Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of
MATHEMATICAL
www.elsevier.com/locate/jmaa

Dynamics of composite functions meromorphic outside a small set

Keaitsuda Maneeruk, Piyapong Niamsup *

Department of Mathematics, Faculty of Science, Chiangmai University, Chiangmai, 50200, Thailand

Received 15 November 2004

Available online 29 January 2005

Submitted by E.J. Straube

Abstract

Let **M** denote the class of functions f meromorphic outside some compact totally disconnected set $E = E(f)$ and the cluster set of f at any $a \in E$ with respect to $E^c = \hat{\mathbb{C}} \setminus E$ is equal to $\hat{\mathbb{C}}$. It is known that class **M** is closed under composition. Let f and g be two functions in class **M**, we study relationship between dynamics of $f \circ g$ and $g \circ f$. Denote by $F(f)$ and $J(f)$ the Fatou and Julia sets of f. Let U be a component of $F(f \circ g)$ and V be a component of $F(g \circ f)$ which contains $g(U)$. We show that under certain conditions U is a wandering domain if and only if V is a wandering domain; if U is periodic, then so is V and moreover, V is of the same type according to the classification of periodic components as U unless U is a Siegel disk or Herman ring.

Keywords: Functions meromorphic outside a small sets; Wandering domain

1. Introduction

Let E be any compact totally disconnected set in $\hat{\mathbb{C}}$, if $z_0 \in E$ and f is a function meromorphic in $E^c = \hat{\mathbb{C}} \setminus E$, then the cluster set $\mathbb{C}(f, E^c, z_0)$ is defined as $\{w\colon\ w = \lim_{n \to +\infty} f(z_n)$ for some $z_n \in E^c$ with $z_n \to z_0\}$. We introduce the class $\mathbf{M} = \{f\colon$ there

* Corresponding author.
E-mail address: scipnmsp@chiangmai.ac.th (P. Niamsup).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.12.047

is a compact totally disconnected set $E = E(f)$ such that f is meromorphic in E^c and $\mathbb{C}(f, E^c, z_0) = \hat{\mathbb{C}}$ for all $z_0 \in E$. If $E = \emptyset$ we make the further assumption that f is neither constant nor univalent in $\hat{\mathbb{C}}$}. The class **M** was first investigated in [1] and [2] where the basic concepts such as Fatou and Julia sets and the basic properties of dynamics of functions in **M** were established. It was proved in [1] that the class **M** is closed under composition and if $f, g \in \mathbf{M}$, then $E(f \circ g) = E(g) \cup g^{-1}(E(f))$. For $f \in \mathbf{M}$, we define f^0 to be the identity function with $E_0 = \emptyset$, and, inductively, $f^1 = f$, $f^n = f \circ f^{n-1}$. We obtain $f^n \in \mathbf{M}$, $n \in \mathbb{N}$, with $E_n = E(f^n) = \bigcup_{j=0}^{n-1} f^{-j}(E) = \{\text{singularities of } f^{-n}\}$. If we set $J_1(f) = \overline{\bigcup_{n=0}^{+\infty} E_n}$ and $F_1(f) = \hat{\mathbb{C}} \backslash J_1(f)$, then $F_1(f)$ is the largest open set in which all f^n are defined and $f(F_1(f)) \subset F_1(f)$. As in [1], for $f \in \mathbf{M}$, we define the Fatou set of f, denoted by $F(f)$, to be the largest open set in which (i) all iterates f^n are meromorphic and (ii) the family $\{f^n\}$ is a normal family; and the Julia set of f, denoted by $J(f)$, is defined to be the complement of $F(f)$. If the set $J_1(f)$ is either empty or contains one point or two points, then f is conjugate to a rational map or entire function or an analytic map of the punctured plane $\mathbb{C}^*$, respectively. In these cases the condition (i) is trivial and the Fatou sets are determined by (ii). In all other cases, by Montel's theorem, we have $F(f) = F_1(f)$ and $J(f) = J_1(f)$. It is easy to see that for $f \in \mathbf{M}$, $F(f)$ is open and completely invariant. Let U be a connected component of $F(f)$, then $f^n(U)$ is contained in a component U_n of $F(f)$. If for some $n \in \mathbb{N}$, $U_n = U$, namely $f^n(U) \subset U$, then U is said to be periodic. If for some pair of $m \neq n$, $U_m = U_n$, but U is not periodic, then U is said to be preperiodic. If whenever $m \neq n$, $U_m \neq U_n$, then U is called a wandering domain of f. For a periodic component of $F(f)$ we have the following classification theorem [1]:

Theorem 1.1. *Let U be a periodic component of the Fatou set of period p. Then precisely one of the following is true*:

- (i) U is a (super)attracting domain of a (super)attracting periodic point a of f of period p such that $f^{np}|_U \to a$ as $n \to +\infty$ and $a \in U$.
- (ii) U is a parabolic domain of a rational neutral periodic point b of f of period p such that $f^{np}|_U \to b$ as $n \to +\infty$ and $b \in \partial U$.
- (iii) U is a Siegel disk of period p such that there exists an analytic homeomorphism $\phi : U \to \Delta$, where $\Delta = \{z\colon\ |z| < 1\}$, satisfying $\phi(f^p(\phi^{-1}(z))) = e^{2\pi\alpha i} z$ for some irrational number α and $\phi^{-1}(0) \in U$ is an irrational neutral periodic point of f of period p.
- (iv) U is a Herman ring of period p such that there exists an analytic homeomorphism $\phi : U \to A$, where $A = \{z\colon\ 1 < |z| < r\}$, satisfying $\phi(f^p(\phi^{-1}(z))) = e^{2\pi\alpha i} z$ for some irrational number α.
- (v) U is a Baker domain of period p such that $f^{np}|_U \to c \in J(f)$ as $n \to +\infty$ but f^p is not meromorphic at c. If $p = 1$, then $c \in E(f)$.

There are several subclasses of the class **M** which are introduced in [1] including those studied by Bolsch in [7,8]. To suit our purpose, we introduce some subclasses and their dynamical properties as follows.

Definition 1.1. Let $f \in \mathbf{M}$. Then

(i) f is in class **K** if there is a compact countable set $E(f) \subset \hat{\mathbb{C}}$ such that f is meromorphic in $\hat{\mathbb{C}} \setminus E(f)$ but in no larger set.
(ii) f is in class $\mathbf{MP}_k$, where k is an integer not less than two, if $E(f) \neq \emptyset$ and for each $z_0 \in E(f)$ and open set U which contains z_0, f takes in $U \setminus E(f)$ every value in $\hat{\mathbb{C}}$ with at most k exceptions.
(iii) f is in class $\mathbf{MA}_k$, where $k \in \mathbb{N}$, if $E(f) \neq \emptyset$ and for each $z_0 \in E(f)$ the function f has the k islands property at z_0, namely for any neighborhood U of z_0 and k simply-connected domains Δi in $\hat{\mathbb{C}}$ which have disjoint closures and which are bounded by sectionally analytic Jordan curves, there is a simply-connected subdomain D in $U \setminus E(f)$ such that f maps D univalently onto one of the Δ_i.
(ii) f is in class **MS** if the set of singular values of f^{-1} is finite.
(iii) f is in class **MSR** if $f \in \mathbf{MS}$ and the complement of $E(f)$ is of class O_{AD} (If W is a domain in the plane and F is a function analytic in W, the Dirichlet integral of F is defined by $D_W(F) = \iint_W |F'(z)|^2 \, dx \, dy$. An analytic function with finite Dirichlet integral is said to be of the class AD. The domain W is said to be of class O_{AD} if the only functions of class AD on W are constants).

The followings results were established in [1]:

Theorem 1.2. *Let $f \in \mathbf{M}$. Then the following statements are true*:

(i) $\mathbf{MA}_k \subset \mathbf{MP}_{k-1}$, $\mathbf{K} \subset \mathbf{MP}_2 \cap \mathbf{MA}_5$, $\mathbf{K} \cap \mathbf{MS} \subset \mathbf{MSR}$.
(ii) *The subclasses* **K**, $\mathbf{MP}_k$, $\mathbf{MA}_k$, *and* **MS** *are closed under composition.*
(iii) *If $f \in \mathbf{MA}_k$ for some $k \geqslant 5$, then the repelling periodic points are dense in $J(f)$.*
(iv) *If $E(f)$ has the local Picard property, namely there exist no open set V with $E \cap V \neq \emptyset$ and no function f meromorphic in $V \setminus E(f)$ with an essential singularity at each point of $E \cap V$ such that f omits three values in $V \setminus E(f)$, then every point of $J(f)$ is a limit point of periodic points of f.*
(v) *If $f \in$* **MS**, *then f has no Baker domains.*
(vi) *If $f \in$* **MSR**, *then f has no wandering domains.*

The following result was given in [2].

Theorem 1.3. *Suppose that $f \in$* **MS**. *If $E(f)$ has an isolated point, then f has at most two completely invariant domains.*

In [3], Baker and Singh studied the dynamics of composite entire functions and showed that if p is a nonconstant entire function and $g(z) = a + be^{2\pi i z/c}$, where a, b and c are nonzero constants and $g \circ p$ has no wandering domains, then neither does $p \circ g$. In [6], Bergweiler and Wang studied the dynamics of composite entire functions without assuming any special forms of functions. The following are results obtained in [6]:

Theorem 1.4. *Let f and g be nonlinear entire functions and $z \in \mathbb{C}$. Then $z \in J(f \circ g)$ if and only if $g(z) \in J(g \circ f)$.*

Theorem 1.5. *Let f and g be nonlinear entire functions. Let U_0 be a component of $F(f \circ g)$ and let V_0 be the component of $F(g \circ f)$ that contains $g(U_0)$. Then*

(i) *U_0 is wandering if and only if V_0 is wandering.*
(ii) *If U_0 is periodic, then so is V_0. Moreover, V_0 is of the same type according to the classification of periodic components as U_0.*

In particular, $f \circ g$ has a wandering domain if and only if $g \circ f$ has a wandering domain.

Several examples of entire functions which have no wandering domains were then constructed by using Theorem 1.5 including an example given earlier in [3]. In [5], Bergweiler and Hinkkanen generalized these results by considering dynamical connection of transcendental entire functions f and h satisfying $g \circ f = h \circ g$, where g is a continuous and open map of the complex plane into itself. Recently, Zheng [9] studied the connections between the Fatou components and the singularities of the inverse function of functions in class **M** and the dynamical connection between f and g in class **M** satisfying the equation $h \circ f = g \circ h$ where h is meromorphic in $\mathbb{C}$. Several examples of Baker domains and wandering domains of transcendental meromorphic functions which have special properties were also given in [9]. In this paper, we extend Theorems 1.4 and 1.5 to functions meromorphic outside a small set which have certain properties such as those in subclasses of class **M** defined above. By using these results, we will give examples of transcendental meromorphic functions and functions in class **M** which do not have wandering domains or Baker domains.

2. Preliminaries

In this section, we give several lemmas which will be used in the proof of our main results. Throughout this paper, we denote $f \circ g$ by fg and $E(f)$ by E_f.

Lemma 2.1. *Let $f, g \in$ **M**. If z_0 is a periodic point of fg, then $g(z_0)$ is a periodic point of gf.*

Proof. Let z_0 be a periodic point of period n of fg, namely $(fg)^n(z_0) = z_0$. Then

$$z_0 \notin E\big((fg)^n\big) = \left(\bigcup_{j=0}^{n-1}\big((fg)^j\big)^{-1}(E_g)\right) \cup \left(\bigcup_{j=0}^{n-1}\big((gf)^j g\big)^{-1}(E_f)\right).$$

Thus, $g(fg)^n(z_0)$ is defined and equal to $g(z_0)$. Since $g(fg)^n(z_0) = (gf)^n(g(z_0))$, it follows that $g(z_0)$ is a periodic point of gf. This completes the proof. □

Recall that the singularities of the inverse function of function f in class **M**, denoted by $\operatorname{sing}(f^{-1})$, is the union of the set of critical values of f, denoted by $CV(f)$, and the set of asymptotic values of f, denoted by $AV(f)$ together with all limit points of $CV(f) \cup AV(f)$. We denote the set of limit points of a set E by E'.

Lemma 2.2. *Let $f, g \in \mathbf{M}$. Assume the following conditions hold*:

(i) $\infty \in E_f \cap E_g$.
(ii) *If for some $z_0 \in E_{fg}$ and for some path $\gamma(t)$, $0 \leqslant t < 1$, we have $\gamma \cap E_{fg} = \emptyset$ and $\gamma \to z_0$ as $t \to 1$, then $g(\gamma)' \cap (E_f \setminus \{\infty\}) = \emptyset$.*

Then we have

$$CV(fg) \subset CV(f) \cup f\big(CV(g)\big), \qquad AV(fg) \subset AV(f) \cup f\big(AV(g)\big),$$

and $\operatorname{sing}(fg)^{-1} \subset \operatorname{sing}(f)^{-1} \cup f(\operatorname{sing}(g)^{-1})$.

Proof. Let α be a critical value of fg. Then there exists z_0 such that $(fg)'(z_0) = f'(g(z_0))g'(z_0) = 0$ and $(fg)(z_0) = \alpha$. Thus, $z_0 \notin E_g \cup g^{-1}(E_f)$. If $f'(g(z_0)) = 0$, then $g(z_0)$ is a critical point for f and we have $(fg)(z_0) \in CV(f)$. If $g'(z_0) = 0$, then z_0 is a critical point of g and so $g(z_0) \in CV(g)$. Thus, $(fg)(z_0) \in f(CV(g))$. Therefore, $CV(fg) \subset CV(f) \cup f(CV(g))$. Now let α be an asymptotic value of fg. Then there exists $z_0 \in E_{fg}$ and a path $\gamma(t)$, $0 \leqslant t < 1$ such that $\gamma \cap E_{fg} = \emptyset$ and $\gamma \to z_0$ as $t \to 1$ and $(fg)(z) \to \alpha$ along γ.

Case 1. z_0 is finite.

Subcase 1.1: $g(z) \to z_0$ along γ.

In this subcase, α is an asymptotic value of f.

Subcase 1.2: $g(z) \nrightarrow z_0$ along γ and $g(z)$ is eventually bounded along γ (namely, there exists $\delta > 0$ such that $|g(z)|$ is bounded on $\{z \in \gamma\colon |z - z_0| < \delta\}$).

In this subcase, there exists a sequence $\{z_n\}$ on γ and a finite point w_0 such that $\lim_{n\to+\infty} z_n = z_0$ and $\lim_{n\to+\infty} g(z_n) = w_0$. By (ii), $w_0 \notin E_f$ and it follows that $f(w_0) = \lim_{n\to+\infty} f(g(z_n)) = \alpha$. By (ii) and the fact that poles of f cannot accumulate at a finite point outside E_f, we can find a neighborhood U_{w_0} of w_0 such that $U_{w_0} \cap (E_f \cup P_f) = \emptyset$, where P_f is the set of poles of f (if there exists a sequence w_n of points in E_f such that $\lim_{n\to+\infty} w_n = w_0$, then $w_0 \in \overline{E_f} = E_f$. This is impossible by (ii)). Thus, f is analytic in U_{w_0}. Let $\rho > 0$ be a fixed sufficiently small positive real number. Then for some $\varepsilon > 0$, we have $|f(w) - \alpha| > \varepsilon$ for $w \in \{w\colon |w - w_0| = \rho\}$. Next, as α is an asymptotic value of $f \circ g$, $|f(g(z)) - \alpha| < \varepsilon$ for all $z \in \{z\colon |z - z_0| < \delta\}$ on γ, for some $\delta > 0$. In particular, if $|z_n - z_0|$ are sufficiently small, then $|f(g(z)) - \alpha| < \varepsilon$ for all z such that $|z - z_0| < |z_n - z_0|$ and $|g(z_n) - w_0| < \rho$. Thus, $|g(z) - w_0| < \rho$ for all z which is arbitrarily closed to z_0 and hence w_0 is an asymptotic value of g. This gives $\alpha \in f(AV(g))$.

Subcase 1.3: $g(z)$ is not eventually bounded along γ.

In this subcase, there exists a sequence $\{z_n\}$ on γ such that $\lim_{n\to+\infty} z_n = z_0$ and $\lim_{n\to+\infty} g(z_n) = \infty$. If there are infinitely many points z_{n_k} of the sequence z_n such that

$g(z_{n_k}) = \infty$, then we modify the path γ slightly so as to avoid the poles of g while preserving all other conditions. Thus, eventually along $\{z_n\}$, g is defined and unbounded; namely, there exists a sequence $\{\alpha_n\}$ on γ such that $\lim_{n\to+\infty} \alpha_n = z_0$, $g(\alpha_n) \neq \infty$ and $\lim_{n\to+\infty} g(\alpha_n) = \infty$. If $g(z) \to \infty$, along γ, then $\alpha \in AV(f)$ since $\infty \in E_f$. Otherwise, there is a sequence β_n on γ such that $\lim_{n\to+\infty} g(\beta_n) = w_0$ for some finite w_0. By (ii), $w_0 \notin E_f$ and it follows that $f(w_0) = \lim_{n\to+\infty} f(g(\beta_n)) = \alpha$. By the same argument as in Subcase 1.2, we can find a neighborhood U_{w_0} of w_0 such that f is analytic in U_{w_0}. Let $\rho > 0$ be a fixed sufficiently small positive real number. Then for some $\varepsilon > 0$, we have $|f(w) - \alpha| > \varepsilon$ for $w \in \{w\colon |w - w_0| = \rho\}$. Next, as α is an asymptotic value of $f \circ g$, $|f(g(z)) - \alpha| < \varepsilon$ for all $z \in \{z\colon |z - z_0| < \delta\}$ on γ, for some constant δ. In particular, if β_n are sufficiently close to z_0 on γ, then $|f(g(z)) - \alpha| < \varepsilon$ for all z beyond β_n on γ and $|g(\beta_n) - w_0| < \rho$. Thus, $|g(z) - w_0| < \rho$ for all z sufficiently close to z_0 on γ. Thus g must be bounded on γ which contradicts to the assumption that $g(z)$ is not eventually bounded along γ. Therefore, this subcase cannot occur at all.

Case 2. $z_0 = \infty$.

Subcase 2.1: $g(z) \to \infty$ along γ.

In this subcase, α is an asymptotic value of f.

Subcase 2.2: $g(z) \nrightarrow \infty$ along γ.

In this subcase, there exists a sequence $\{z_n\}$ on γ and a finite point w_0 such that $\lim_{n\to+\infty} z_n = \infty$ and $\lim_{n\to+\infty} g(z_n) = w_0$. By (ii), $w_0 \notin E_f$ and it follows that $f(w_0) = \lim_{n\to+\infty} f(g(z_n)) = \alpha$. The same argument as in Subcase 1.2 gives $\alpha \in f(AV(g))$.

From Cases 1 and 2, we conclude that $AV(fg) \subset AV(f) \cup f(AV(g))$. This completes the proof. □

Remark 2.1. If f and g are transcendental entire functions, then all assumptions in Lemma 2.2 hold.

Lemma 2.3 (Denjoy–Carleman–Ahlfors Theorem [8]). *If the inverse function of a meromorphic function f has n direct singularities, $n \geqslant 2$, then*

$$\liminf_{r\to+\infty} \frac{T(r,f)}{r^{\frac{n}{2}}} > 0.$$

Consequently, the inverse function to a meromorphic function of finite order ρ has at most $\max\{2\rho, 1\}$ *direct singularities. Moreover, an entire function of finite order ρ has at most* 2ρ *finite asymptotic values.*

The following lemma is proved in [4].

Lemma 2.4. *For a meromorphic function f of finite order, every indirect singularity of f^{-1} is a limit of critical values.*

3. Results

We are now ready to state and prove our main results.

Theorem 3.1. *Let $f, g \in \mathbf{M}$. Assume that $\infty \in E_f \cap E_g$ and every point in $J(fg)$ and $J(gf)$ is a limit point of periodic points of fg and gf, respectively. Then the following statements hold*:

(i) *If $z \in J(fg) \setminus E_g$, then $g(z) \in J(gf)$.*
(ii) *If $g(z) \in J(gf) \setminus E_f$, then $z \in J(fg)$.*

Proof. Let $z \in J(fg) \setminus E_g$. By assumption, there exist periodic points z_k of fg say $(fg)^{n_k}(z_k) = z_k$ where $z_k \neq z$ such that $z_k \to z$ as $k \to +\infty$. By Lemma 2.1, $g(z_k)$ are periodic points of gf and $g(z_k) \neq g(z)$ for all but finitely many k (otherwise, the set $\{w\colon g(w) - g(z) = 0\}$ has a limit point and hence g is a constant). As $z, z_k \notin E_g$ we have $g(z_k) \to g(z)$ as $k \to +\infty$ and hence $g(z)$ is a limit point of periodic points of gf. It follows that $g(z) \in J(gf)$. Similarly, by interchanging the role of f and g, if $w \in J(gf) \setminus E_f$, then $f(w) \in J(fg)$. Conversely, assume that $g(z) \in J(gf) \setminus E_f$, then $f(g(z)) \in J(fg)$ and by the complete invariance property of the Julia set we obtain $z \in J(fg)$. This completes the proof. □

From Theorem 3.1, we have

Corollary 3.1. *If U is a component of $F(fg)$, then $g(U)$ is contained in a component V of $F(gf)$.*

Proof. Let U be a component of $F(fg)$. Then $U \cap J(fg) = \emptyset$. We claim that $g(U) \cap J(gf) = \emptyset$. Suppose that $g(U) \cap (J(gf) \setminus E_f) \neq \emptyset$. Then there exists $z_0 \in U$ such that $g(z_0) \in (J(gf) \setminus E_f)$. By Theorem 3.1(ii), we have $z_0 \in J(fg)$ which is impossible. Now if $g(U) \cap E_f \neq \emptyset$, then there exists $z_0 \in U$ such that $g(z_0) \in E_f$. Thus, $z_0 \in g^{-1}E_f \subset E_{fg} \subset J(fg)$ which is impossible. Therefore, $g(U) \cap J(gf) = \emptyset$ and hence $g(U)$ is contained in a component V of $F(gf)$. This completes the proof. □

Theorem 3.2. *Let $f, g \in \mathbf{M}$. Assume that $\infty \in E_f \cap E_g$ and every point in $J(fg)$ and $J(gf)$ is a limit point of periodic points of fg and gf, respectively. Let U be a component of $F(fg)$ and let V be the component of $F(gf)$ which contains $g(U)$. Then*

(i) *U is a wandering domain if and only if V is a wandering domain.*
(ii) *If U is periodic, then so is V. Moreover, V is of the same type according to the classification of periodic components as U unless U is a Siegel disk or Herman ring where in this case V is either a Siegel disk or Herman ring.*

Proof. For each $n \in \mathbb{N}$, let U_n be the component of $F(fg)$ which contains $(fg)^n(U)$ and let V_n be the component of $F(gf)$ which contains $(gf)^n(V)$. As $U \cap E_g = \emptyset$ we see that $g((fg)^n(U)) = (gf)^n(g(U))$ which gives $g(U_n) \subset V_n$. By a similar argument used in the proof of Corollary 3.1, we may show that $f(V_n) \subset U_{n+1}$. As a result, if $U_m = U_n$, then $V_m = V_n$ and if $V_m = V_n$, then $U_{m+1} = U_{n+1}$. This gives the statement (i) of the theorem. Moreover, if $U_n = U$, then $V_n = V$, namely if U is periodic, then so is V. Assume that $U_n = U$ and for some sequence $\{n_j\}$ we have $(fg)^{n_j}|_U \to \phi$ as $j \to +\infty$ where $\phi \notin E_{fg}$. Let V^* be a domain in V such that a branch $g_V^{-1}: V^* \to U^* \subset U$ of the inverse function of g is defined. Then $(gf)^n|_{V^*} = g(fg)^n g_V^{-1}|_{V^*}$ and hence $(gf)^n(V^*) \to \psi = g\phi g_V^{-1}$. If U is a Siegel disk or Herman ring, then ϕ is a nonconstant limit function of $\{(fg)^n\}$ on U, hence ψ is also a nonconstant limit function of $\{(fg)^n\}$ on V and hence V is either a Siegel disk or Herman ring. If U is an attracting domain, then ϕ is a constant limit function lying in $F(fg)$, hence ψ is also a constant limit function lying in $F(gf)$ and V must be an attracting domain. Similarly, if U is a parabolic domain, then so is V. By the same arguments, if V is an attracting or parabolic domain, then so is U_1; and if V is a Siegel disk or Herman ring, then U_1 is either a Siegel disk or Herman ring. It follows that if U is a Baker domain, then so is V. This completes the proof. □

We now give an example of transcendental meromorphic function and function in class **M** which do not have wandering domains or Baker domains.

Example 3.1. Let $f(z) = e^{iz} + z$ and $g(z) = \tan z$. Then g has finite order and has no critical values; hence, by Lemmas 2.3 and 2.4, g has only finitely many asymptotic values. In fact, $AV(g) = \{-i, i\}$. For f we may easily show that $CV(f) = \{i + (\frac{\pi}{2} + 2k\pi)\colon k \in \mathbb{Z}\}$ and f has no finite asymptotic values. We may show that $g(CV(f)) = \{-\cot i\}$, hence, by Lemma 2.2, $AV(gf) \subset \{-i, i\}$ and $CV(gf) \subset \{-\cot i\}$. Since $E_{gf} = E_f \cup f^{-1}(E_g) = \{\infty\}$, gf is a transcendental meromorphic function on $\mathbb{C}$ and $gf \in \mathbf{K} \cap \mathbf{MS} \subset \mathbf{MSR}$. By Theorem 1.1, gf has no wandering domains or Baker domains. We conclude from Theorem 3.2 that $fg = e^{i \tan z} + \tan z$ has no wandering domains or Baker domains. Note that $CV(fg) = \{i + \frac{\pi}{2} + 2k\pi\colon k \in \mathbb{Z}\}$, hence $fg \notin \mathbf{MS}$ or not even of bounded type. □

Remark 3.1. Theorem 3.2 generalizes Theorem 1.5 obtained in [6] and in fact we may find other examples of transcendental entire or meromorphic functions which have no wandering domains or Baker domains.

Acknowledgments

The first author is supported by the Royal Golden Jubilee program grant number PHD/0195/2544 and the second author is supported by the Thailand Research Fund grant number RSA4780012. We thank Prof. Zheng Jian-Hua for his hospitality during our visit to Tsinghua University and his suggestions on the topics. We also thank the referees for their invaluable comments.

References

[1] I.N. Baker, P. Domínguez, M.E. Herring, Dynamics of functions meromorphic outside a small set, Ergodic Theory Dynam. Systems 21 (2001) 647–672.

[2] I.N. Baker, P. Domínguez, M.E. Herring, Functions meromorphic outside a small set: completely invariant domains, Complex Variables Theory Appl. 49 (2004) 95–100.
[3] I.N. Baker, A.P. Singh, Wandering domains in the iteration of compositions of entire functions, Ann. Acad. Sci. Fenn. Ser. A I 20 (1995) 149–153.
[4] W. Bergweiler, A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoamericana 11 (1995) 355–373.
[5] W. Bergweiler, A. Hinkkanen, On semiconjugation of entire functions, Math. Proc. Cambridge Philos. Soc. 126 (1999) 565–574.
[6] W. Bergweiler, Y. Wang, On the dynamics of composite entire functions, Ark. Mat. 36 (1998) 131–139.
[7] A. Bolsch, Repulsive periodic points of meromorphic function, Complex Variables Theory Appl. 31 (1996) 75–79.
[8] A. Bolsch, Iteration of meromorphic functions with countably many singularities, Dissertation, Technische Universität, Berlin, 1997.
[9] J.H. Zheng, Iteration of functions meromorphic outside a small set, Tôhoku Math. J., submitted for publication.