Matrices with multiple symmetry properties: applications of centrohermitian and perhermitian matrices

Irwin S. Pressman a,b,2

a School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, 4302 Herzberg Laboratories, Ottawa, Ont., Canada K1S 5B6
b The Fields Institute for Research in Mathematical Sciences, Toronto, Ont., Canada

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Abstract

Twelve known symmetry patterns of matrices are combined with three modest patterns to form a steiner triple system. We investigate matrices satisfying more than one symmetry pattern. We show how a group of operators on GL(n, C) gives rise to distinct types of matrices which satisfy sets of patterns, and which give unique decompositions of matrices into components of each type. These give a new characterization of normal and unitary matrices. We extend symmetry patterns to vectors to study spectral properties of these matrices. When a (skew) symmetric basis of eigenvectors exist, we can infer symmetry properties of these matrices. © 1998 Elsevier Science Inc. All rights reserved.

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2 Address for correspondence: School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, 4302 Herzberg Laboratories, Ottawa, Ont., Canada K1S 5B6. E-mail: ipress@math.carleton.ca.
1. Summary

Many problems in engineering and statistics have intrinsic symmetry that is used to derive their solution [5,7,8]. Important constructs in Physics contain higher levels of symmetry [4,9]. Various patterns of symmetry of matrices have been studied [22,20]. Centrosymmetric matrices are found in early texts and papers [1,10,17] and Collar published many results about them that were subsequently rediscovered [6]. Persymmetric matrices are defined in several ways in the literature [1]; we use the definition of Zohar [23]. Centrohermitian and skew-centrohermitian matrices were introduced in [16] and described in [3,11]. Hill et al. organized centrohermitian results and introduced (skew) perhermitian matrices [11,12]. They generalized their results to $\kappa$-hermitian matrices [13]. Centrosymmetry corresponds to symmetry about the centre of the matrix; persymmetry is symmetry across the secondary main diagonal.

We bring a set of symmetric patterns together to investigate their interactions, and to examine matrices with multiple patterns. We find that these patterns form a Steiner triple system, or, a 2-(15, 3, 1) balanced incomplete block design [15]. We study decomposition of matrices into multiply (skew) symmetric parts and derive useful results from this. We characterize normal and unitary matrices in terms of components that arise from their unique factorizations into pieces that are simultaneously of two patterns. Each component is connected to a single block of the design. A symmetric basis of the space of $4 \times 4$ matrices is given as an instance of how generalized "symmetric" matrices span spaces of dimension $2^n$. We study the spectral properties of such matrices by extending ideas of symmetry to vectors. We show that we can infer symmetry properties of the matrices in cases where there is a basic of symmetric eigenvectors. We show that matrices with two particular symmetry patterns have quadruples of orthogonal eigenvectors. We develop some connections between algebra and these symmetry patterns.

We consider groups of operations and the decomposition they induce on the set of $n \times n$ matrices. The weighted averages of the actions of this group are used to find our particular decompositions. We study two such groups here. We also examine the symmetry patterns of vectors and eigenvectors.

For conciseness, we omit straightforward proofs or provide just one proof of a set of similar ones.

2. The codiagonal symmetries of a square matrix

2.1. A classification of symmetry and skew-symmetry patterns

All matrices and vectors here have coefficients in the field $\mathbb{C}$ of complex numbers, unless otherwise indicated, where $i = \sqrt{-1}$. Let $m = m_n(\mathbb{C})$ denote
the set of $n \times n$ matrices over $\mathbb{C}$. Denote the field of real numbers by $\mathbb{R}$. Let $J_n = [\delta_{i,j+1-n}]$ denote the $n \times n$ permutation matrix, where $\delta_{i,j}$ is the Kronecker delta. We will write $J$ for $J_n$ when the context is clear. $J$ has 1’s along the secondary main diagonal or codiagonal. $J$ is symmetric and involutary and is called the coidentity matrix. We consider twelve known patterns of symmetry of matrices in Table 1.

These element wise definitions are difficult to use. The complex conjugate $\bar{A}$ of $A$ will usually be denoted by $A^c$ and the transpose of $A$ by $A^T$. These give a commuting pair of operations $(-)^c$ and $(-)^T$ on $m$. The Hermitian adjoint operation can be thought of as the composition of the two operators: $(-)^H = ((-)^c)^T$; i.e., $A^H = A^{cT} = A^{TC}$ for any matrix $A$. For $J$ the $n \times n$ coidentity matrix, we have $J = J^T = J^c = J^H = J^{-1}$.

We will often use an abbreviated notation, e.g., we say that “$A$ is $p$” instead of “$A$ is persymmetric”. The proof of the following lemma is straightforward.

**Lemma 2.1.1.** The following 12 statements are true for any matrix $A$ in $m$:

- $A$ is symmetric $\iff A = A^T$
- $A$ is skew-symmetric $\iff A = -A^T$
- $A$ is centrosymmetric $\iff A = JAJ$
- $A$ is skew-centrosymmetric $\iff A = -JAJ$
- $A$ is persymmetric $\iff A = JA^TJ$
- $A$ is skew-persymmetric $\iff A = -JA^TJ$
- $A$ is hermitian $\iff A = A^H$
- $A$ is skew-hermitian $\iff A = -A^H$
- $A$ is centrohermitian $\iff A = JACJ$
- $A$ is skew-centrohermitian $\iff A = -JACJ$
- $A$ is perhermitian $\iff A = JA^HJ$
- $A$ is skew-perhermitian $\iff A = -JA^HJ$
It is easy to confirm that when two of the three patterns \{s, c, p\} are valid for a matrix, then so is the third. For instance, if \(A\) is both \(c\) and \(s\), then \(JAJ = A = A^T\). Hence, \(A = JA^T\) so \(A\) is \(p\). We introduce the "&" relation for symmetry patterns: \(c& s \Rightarrow p\) means that properties \(c\) and \(s\) imply \(p\). Similarly \(c&p \Rightarrow s\); \(p&s \Rightarrow c\). In the same way we can show, for example, that \(c&s^{-H} \Rightarrow p^{-H}\). We add three additional matrix patterns to the 12 symmetry patterns given in Table 1: (1) \(M^R\): the matrix is real; (2) \(iM^R\): the matrix is pure imaginary; (3) \(O\): all entries are 0.

Note that the relation & in Table 2 is idempotent in that \(p& p \Rightarrow p\). We impose a unique definition on \(O & x\) for each pattern \(x\) to give a commutative & relation in Table 2. The proof of the Theorem 2.1.2 consists of checking Table 2 for each pair and triple of patterns. Every triple \{\(c, p\), \(c& p\)\} is a block of the design.

**Theorem 2.1.2.** If \(\pi\) and \(\pi'\) are two distinct patterns of symmetry, and if \(\pi & \pi' \Rightarrow \pi''\), then we also have \(\pi'' & \pi' \Rightarrow \pi'\) and \(\pi' & \pi'' \Rightarrow \pi\). Table 2 is commutative. The 15 patterns, treated as points, give a 2-(15, 3, 1) balanced incomplete block design (bibd) which is a Steiner triple system.

**Examples.** 1. The Lorentz transformation of physics is \(s^H, c\) and \(p^H\) ([9], p. 69).

2.

\[
M_1 = \begin{pmatrix}
-7 - 8i & 5 + 8i & 6 - 3i & 11 \\
7 - 2i & -4 - i & 22 & 6 + 3i \\
8 + 13i & 33 & -4 + i & 5 - 8i \\
44 & 8 - 13i & 7 + 2i & -7 + 8i
\end{pmatrix}
\]

is \(p^H\).

\[
M_2 = \begin{pmatrix}
-7 & 5 + 8i & 6 - 3i & 11 \\
5 - 8i & -4 & 22 & 6 + 3i \\
6 + 3i & 22 & -4 & 5 - 8i \\
11 & 6 - 3i & 5 + 8i & -7
\end{pmatrix}
\]

is \(s^H, c \& p^H\).

2.2. **Involutary operators and stability**

We expand the set of operators on \(m\). The matrix \(P\) will always be used to denote an involutary real, symmetric matrix; so \(P^2 = I\) and \(P = P^c = P^T = P^H = P^{-1}\) hold. Define [19] the operator \((-)^P : m \to m\)

\[(A)^P = PAP\]
The & relation for matrices satisfying two given matrix symmetry patterns

Table 2
Note that $A^{P|C} - (A^{P|})^C = PA^{C|P} - A^{C|P}$; i.e., $(-)^C$ and $(-)^{P|}$ commute for every $A$. We also have $A^{P|T} = A^{T|P}$; $A^{P|H} = A^{H|P}$; and $(A^{P|})^{P|} = A$. $(-)^{J|}$ is an instance of this operator, where $J$ is the matrix of Section 2.1.

**Definition 2.2.1.** $A \in \mathbb{F}$ is called $P$-fixed if $A^{P|} = A$. $A \in \mathbb{F}$ is called skew-$P$-fixed if $A^{P|} = -A$.

**Remark 2.2.2.** The matrices $(1 + P)A(1 + P)$ and $(1 - P)A(1 - P)$ are $P$-fixed. The matrices $(1 - P)A(1 + P)$ and $(1 + P)A(1 - P)$ are skew-$P$-fixed. For instance: $((1 - P)A(1 + P))^{P|} = P(1 - P)A(1 + P)P = (P - P^2)A(P^2 + P) = -(1 - P)A(1 + P)$ so $(1 - P)A(1 + P)$ is skew-$P$-fixed. The other results follow in a similar fashion.

**Lemma 2.2.3.** The following 12 statements are true for any matrix $A$ in $\mathbb{F}$:

- $A$ is $s$ $\iff$ $A$ is $T$-fixed
- $A$ is $c$ $\iff$ $A$ is $[J]$-fixed
- $A$ is $p$ $\iff$ $A$ is $[J]$ $T$-fixed
- $A$ is $s^H$ $\iff$ $A$ is $H$-fixed
- $A$ is $c^H$ $\iff$ $A$ is $[J]$ $C$-fixed
- $A$ is $p^H$ $\iff$ $A$ is $[J]$ $H$-fixed

Lemma 2.2.3 restates Lemma 2.1.1 using the “fixed” convention. The fixed points of each of these operations give the matrices of one of these patterns: e.g., the hermitian matrices are the fixed points of $(-)^H$; the centrohermitian matrices are fixed under $(-)^{J|}$; and the pure imaginary matrices are skew fixed under $(-)^C$. We next generalize a well-known results about symmetric matrices that will be useful here.

**Lemma 2.2.4.** Let $(-)^E$ be an involutary Operator on $\mathbb{F}$. For any matrix $A \in \mathbb{F}$, $A$ has a unique decomposition into a sum $A = A' + A''$, where $A' = \frac{1}{2}(A + AE)$ is $E$-fixed, and where $A'' = \frac{1}{2}(A - AE)$ is skew-$E$-fixed.

**Example 2.2.5.** If $E = [J]H$, $A' = \frac{1}{2}(A + A^[J]H)$ is $p^H$ and $A'' = \frac{1}{2}(A - A^[J]H)$ is $p^{-H}$.

**Proposition 2.2.6.** For any $A, B \in \mathbb{F}$ and every real symmetric involution $P$:

For all $s, t \in \mathbb{C}$

- (i) $(sA + tB)^{P|} = sA^{P|} + tB^{P|}$,
- (ii) $(AB)^{P|} = A^{P|}B^{P|}$,
- (iii) $(A^{P|})^{-1} = (A^{-1})^{P|}$,

If $A$ is nonsingular, then

If $P$ is centrosymmetric, then
The proof of the above proposition is evident. The linearity of the operator \((-)^{[P]}\) is shown in (i). The \((-)^{[P]}\) operator is an isomorphism of \(m\), which is its own inverse, as shown in (ii), where \(m\) can be thought of as a matrix algebra. The operators \((-)^{H}\), \((-)^{T}\) and each of their composites with the operator \((-)^{[P]}\) are anti-isomorphisms in the sense of ([18], p. 117); i.e., \((AB)^{H} = B^{H}A^{H}\). From [iii] we see that the isomorphism is compatible with the group property of \(GL(n,C)\). We see from (iv) that additional conditions on \(P\) are needed to ensure commutativity with \((-)^{[J]}\).

2.3. Construction of symmetric involutary matrices and groups \(\Gamma\)

We can find many groups of operators on \(m\). Using the commuting pair of involutary operators \((-)^{H}\) and \((-)^{[J]}\) as generators, we build the abelian group \(\Gamma_2\) of operators on \(m\). The order of \(\Gamma_2\) is \(|\Gamma_2| = 2^2 = 4\). Denote the identity element by \(e\).

Construct the \(2n \times 2n\) block \(K\) matrix in \(m_{2n}(C)\),

\[
K_{2n} = \begin{pmatrix}
J_n & 0 \\
0 & J_n
\end{pmatrix}
\]

and note that \(K = K^{T} = K^{C} = K^{H} = K^{-1}\) and \(KJ_{2n} = J_{2n}K\). Define the operation \((-)^{[K]}\). Since \(A^{H[K]} = A^{[K]H}\) and \(A^{[K][J]} = A^{[J][K]}\), the involutary operators \((-)^{H}\), \((-)^{[J]}\) and \((-)^{[K]}\) commute with one another. Build the abelian group \(\Gamma_3\) of operators on \(m_{2n}(C)\) using these three generators. The order of this group is \(|\Gamma_3| = 2^3 = 8\).

**Example.** For

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{pmatrix}
\]

we have

\[
A^{[K]H} = (K_4AK_4)^{H} = \begin{pmatrix}
6 & 2 & 14 & 10 \\
5 & 1 & 13 & 9 \\
8 & 4 & 16 & 12 \\
7 & 3 & 15 & 11
\end{pmatrix}.
\]

Next define \(L = \text{diag}(J_n, J_n, J_n, J_n)\) in \(m_{4n}(C)\), e.g.,

\[
L_{4n} = \begin{pmatrix}
K_{2n} & 0 \\
0 & K_{2n}
\end{pmatrix}
\]
and check that \( LJ - JL, LK - KL \) and \( L - L^T = L^C - L^H = L^{-1} \). The involutary operators \((-)^H, (-)^J, (-)^K\), and \((-)^L\) commute. We next build the abelian group \( \Gamma_4\) generated by these four operators; \( |\Gamma_4| = 2^4 = 16 \). We could introduce other new types of symmetry here which give patterns beyond those discussed in Tables 1 and 2 by using \( K \) and \( L \).

2.4. Decomposition of matrices

Let \( \chi : \Gamma \rightarrow \{+1, -1\} \) denote an abelian group homomorphism, where \( \Gamma \) is a given finite multiplicative abelian group of involutary operators on \( m \). Hence \( \chi(e) = +1 \). Let \( \Phi \) denote the character group of all such functions. Form the "average" values of the action of the operators \( \pi \in \Gamma \) on a matrix \( M \in \mathbb{m} \). For each \( \chi : \Gamma \rightarrow \{+1, -1\} \) we define the \( \chi \)-weighted average of

\[
M_\chi = \frac{1}{|\Gamma|} \sum_{\pi \in \Gamma} \chi(\pi) M^\pi.
\]

We first study the group \( \Gamma_2 \) generated by \((-)^J\) and \((-)^H\) where we abbreviate our reference to the operators by \{J, H\}. The four possibilities for \( \{\chi(J), \chi(H)\} \) are given by \{\{+1, +1\}; \{-1, +1\}; \{-1, -1\}; \{+1, -1\}\}, which we abbreviate to \( \Phi_2 = \{\{+, +\}; \{-, +\}; \{-, -\}; \{+, -\}\} \).

Denote the four types of \( \chi \)-weighted averages by the Roman numerals 1–4. Call \( A = A_I + A_{II} + A_{III} + A_{IV} \) the type decomposition of \( A \). The signs of the \( \chi \)-form coincide with standard \( X-Y \) quadrant usage. The term type refers to the Roman numeral of the particular \( \chi \)-form used (Table 3). The Lemma 2.4.1 shows that each type corresponds to a single block of the design. The types \{c, c^-, s^H, s^{-H}, p^H, p^{-H}\} correspond to the vertices and the \( A_K \) blocks to the sides of a complete quadrangle of a projective plane.

**Lemma 2.4.1.** \( A_I \) is c, \( s^H \) and \( p^H \); \( A_{II} \) is \( c^- \), \( s^H \) and \( p^{-H} \); \( A_{III} \) is \( c^- \), \( s^{-H} \) and \( p^H \); \( A_{IV} \) is c, \( s^{-H} \) and \( p^{-H} \).

**Proof.** We verify these results for \( A_{IV} \). The other proofs are similar.

(i) \( A_{IV}^{[j]} = \frac{1}{4} (A^{[j]} + A^{[j][j]} - A^{[j][H]} - A^{[j][H][j]}) = \frac{1}{4} (A^{[j]} + A - A^{[j][H]} - A^H) = A_{IV} \).

Hence any matrix of type IV is \( J \)-fixed so it is centrosymmetric (c) by Lemma 2.2.3.

**Table 3**

The definition of the four types of the decomposition of \( A \)

<table>
<thead>
<tr>
<th>Type</th>
<th>Equation</th>
<th>( \chi )-Form</th>
<th>Diagonal</th>
<th>Codiagonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( A_I = \frac{1}{4} (A + A^{[j]} + A^H + A^{[j][H]}) )</td>
<td>{+, +}</td>
<td>Real</td>
<td>Real</td>
</tr>
<tr>
<td>II</td>
<td>( A_{II} = \frac{1}{4} (A - A^{[j]} + A^H - A^{[j][H]}) )</td>
<td>{-, +}</td>
<td>Real</td>
<td>Pure imaginary</td>
</tr>
<tr>
<td>III</td>
<td>( A_{III} = \frac{1}{4} (A - A^{[H]} - A^H + A^{[j][H]}) )</td>
<td>{-, -}</td>
<td>Pure imaginary</td>
<td>Real</td>
</tr>
<tr>
<td>IV</td>
<td>( A_{IV} = \frac{1}{4} (A + A^{[H]} - A^H - A^{[j][H]}) )</td>
<td>{+, -}</td>
<td>Pure imaginary</td>
<td>Pure imaginary</td>
</tr>
</tbody>
</table>
That is, $A_{IV}$ is skew-H-fixed, so it is skew-hermitian (s$^{-H}$) by Lemma 2.2.3.
(iii) $A_{IV}$ is s$^{-H}$ & c so by Table 2 it is skew-perhermitian (p$^{-H}$) too. □

Theorem 2.4.2. Every square matrix $A$ can be uniquely decomposed into a sum $A = A_1 + A_{II} + A_{III} + A_{IV}$ where the matrices $A_j$ are of type $I \leq j \leq IV$.

Proof. We only confirm the uniqueness of the decomposition. Suppose that $A = B_1 + B_{II} + B_{III} + B_{IV}$ is a second decomposition. $(A_I + A_{II}), (B_I + B_{II})$ and their difference are hermitian; $(B_{III} + B_{IV}), (A_{III} + A_{IV})$ and their difference are skew-hermitian. By Lemma 2.2.4, it follows that $(A_I + A_{II}) - (B_I + B_{II}) = (B_{III} + B_{IV}) - (A_{III} + A_{IV}) = 0$. That is, $(A_I + A_{II}) = (B_I + B_{II})$ and $(B_{III} + B_{IV}) = (A_{III} + A_{IV})$. $B_{III}$ and $A_{III}$ are perhermitian while $B_{IV}$ and $A_{IV}$ are both skew-perhermitian. By Example 2.2.5, $A_{III} = B_{III}$ and $A_{IV} = B_{IV}$. Similarly $A_I = B_I$ and $A_{II} = B_{II}$. □

Corollary 2.4.3. If $A^{[J]} = A$ then $A_{II} = A_{III} = O$; if $A^{[J]} = -A$ then $A_I = A_{IV} = O$.

Proof. $A^{[J]} = J(A_I + A_{II} + A_{III} + A_{IV})J = A_I - A_{II} - A_{III} + A_{IV} = A_I + A_{II} + A_{III} + A_{IV} = A$. By the uniqueness of the decomposition of Theorem 2.4.2, $A_{II} = A_{III} = O$. □

Theorem 2.4.2 can be generalized for any commuting pair of involutions. Other symmetries could be chosen to give analogous, but different, decompositions of matrices into types other than those that are described here.

Examples. 1. The angular momentum operators ([4], p. 38) have Pauli matrices for eigenvalue $\mu = 1$ that are respectively of types \{I, II, III\}:

$$J_1^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$J_3^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

2. The quaternions are generated as a four-dimensional division algebra over $\mathbb{R}$ by $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$ which are respectively of type \{I, III, III, II\}. 
Proposition 2.4.4 [11,12]. For $A \in \mathbb{U}^n$
(a) $A$ is $s_H$ (respectively $s^{-H}$) \iff $JA$ is $p_H$ (respectively $p^{-H}$),
(b) $A$ is $p_H$ (respectively $s_H$) \iff $iA$ is $p^{-H}$ (respectively $s^{-H}$).

Notation. Let $S_k = \{\text{set of all } n \times n \text{ matrices of type } k, \; 1 \leq k \leq IV\}$. For $1 \leq k \leq IV$, $S_k$ is a real subspace of the real vector space $\mathbb{R}^m$ since $(-)^H$ and $(-)^H$ both preserve real linear combinations.

Lemma 2.4.5. The linear transformations induced by left multiplication by $i$, $A \rightarrow iA$, and left multiplication by $J$, $A \rightarrow JA$, \{i\}: $S_I \rightarrow S_{IV}$; \{i\}: $S_{II} \rightarrow S_{III}$; \{i\}: $S_{IV} \rightarrow S_{I}$; \{J\}: $S_I \rightarrow S_I$; \{J\}: $S_{II} \rightarrow S_{III}$; \{J\}: $S_{III} \rightarrow S_{II}$; \{J\}: $S_{IV} \rightarrow S_{IV}$ are vector space isomorphisms. The dimensions are pairwise equal: $\dim_{\mathbb{R}}(S_I) = \dim_{\mathbb{R}}(S_{IV})$ and $\dim_{\mathbb{R}}(S_{II}) = \dim_{\mathbb{R}}(S_{III})$.

Proof. We verify this for the third type. By Lemma 2.4.1 $A_{III}$ is skew-hermitian and perhermitian. If $B = iA_{III}$ then $B^{[H]} = (JB)H = i^HJ A_{III}^HJ = (-i)(A_{III}) = -B$. By Lemma 2.2.3, $B$ is skew-perhermitian. Similarly, $B^H = i^H A_{III}^H = (-i)(-A_{III}) = B$ so $B$ is hermitian. By Lemma 2.4.1, $B$ is in $S_{II}$. These are vector spaces isomorphisms since \{-i\} and \{J\} are respectively the inverse linear transformations to \{i\} and \{J\}. \qed

Corollary 2.4.6. (a) $\mathbb{M}$ decomposes into a direct sum of two disjoint vector spaces: $\mathcal{V}^c = S_I \oplus S_{IV}$ is the vector space of all centrosymmetric matrices, and $\mathcal{V}^{-c} = S_{II} \oplus S_{III}$ is the vector space of all skew-centrosymmetric matrices.
(b) $\mathcal{V}^c \oplus \mathcal{V}^{-c} = \mathbb{M}$, $\mathcal{V}^c \cap \mathcal{V}^{-c} = \{0\}$. If $n = 2k$, $\dim_{\mathbb{C}}(\mathcal{V}^c) = \dim_{\mathbb{C}}(\mathcal{V}^{-c}) = k$; if $n = 2k + 1$, $\dim_{\mathbb{C}}(\mathcal{V}^c) = k + 1$ and $\dim_{\mathbb{C}}(\mathcal{V}^{-c}) = k$. These vector spaces are orthogonal under the Frobenius inner product, where for $A = (a_{ij})$, $B = (b_{ij})$, the product is $\langle A, B \rangle = \text{Tr}(A^T B) = \sum \sum a_{ij}b_{ij}$ for $1 \leq i \leq n$, $1 \leq j \leq n$.

Proof. (a) $\mathcal{V}^c$ and $\mathcal{V}^{-c}$ are closed under multiplication by scalars in $\mathbb{C}$ by Proposition 2.2.6(i) and Lemma 2.4.5. $(A_1' + A_1^{IV}) + (A_2'' + A_2''') = (A_1' + A_2'') + (A_1^{IV} + A_2^{IV})$ so $\mathcal{V}^c$ is closed under addition. Therefore $\mathcal{V}^c$ is a complex vector space. A similar proof works for $\mathcal{V}^{-c}$.
(b) The unique decomposition of $\mathbb{M}$ into the direct sum of two vector spaces follows from Lemma 2.2.4. The orthogonality is checked by using the definitions of Table 1. \qed

2.5. A particular orthogonal basis of $\mathbb{M}_k(\mathbb{C})$

Any matrix in $\mathbb{M}_k(\mathbb{C})$ can be written as a unique linear combination over $\mathbb{C}$ of the 16 matrices $\Xi_k$ in Table 4. All the determinants $|\Xi_k|$ and $|(-1)\Xi_k|$ equal 1.
<table>
<thead>
<tr>
<th>Table 4</th>
<th>An orthogonal basis of $m_n(\mathbb{C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Matrix Diagram" /></td>
<td><img src="image" alt="Matrix Diagram" /></td>
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<tr>
<td><img src="image" alt="Matrix Diagram" /></td>
<td><img src="image" alt="Matrix Diagram" /></td>
</tr>
</tbody>
</table>
The set \( \{ \Xi_k, (-1)\Xi_k : 1 \leq k \leq 16 \} \) forms a noncommutative group of order 32. All the Lie brackets \([\Xi_1, \Xi_2] = 0, 2\Xi_k \text{ or } -2\Xi_k\); e.g., \([\Xi_7, \Xi_5] = \Xi_7\Xi_5 - \Xi_5\Xi_7 = 2\Xi_{12}\), where \( \{ \Xi_7, \Xi_5, \Xi_{12} \} \) are respectively from \( S_{III}, S_{II}, \) and \( S_I \).

Each matrix is of one of the four basic types: \( \{ \Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_{11}, \Xi_{12} \} \in S_I; \{ \Xi_5, \Xi_6, \Xi_{13}, \Xi_{14} \} \in S_{II}; \{ \Xi_7, \Xi_8, \Xi_{15}, \Xi_{16} \} \in S_{III}; \{ \Xi_9, \Xi_{10} \} \in S_{IV} \). Note that \( i\Xi_1 \in S_{III} \) and \( i\Xi_{16} \in S_I \). Under Frobenius inner product, these form an orthogonal basis of the 16 dimensional vector space of \( 4 \times 4 \) matrices over \( \mathbb{C} \). The set \( \{ \Xi_k, i\Xi_k : 1 \leq k \leq 16 \} \) give a basis of the 32 dimensional real vector space of \( 4 \times 4 \) matrices with coefficients in \( \mathbb{C} \).

Any \( A \in S_I \) has the form

\[
\begin{pmatrix}
    a & u + vi & s + ti & c \\
    u - vi & b & d & s - ti \\
    s - ti & d & b & u - vi \\
    c & s + ti & u + vi & a
\end{pmatrix}
\]

with \( \{a, b, c, d, s, t, u, v\} \subseteq \mathbb{R} \). This can be generalized for any space of dimension \( 2^n \). There would be 64 such permutation matrices for the \( 8 \times 8 \) case; e.g., there are 16 diagonal matrices generated by all ordered pairs of \( \Xi_1, \Xi_3, \Xi_{11}, \Xi_{13} \) of the form

\[
\begin{pmatrix}
    \Xi & 0 \\
    0 & \Xi
\end{pmatrix}
\]

### 2.6. \( K \)-fixed matrices

\( K_{2n} \) is the matrix defined in Section 2.3. \( A \in m_{2n}(\mathbb{C}) \) is \( K \)-fixed, if \( A = KAK \). By Lemma 2.2.4, every \( M \subseteq m_{2n}(\mathbb{C}) \) is the (unique) sum of a \( K \)-fixed matrix and a \( K \)-skew-fixed matrix. Recall that \( \Gamma_3 \) is the group of operators generated by

---

**Table 5**

The eight basic types of \([J,H,K]\)-fixed matrices

<table>
<thead>
<tr>
<th></th>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( A_1{(+,+,+)} )</td>
<td>( c &amp; s^H &amp; K )-fixed</td>
</tr>
<tr>
<td>2.</td>
<td>( A_2{(-,+,+)} )</td>
<td>( c^- &amp; s^H &amp; K )-fixed</td>
</tr>
<tr>
<td>3.</td>
<td>( A_3{(-,-,+)} )</td>
<td>( c^- &amp; s^{-H} &amp; K )-fixed</td>
</tr>
<tr>
<td>4.</td>
<td>( A_4{(+,-,+)} )</td>
<td>( c &amp; s^{-H} &amp; K )-fixed</td>
</tr>
<tr>
<td>5.</td>
<td>( A_5{(+,+,-)} )</td>
<td>( c &amp; s^H &amp; \text{skew} - K )-fixed</td>
</tr>
<tr>
<td>6.</td>
<td>( A_6{(-,+,-)} )</td>
<td>( c^- &amp; s^H &amp; \text{skew} - K )-fixed</td>
</tr>
<tr>
<td>7.</td>
<td>( A_7{(-,-,-)} )</td>
<td>( c^- &amp; s^{-H} &amp; \text{skew} - K )-fixed</td>
</tr>
<tr>
<td>8.</td>
<td>( A_8{(+,-,-)} )</td>
<td>( c &amp; s^{-H} &amp; \text{skew} - K )-fixed</td>
</tr>
</tbody>
</table>
Let $x : \Gamma_3 \rightarrow \{\pm 1\}$ be a character, and let $\Phi_3$ denote the character group. The eight possibilities for $x(J)$, $x(H)$, $x(K)$ are $\{\pm 1, \pm i, \pm j\}$ which we abbreviate to $\Phi_3 = \{\pm, \pm i, \pm j\}$. The eight components form new types that are described using the $\pm$ notation (see Table 5).

**Theorem 2.6.1.** Every square $2n \times 2n$ matrix $A$ can be uniquely decomposed into a sum $A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8$ where the square matrices $A_k$ are respectively of type $k$, $1 \leq k \leq 8$, of the eight basic types listed in Table 5.

The proof is analogous to the proof of Theorem 2.4.2. The matrices of Table 4 can be categorized into seven of the eight types given here: type 1 $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$; type 2 $\{\Xi_5, \Xi_6\}$; type 3 $\{\Xi_8, \Xi_9\}$; type 5 $\{\Xi_{11}, \Xi_{12}\}$; type 6 $\{\Xi_{13}, \Xi_{14}\}$; type 7 $\{\Xi_7, \Xi_{15}\}$; type 8 $\{\Xi_9, \Xi_{10}\}$. By the uniqueness result of Theorem 2.6.1, there cannot be any $4 \times 4$ matrices of type 4. An example of a $6 \times 6$ matrix of type 4 is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$

### 3. Products of symmetric matrices

#### 3.1. Multiplication of centrosymmetric matrices

In general, a product of hermitian matrices is not hermitian. A similar observation is true for perhermitian matrices, and for matrices that are both centrosymmetric and symmetric [6]. However, the vector space $\mathcal{V}^c$ of Corollary 2.4.6 is closed under multiplication. We introduce a notation for operations on sets of matrices. Given sets $\mathcal{X}, \mathcal{Y} \subset m$, $\mathcal{X} \mathcal{Y} = \{XY : X \in \mathcal{X}, Y \in \mathcal{Y}\}$ and $\mathcal{X} + \mathcal{Y} = \{X + Y : X \in \mathcal{X}, Y \in \mathcal{Y}\}$. To illustrate $\mathcal{X} \mathcal{Y} + \mathcal{Y} \mathcal{X}$, if $\mathcal{X} = S_{III}$ and $\mathcal{Y} = S_{IV}$, then $\mathcal{X} \mathcal{Y} + \mathcal{Y} \mathcal{X}$ denotes the set $\{AB + BA : A \in S_{III} \text{ and } B \in S_{IV}\}$.

**Proposition 3.1.1.** If $A \in S_I$ and $B \in S_I$ then $(AB + BA) \in S_I$, and $(AB - BA) \in S_{IV}$. Furthermore, $AB = \frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA)$; or $S_I S_I \subseteq \mathcal{V}^c = S_I \oplus S_{IV}$. 

Table 6
Three type multiplication tables

<table>
<thead>
<tr>
<th>A\B</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>A\B</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>A\B</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>II</td>
<td>III</td>
<td>IV</td>
<td>I</td>
<td>I</td>
<td>IV</td>
<td>III</td>
<td>II</td>
<td>I</td>
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<td>II</td>
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<td>IV</td>
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<td>IV</td>
<td>I</td>
<td>IV</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Proof. $(AB - BA)^{[J]} = (A^{[J]}B^{[J]} - B^{[J]}A^{[J]}) = (AB - BA)$; and $(AB - BA)^H = -(AB - BA)^H$. By Lemmas 2.2.3 and 2.4.1, $(AB - BA) \in S_{IV}$. The rest of the proof is straightforward.

We use the methods of Proposition 3.1.1 to form Table 6 which evaluates sums and products of all pairs of types from Section 2.4 of the form $(S_i S_j + S_i S_j), (S_i S_j + S_i S_j)$, and $(S_i S_j)$. The resulting type is recorded. The verification of the table is not hard. The third table has a + or − according to whether the set of products $A_j B_k \subset V^c$ or $V^c$.

Using Table 6, we see that $V^c$ is closed under multiplication. It follows that $V^c$ is a subalgebra of $m$. Related results were observed in [10,11,16,19]. There is a $Z_2$-graded algebra structure on $m$ since $V^c \subset V^c$; $V^c V^c \subset V^c$; $V^c V^c \subset V^c$. The nonsingular matrices in $V^c$ form a subgroup of the general linear group $GL(n, C)$ which is the centralizer $Z(J)$ of $J$.

Proposition 3.1.2. The centralizer $Z(J)$ of $J$ in $GL(n, C)$ is $GL(n, C) \cap V^c$.

Proof. By Lemma 2.2.3, every matrix in $V^c$ commutes with $J$. By Theorem 2.4.2, every matrix $A$ in the centralizer $Z(J)$ has the unique decomposition $A = A_1 + A_2 + A_3 + A_4$. By Lemma 2.1.1, $A \in Z(J) \iff JA = AJ \iff JAJ = A \iff A_1 = A_2 = O$.

Corollary 3.1.3. If $A \in V^c$ is nonsingular, then $A^{-1} \in V^c$.

Proof. Since $Z(J)$ is a group, $A$ is invertible and $A^{-1} \in Z(J) \subset V^c$.

The above result is noted in [8,10] but our proof is briefer.

3.2. Applications of unique decomposition

Example 1. A matrix $A = A_1 + A_2 + A_3 + A_4$ is normal if $AA^H = A^HA$. By multiplication and elimination of like terms, we have the equation:
\[(A_I A_{IV} - A_{IV} A_I) + (A_{III} A_{II} - A_{II} A_{III}) + (A_I A_{III} - A_{III} A_I)\]
\[+ (A_{II} A_{IV} - A_{IV} A_{II}) = 0.\]

From Table 6, these terms are of types \{I, I, II, II\}. By the unique decomposition of Theorem 2.4.2, the type I and type II components are themselves 0, or with rearrangements:
\[(A_I A_{IV} - A_{IV} A_I) = (A_{III} A_{II} - A_{II} A_{III}) \quad \text{and} \quad (A_{III} A_I - A_I A_{III}) = (A_{II} A_{IV} - A_{IV} A_{II}).\]

Using Lie brackets, the components of a normal matrix can be related. This extends a characterization of normal matrices of Horn ([14], p. 109 # 8).

A is normal iff \([A_I, A_{IV}] = [A_{III}, A_{II}]\) and \([A_I, A_{III}] = [A_{IV}, A_{II}]\).

**Example 2.** \(U\) is unitary if \(U^U U^H = U = U^H U\). Write \(U = U_I + U_{II} + U_{III} + U_{IV}\). Then, \(U^H = U_I + U_{II} - U_{III} - U_{IV}\). Expanding \(U^U U^H\) gives 16 product terms:
\[1 = U_I U_I + U_{II} U_I + U_{III} U_I + U_{IV} U_I + U_I U_{II} + U_{II} U_{II} + U_{III} U_{II} - U_{IV} U_{II} + U_{II} U_{IV} - U_{III} U_{IV} - U_{IV} U_{IV} + U_{II} U_{III} - U_{III} U_{III} - U_{IV} U_{III} - U_{III} U_{IV} - U_{IV} U_{IV} + U_{III} U_{IV}
\]

Since a unitary matrix is normal, we use the above results to cancel terms. Thus
\[1 = U_I U_I + U_{II} U_I + U_{III} U_I + U_{IV} U_I + U_I U_{II} + U_{II} U_{II} + U_{III} U_{II} - U_{IV} U_{II} + U_{II} U_{IV} - U_{III} U_{IV} - U_{IV} U_{IV} + U_{II} U_{III} - U_{III} U_{III} - U_{IV} U_{III} - U_{III} U_{IV} - U_{IV} U_{IV} + U_{III} U_{IV}
\]
or, \(1 = (U_I U_I + U_{II} U_I - U_{III} U_{III} - U_{IV} U_{IV} + U_{IV} U_{III} - U_{III} U_{IV})\) and \(1 = U_I U_I + U_{II} U_{II} - U_{III} U_{III} - U_{IV} U_{IV} = U_I^2 + U_{II}^2 - U_{III}^2 - U_{IV}^2\).

The identity matrix is of type I. Reorganize the right-hand side as the sum of two matrices of types I and II respectively. By Theorem 2.4.2, it follows that for any matrix \(U\), \(U\) is unitary iff \(U\) is normal, \((U_I U_{II} + U_{II} U_I) = (U_{III} U_{IV} + U_{IV} U_{III})\) and \(1 = U_I U_I + U_{II} U_{II} - U_{III} U_{III} - U_{IV} U_{IV} = U_I^2 + U_{II}^2 - U_{III}^2 - U_{IV}^2\).

### 4. Properties of eigenvectors

#### 4.1. Symmetric and skew-symmetric vectors

All vectors will be in \(C^n\) and matrices \(A\) in \(M\). The hermitian operator extends to vectors as \(V^H = V^C\), where we write \(V^C\) as the complex conjugate of the vector \(V\) in order to emphasize the operator aspect of complex conjugation. Two vectors \(V, W\) are orthogonal if \(V^H W = 0\). In this section, \(P\) and \(Q\) denote two given symmetric involutary permutation matrices in \(M\). \(V\) is called \(P\)-symmetric if \(PV = V\); e.g., if \(P = J\), \(JV\) has coefficients in reverse order to \(V\).

**Definition.** A vector \(V\) is defined to be symmetric if \(V = JV\); \(cc\)-symmetric if \(V = JV^C\); skew-symmetric if \(V = -JV\); skew-cc-symmetric if \(V = -JV^C\).

Lemma 4.1.2. Let \( V \) be an eigenvector of \( A \) with eigenvalue \( \lambda \).

(a) If \( A \) is \( P \)-fixed (respectively skew-\( P \)-fixed) then \( PV \) is an eigenvector of \( A \) with eigenvalue \( \lambda \) (respectively \( -\lambda \)).

(b) If \( A \) is \( P \)-fixed and \( Q \)-fixed (or skew-\( P \)-fixed and skew-\( Q \)-fixed), then \( PQV \) is an eigenvector of \( A \) with eigenvalue \( \lambda \). If \( A \) is skew-\( P \)-fixed and \( Q \)-fixed (or \( P \)-fixed and skew-\( Q \)-fixed), then \( PQV \) is an eigenvector of \( A \) with eigenvalue \( -\lambda \).

(c) If \( A \) is \( PH \)-fixed (respectively skew-\( PH \)-fixed) and if \( V \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), then \( PV^C \) is an eigenvector of \( A^\dagger \) with eigenvalue \( \bar{\lambda} \) (respectively \( -\bar{\lambda} \)). All eigenvalues with nonzero imaginary part appear in conjugate pairs.

Proof. (a) If \( AV = \lambda V \), then, since \( PAP = A \), we have \( PAPV = \lambda V \) or \( APV = \lambda PV \).

(b) If \( QAQ = A \), then by (a) we have, \( QAQPV = \lambda PV \), or, \( AQPV = \lambda QPV \).

(c) Since \( PA^H P = A \), we have \( A^H P V = PAV = \lambda PV \) or \( A^T PV^C = \bar{\lambda} PV^C \). But, since \( A^T \) and \( A \) have the same eigenvalues, \( \bar{\lambda} \) is also an eigenvalue of \( A \). That is, if the imaginary part of \( \lambda \neq 0 \), then both \( \lambda \) and \( \bar{\lambda} \) are eigenvalues of \( A \). □

Andrew [2] shows that every eigenspace of a centrosymmetric matrix has a basis consisting of vectors that are either symmetric or skew-symmetric. He shows further that an \( n \times n \) matrix with \( n \) such linearly independent eigenvectors must be centrosymmetric. We extend these results here.

Theorem 4.1.3. If \( A \) has real eigenvalues and there is a basis of \( n \) eigenvectors of \( A \) which are either cc-symmetric or skew-cc-symmetric, then \( A \) is centrohermitian. If \( A \) is normal too, then \( A \) is persymmetric.

Proof. Suppose that \((V, \lambda)\) is a skew-cc-symmetric eigenvector eigenvalue pair for \( A \), i.e., \( V = -JV^C \). Since \( \lambda \) is real, \(-AJV^C = -\lambda JV^C \); or \( JAJV^C = \lambda JV^C \). Take the complex conjugate of the last equation to get: \( JA^C JV = \lambda V \). A similar result holds for cc-symmetric eigenvectors. Thus \( JA^C JW = WA \), for \( W \) the nonsingular matrix of \( n \) eigenvectors. We also have \( AW = WA \), so \( JA^C J = A \). By Lemma 2.1.1 \( A \) is cc-

By [14] (Theorem 4.1.4, p. 171), if \( A \) is normal and its eigenvalues are all real, then \( A \) is hermitian. It follows from Table 2 that \( A \) is persymmetric. □

Proposition 4.1.4. If \( A \) is nonsingular and \( P \)-fixed (or skew-\( P \)-fixed), then so is \( A^{-1} \).
Proof. $A^{[P]} = A$, then $A^{-1} = (A^{[P]})^{-1} = (PAP)^{-1} = PA^{-1}P = (A^{-1})^{[P]}$. □

Proposition 4.1.5. Let $B$ be a symmetric vector (respectively skew-symmetric). If $A$ is nonsingular and $J$-fixed, then the solution $X$ of $AX = B$ is symmetric too (respectively skew-symmetric).

Proof. $JA^{-1} = A^{-1}J$ by Proposition 4.1.4. Thus $JX = JA^{-1}B = A^{-1}JB = A^{-1}B = X$. □

We define a vector $B$ of length $2n$ to be $\frac{1}{2}$-symmetric if $KB = B$, where $K$ is the matrix defined in Section 2.3. The two sub-vectors of the first (and last) $n$ coefficients are both symmetric vectors. A vector $B$ is called doubly symmetric if $JB = B = KB$.

Example. $\{[1, 2, 1, 1, 2, 1, 1, 2, 1]^{T}; [7, 3, -3, -7, -3, 3, 7]^{T}; [8, 4, 8, -8, -4, -8]^{T}; [9, 5, -5, -9, 9, -5, -9]^{T}\}$ are doubly symmetric; symmetric and skew-$\frac{1}{2}$-symmetric; skew-symmetric and $\frac{1}{2}$-symmetric; and skew-symmetric and skew-$\frac{1}{2}$-symmetric vectors respectively.

Corollary 4.1.6. If $B$ is a doubly symmetric vector of length $2n$, and $A$ a $2n \times 2n$ nonsingular $J$-fixed and $K$-fixed matrix, then $A^{-1}B$ is a doubly symmetric vector.

Define the four vectors $V_i$, $1 \leq i \leq 4$, that correspond to the first four classes of Table 7:

$V_1 = \frac{1}{4}(V + JV + KV + KJV) = \frac{1}{4}(I + J)(I + K)V,$
$V_2 = \frac{1}{4}(V + JV - KV - KJV) = \frac{1}{4}(I + J)(I - K)V,$
$V_3 = \frac{1}{4}(V - JV + KV - KJV) = \frac{1}{4}(I - J)(I + K)V,$
$V_4 = \frac{1}{4}(V - JV - KV + KJV) = \frac{1}{4}(I - J)(I - K)V.$

These sum to $V$ and are orthogonal by Lemma 4.1.1. The following theorem shows that matrices with two symmetric patterns may have eigenvectors in an eigenspace occurring in orthogonal sets of four.

Table 7

<table>
<thead>
<tr>
<th>Symmetry classification of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $V = JV$ and $KV = V$, $V$ is symmetric and $\frac{1}{2}$-symmetric (doubly symmetric)</td>
</tr>
<tr>
<td>(ii) $V = JV$ and $KV = -V$, $V$ is symmetric and skew-$\frac{1}{2}$-symmetric</td>
</tr>
<tr>
<td>(iii) $V = -JV$ and $KV = V$, $V$ is skew-symmetric and $\frac{1}{2}$-symmetric</td>
</tr>
<tr>
<td>(iv) $V = -JV$ and $KV = -V$, $V$ is skew-symmetric and skew-$\frac{1}{2}$-symmetric</td>
</tr>
<tr>
<td>(v) $V = JV^C$ and $KV = V$, $V$ is cc-symmetric and $\frac{1}{2}$-symmetric</td>
</tr>
<tr>
<td>(vi) $V = JV^C$ and $KV = -V$, $V$ is cc-symmetric and skew-$\frac{1}{2}$-symmetric</td>
</tr>
<tr>
<td>(vii) $V = -JV^C$ and $KV = V$, $V$ is skew-cc-symmetric and $\frac{1}{2}$-symmetric</td>
</tr>
<tr>
<td>(viii) $V = -JV^C$ and $KV = -V$, $V$ is skew-cc-symmetric and skew-$\frac{1}{2}$-symmetric</td>
</tr>
</tbody>
</table>
Theorem 4.1.7. If \( V \) is an eigenvector of a \( J \)-fixed (centrosymmetric) and \( K \)-fixed matrix \( A \), with eigenvalue \( \lambda \neq 0 \), then \( V \) can be decomposed into a linear combination of mutually orthogonal vectors of the first four classes of Table 1. This decomposition is unique and \( V_1, V_2, V_3 \) and \( V_4 \) have eigenvalue \( \lambda \).

Proof. To confirm the uniqueness of the decomposition, let \( V = W_1 + W_2 + W_3 + W_4 \) be a corresponding second decomposition. \((V_1 + V_2), (W_1 + W_2)\) and their difference are symmetric vectors; \((V_3 + V_4), (V_3 + V_4)\) and their difference are skew-symmetric. Hence \((V_1 + V_2) - (W_1 + W_2) = (W_3 + W_4) - (V_3 + V_4) = 0\), i.e., \((V_1 + V_2) = (W_1 + W_2)\) and \((W_3 + W_4) = (V_3 + V_4)\).

\( W_3 \) and \( V_3 \) are skew-symmetric and \( \frac{1}{2} \) symmetric while \( W_4 \) and \( V_4 \) are both skew-symmetric and skew-\( \frac{1}{2} \)-symmetric. Therefore, \((V_3 - W_3) = -(V_4 - W_4)\) is both a skew-\( \frac{1}{2} \)-symmetric and skew-\( \frac{1}{2} \)-symmetric vector, so it must be \( 0 \) too. Therefore, \( V_3 = W_3 \) and \( V_4 = W_4 \). Similarly, \( V_1 = W_1 \) and \( V_2 = W_2 \). Therefore the decomposition is unique.

If \( AV = \lambda V \), then by Lemma 4.1.2 \( JV, KV, \) and \( JKV \) are eigenvectors of \( A \) with eigenvalue \( \lambda \) too. Hence \( AV_1 = A \frac{1}{4} (I + J)(I + K)V = \frac{1}{4} (AV + AJV + AKV + AKJV)V = \lambda V_1 \) and \( AV_2 = A \frac{1}{4} (I + J)(I - K)V = \frac{1}{4} (AV + AJV - AKV - AKJV)V = \lambda V_2 \). Similarly, \( V_3 \) and \( V_4 \) have eigenvalue \( \lambda \). \( \square \)

Weaver proved that \((\lambda, V)\) is the eigenvalue eigenvector pair of a \( c \) matrix if and only if \((\lambda, JV)\) is a pair also [20,21]. The eigenvalues of a \( c^H \) matrix come in conjugate pairs [16]. If \( A \) is \( c^H \), then \((\lambda, V)\) is an eigenvalue eigenvector pair if and only if \((\lambda, JV^C)\) is an eigenvalue eigenvector pair [11]. We extend this to skew-centrosymmetric matrices.

Theorem 4.1.8. Let \((\lambda, V)\) be an eigenvalue eigenvector pair of \( A \in V^c \).

(a) If \( JV = \pm V \), i.e., \( V \) is symmetric or skew-symmetric, then \( \lambda = 0 \).

(b) If \( JV \neq \pm V \), then \(-\lambda \) is an eigenvalue of \( A \) with eigenvector \( JV \).

(c) The characteristic polynomial \( p(x) \) of \( A \) has the form \( p(x) = \prod (x^2 - a_k^2) \) for \( n \) even, and \( p(x) = x^{\frac{n+1}{2}} \prod (x^2 - a_k^2) \) for \( n \) odd and \( s \) an integer, where the eigenvalues \( a_k \) are all real if \( A \in S_{II} \), and all pure imaginary if \( A \in S_{III} \). If \( n \) is odd, then \( A \) is singular.

Proof. (a) By Lemma 2.1.1, \( AJ = -JA \) since \( A \in V^c \). Hence, \( AJV = -JAV = -JAV - (-\lambda)JV \). If \( JV = V \), then \( \lambda V = -V \), so \( \lambda = 0 \). Similarly, if \( JV = -V \), then \( -\lambda V = V \), so \( \lambda = 0 \).

(b) If \( JV \neq \pm V \), then by (a) \((\lambda, JV)\) is an eigenvalue eigenvector pair.

(c) If \( A \) is of type \( II \), then it is hermitian and each eigenvalue \( \lambda = a_k \) is real. If \( A \) is of type \( III \), then it is skew-hermitian and each eigenvalue \( \lambda = a_k \) is pure imaginary. The characteristic polynomial has an \((x - \lambda)(x + \lambda) = x^2 - \lambda^2 \) fac-
tor. If $n$ is odd, then there is at least one unpaired eigenvalue which must be zero, so $A$ is singular. □

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