

3. $d(v) = 4$. Then $w'(v) = w(v) = d(v) - 4 = 0$.

4. $d(v) = 5$. Then $w(v) = 1$. By (C4.1), $n_2(v) \leq 1$ and $t(v) + p_3(v) \leq 1$. Assume that $n_3(v) = 0$. If $p_3(v) = 0$, then $t(v) \leq 1$ and $w'(v) \geq 1 - \frac{1}{3} + \frac{1}{6} \times 2 = \frac{1}{3}$ by (R1) and (R3). If $p_3(v) = 1$, then $t(v) = 0$ and $w'(v) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by (R3). Now assume that $n_2(v) = 1$. By (C4.2), $p_3(v) = 0$. If $t(v) = 0$, then $w'(v) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by (R2). If $t(v) = 1$, suppose that $v_1$ is a 2-vertex and $f_2 = [vv_2v_3]$ is a 3-face. From Lemma 3, we see that $f_2$ is a $(4^+, 4^+, 5)$-face. Thus, $w'(v) \geq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ by (R1) and (R2).

Given a rational number $r$, let $\sigma_r(v)$ denote the number of 2-vertices each of which gets $r$ from $v$ according to the rules. By definition, $\sigma_{\frac{1}{2}}(v) + \sigma_{\frac{1}{3}} + \sigma_{\frac{2}{3}}(v) \leq n_2(v)$. Moreover, we have the following:

**Claim 3.** Suppose that $v$ is a $6^+$-vertex. Then

1. $\sigma_{\frac{1}{2}}(v) \leq n_3(v)$;
2. $\sigma_{\frac{2}{3}}(v) \leq \lfloor n_3(v)/2 \rfloor$.

**Proof.** Assume to the contrary that $\sigma_{\frac{1}{2}}(v) > n_3(v)$. Then there exists a 3-vertex, say $v_2$, incident to two faces $f_1 = [v_1v_2vx \cdots]$ and $f_2 = [v_3v_2vy \cdots]$ such that $d(v_1) = d(v_3) = 2$, $\tau(v \to v_1) = \tau(v \to v_3) = \frac{2}{3}$, $f_1$ is a heavy face of $vv_1$ and $f_2$ is a heavy face of $vv_3$. Combining (C3.1) and the definition of a heavy face, it follows that $d(x) = d(y) = 4$, $xx^*$ lies on a 3-face $[xx^*x']$, and $yy^*$ lies on a 3-face $[yy^*y']$. However, the distance between $[xx^*x']$ and $[yy^*y']$ is 2, contradicting the assumption on $G$. This proves (1). With a similar argument, we can prove (2). □

5. $d(v) = 6$. Then $w(v) = 2$, $n_2(v) \leq 4$ by (C5.1), and $t(v) + p_3(v) \leq 1$. We only consider the following four cases in the light of the size of $n_2(v)$.

5.1. $n_2(v) = 4$. By (C5.2), $n_3(v) = 0$, implying that $p_3(v) = 0$ and $\sigma_{\frac{2}{3}}(v) = \sigma_{\frac{1}{2}}(v) = 0$. By (B3), $t(v) = 0$.

Thus, $w'(v) \geq 2 - \frac{1}{6} \times 4 = 0$ by (R2).

5.2. $n_2(v) = 3$. By (B2), $n_3(v) \leq 1$, and hence $\sigma_{\frac{2}{3}}(v) = 0$ and $\sigma_{\frac{1}{2}}(v) \leq 1$ by Claim 3. If $n_3(v) = 0$, then $\sigma_{\frac{2}{3}}(v) = p_3(v) = 0$, thus $w'(v) \geq 2 - \frac{1}{6} \times 3 = \frac{5}{6}$ by (R2). So suppose that $n_3(v) = 1$. We consider two situations as follows:

Assume that $t(v) = 0$. If $\sigma_{\frac{1}{2}}(v) = 0$, then $w'(v) \geq 2 - \frac{1}{6} \times 3 = 0$ by (R2). Suppose that $\sigma_{\frac{2}{3}}(v) = 1$ and $d(v_k) = 3$. If $v_k$ is not a pendant light 3-vertex of $v$, then it is obvious that $\tau(v \to v_k) \leq \frac{1}{2}$ by (R3).
Otherwise, it is easy to derive that $v_k$ must be incident to a $(3, 3, 6^+)$-face by the definition of a heavy face, we also have $\tau(v \rightarrow v_k) = \frac{1}{3}$. Consequently, $w'(v) \geq 2 - \frac{2}{3} - \frac{1}{3} \times 2 = \frac{1}{3} = 0$ by (R2) and (R3).

Assume that $t(v) = 1$. Let $f_1 = [vu_1v_2]$ be a 3-face. Then $p_3(v) = 0$. If $f_1$ is a $(4^+, 4^+, 6)$-face, then by (R2) and (R1), $w'(v) \geq 2 - \frac{2}{3} - \frac{1}{3} \times 2 = \frac{1}{3} = 0$. If $f_1$ is a $(3, 4^+, 6)$-face, then $\sigma_2(v) = 0$ by (C3.2), $\tau(v \rightarrow f_1) = \frac{1}{3}$ by (R1), and $w'(v) \geq 2 - \frac{1}{3} \times 3 = \frac{1}{3} = 0$ by (R1) to (R3).

Claim 2: $\sigma_2(v) \leq n_3(v)/2 \leq (d(v) - n_2(v))/2 = (6 - 4)/2 = 2$. In terms of the size of $t(v)$, we consider two cases as follows:

Assume that $t(v) = 0$. If $p_3(v) = 0$, then $w'(v) \geq 2 - \frac{5}{6} \times 2 = \frac{1}{3} = 0$ by (R2). Suppose that $p_3(v) = 1$ and let $v_1$ be a light pendant 3-vertex of $v$. If $\sigma_2(v) = 1$, then $w'(v) \geq 2 - \frac{5}{6} \times \frac{3}{2} = \frac{1}{3} = 0 = 0$ by (R2) and (R3). If $\sigma_2(v) = 2$, then $v_1$ is incident to a $(3, 3, 6^+)$-face, hence $\tau(v \rightarrow v_1) \leq \frac{1}{3} = 0$ and $w'(v) \geq 2 - \frac{5}{6} \times 2 = \frac{1}{3} = 0$ by (R2) and (R3).

Assume that $t(v) = 1$. Then $p_3(v) = 0$. Let $f_1 = [vu_1v_2]$ be a 3-face. If $f_1$ is a $(3, 3, 6)$-face, then $n_3(v) \leq 2$ by (B1). This implies that $d(v_i) \neq 3$ for all $i = 3, 4, 5, 6$. Furthermore, it is easy to show that $\sigma_2(v) = 2$ and hence $w'(v) \geq 2 - \frac{1}{2} \times 2 - \frac{5}{6} \times 2 = \frac{1}{3} = 0$ by (R1) to (R3). If $f_1$ is not a $(3, 3, 6)$-face, then $\sigma_2(v) \leq 1$ and $\sigma_2(v) \leq 2$ by Claim 3. If $\sigma_2(v) = 1$, then $\sigma_2(v) = 0$, and $w'(v) \geq 2 - \frac{5}{6} \times \frac{3}{2} = \frac{1}{3} = 0$ by (R1) to (R3).

Claim 2: $\sigma_2(v) \leq 1$. If $t(v) = 1$, then $w'(v) \geq 2 - \frac{5}{6} \times \frac{3}{2} = \frac{1}{3} = 0$. Otherwise,

\[
6. d(v) = 7. \text{ Then } w(v) = 3, n_2(v) \leq 5 \text{ by (C6). In view of the value of } n_2(v), \text{ we consider the following four subcases:}
\]

(6.1) $n_2(v) = 5$. By (F2), $n_3(v) = t(v) = 0$, implying $\sigma_2(v) = 0$ and $w'(v) \geq 3 - \frac{5}{6} \times 5 = \frac{1}{6}$ by (R2).

(6.2) $n_2(v) = 4$. By (F1), $n_3(v) \leq 2$, so $\sigma_2(v) \leq 1$ and $\sigma_2(v) \leq 2$ by Claim 3.

Assume that $t(v) = 0$. If $\sigma_2(v) = 0$, then $w'(v) \geq 3 - \frac{5}{6} \times 2 = \frac{1}{6}$ by (R2) and (R3). If $\sigma_2(v) = 1$, then $\sigma_2(v) = 0$ and $w'(v) \geq 3 - \frac{5}{6} \times \frac{1}{2} = \frac{1}{6}$ by (R2) and (R3).

Assume that $t(v) = 1$. Let $f_1 = [vu_1v_2]$ be a 3-face. Then $p_3(v) = 0$ and $d(v_1), d(v_2) \geq 3$ by (C7.1). This implies that at most one of $u_3, \ldots, v_7$ is a 3-vertex. Thus $\sigma_2(v) = 0$ and $\sigma_2(v) \leq 1$ by Claim 3. If $\sigma_2(v) = 0$, then $w'(v) \geq 3 - \frac{5}{6} \times \frac{1}{2} = \frac{1}{6} \times 2 = 0$. If $\sigma_2(v) = 1$, it is easy to see that $f_1$ is not a $(3, 3, 7)$-face and hence $w'(v) \geq 3 - \frac{5}{6} \times \frac{3}{2} = \frac{1}{6} \times 3 = \frac{1}{6}$ by (R1) to (R3).

(6.3) $n_2(v) = 3$. Since $n_3(v) \leq 4$, $\sigma_2(v) \leq 2$ by Claim 3.

If $\sigma_2(v) = 2$, then $\sigma_2(v) = t(v) = 0$ and $w'(v) \geq 3 - \frac{5}{6} \times 2 = \frac{1}{6} = 0$.

Assume that $\sigma_2(v) = 1$. Then $\sigma_2(v) \leq 2$. If $\sigma_2(v) = 2$, then $t(v) = 0$ and $w'(v) \geq 3 - \frac{5}{6} \times 2 = \frac{1}{6}$ by (R1) to (R3). Suppose that $\sigma_2(v) = 1$. When $t(v) = 0$, $w'(v) \geq 3 - \frac{5}{6} \times \frac{1}{2} = \frac{1}{6} \times 2 = \frac{1}{6}$; When $t(v) = 1$, $w'(v) \geq 3 - \frac{5}{6} \times \frac{1}{2} = \frac{1}{6} \times 2 = \frac{1}{6}$.

Assume that $\sigma_2(v) = 0$. If $t(v) = 0$, then $w'(v) \geq 3 - \frac{5}{6} \times 3 - \frac{1}{2} = \frac{1}{6}$. If $t(v) = 1$, then $w'(v) \geq 3 - \frac{5}{6} \times 3 - \frac{1}{2} = \frac{1}{6}$.

(6.4) $n_2(v) \leq 2$. Clearly, $w'(v) \geq 3 - \frac{5}{6} \times 2 = \frac{1}{6} \times 2 = \frac{1}{6}$ by (R1) to (R3).

7. $d(v) \geq 8$. In what follows, we write simply $\sigma_i$ for $\sigma_i(v)$. We need to consider the following two cases:

(7.1) $t(v) = 0$. It is easy to show that $\sigma_1 + 2\sigma_2 + 3\sigma_3 \leq d(v)$ by definition. Using this fact, we derive

\[
w'(v) \geq \frac{d(v) - 4}{2} - \frac{5}{6} \sigma_3 - \frac{2}{3} \sigma_2 - \frac{1}{2} \sigma_1 - \frac{1}{2} \\
= \frac{1}{2}d(v) - \frac{9}{2} + \frac{2}{3} \sigma_3 + \frac{1}{3} \sigma_2 \equiv w^*.\]
If \( d(v) \geq 9 \), then \( w^* \geq 0 \).

Assume that \( d(v) = 8 \). If \( \sigma_2 \geq 1 \), then \( w^* \geq \frac{1}{2} \times 8 - \frac{9}{2} + \frac{2}{3} = \frac{1}{6} \). If \( \sigma_2 = 0 \), we have two possibilities:

- When \( \sigma_2 \geq 2 \), \( w^* \geq \frac{1}{2} \times 8 - \frac{9}{2} + \frac{2}{3} \times 2 = \frac{1}{6} \). When \( \sigma_2 \leq 1 \), since \( n_2(v) \leq 6 \) by Lemma 6, we have

\[
 w'(v) \geq 4 - \frac{2}{3} - \frac{1}{2} \times 5 - \frac{1}{2} = \frac{1}{2} \text{ by (R2) and (R3)}. 
\]

(7.2) \( t(v) = 1 \). Let \( f_1 = [vv_1v_2] \) be a 3-face. Then \( p_3(v) = 0 \). By (C2), \( d(v_1), d(v_2) \geq 3 \). Moreover, \( f_2 \) cannot be a heavy face of \( vv_2 \) and \( f_d(v) \) cannot be a heavy face of \( vv_d(v) \). This implies that

\[
 \sigma_2 + 2\sigma_3 + 3\sigma_5 \leq d(v) - 2. 
\]

Therefore,

\[
 w'(v) \geq d(v) - 4 - \frac{5}{6}\sigma_5 - \frac{2}{3}\sigma_2 - \frac{1}{2}\sigma_4 - \frac{2}{3} - \frac{1}{6} \times 2 
\]

\[
 \geq d(v) - 5 - \frac{5}{6}\sigma_5 - \frac{2}{3}\sigma_2 - \frac{1}{2}(d(v) - 2 - 3\sigma_5 - 2\sigma_5) 
\]

\[
 = \frac{1}{2}d(v) - 4 + \frac{2}{3}\sigma_5 + \frac{1}{3}\sigma_2 \geq 0. \quad \square 
\]

References