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Distortion of Univalent Functions

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TO LEON MIRSKY ON HIS 60TH BIRTHDAY

1

For a complex function f which is analytic and univalent on the open unit disk $\Delta = \{z: |z| < 1\}$ and normalized with $f(0) = 0$, $f'(0) = 1$, the classical Koebe distortion theorem [1, 3] states that

$$\frac{|z|}{(1+|z|)^2} \leqslant |f(z)| \leqslant \frac{|z|}{(1-|z|)^2}, \tag{1}$$

$$\frac{1-|z|}{(1+|z|)^3} \leqslant |f'(z)| \leqslant \frac{1+|z|}{(1-|z|)^3} \tag{2}$$

hold for $z \in \Delta$. As these inequalities involve a single point of Δ, it is natural to consider two points u, v of Δ and to look for bounds for the ratios $|f(u) - f(v)|/|(u-v)f'(v)|$ and $|f'(u)|/|f'(v)|$. It is reasonable to expect the bounds to depend on the non-Euclidean distance

$$D(u, v) = \frac{1}{2} \log \frac{|1 - u\bar{v}| + |u - v|}{|1 - u\bar{v}| - |u - v|}. \tag{3}$$

Such bounds (4), (6) are given in the following theorem, which is valid for all f analytic and univalent on Δ (no need for normalization). We also include (5), which is less sharp than (4) but is simpler.

THEOREM 1. *If f is analytic and univalent on the open unit disk Δ, then inequalities*

$$\frac{|1-u\bar{v}|}{1-|u|^2} e^{-2D(u,v)} \leqslant \left| \frac{f(u)-f(v)}{(u-v)f'(v)} \right| \leqslant \frac{|1-u\bar{v}|}{1-|u|^2} e^{2D(u,v)}, \tag{4}$$

$$\frac{|1-u\bar{v}|}{(1+|u|)^2} \cdot \frac{1-|v|}{1+|v|} \leqslant \left| \frac{f(u)-f(v)}{(u-v)f'(v)} \right| \leqslant \frac{|1-u\bar{v}|}{(1-|u|)^2} \cdot \frac{1+|v|}{1-|v|}, \tag{5}$$

$$\left| \frac{f'(u)}{f'(v)} \right| \leqslant \frac{1-|v|^2}{1-|u|^2} e^{4D(u,v)} \tag{6}$$

hold for u, v in Δ.

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Before proving this result, let us first identify some of its special cases. For functions f normalized with $f(0) = 0, f'(0) = 1$, the classical double inequality (1) is the case $v = 0$ of (4) or (5), while the case $u = 0$ of (4) or (5) is another known result [1, pp. 88–89; 2, p. 224]:

$$\frac{|v|(1-|v|)}{1+|v|} \leqslant \left|\frac{f(v)}{f'(v)}\right| \leqslant \frac{|v|(1+|v|)}{1-|v|}. \tag{7}$$

The double inequality (2) is clearly a special case of (6). As an immediate consequence of inequality (16) (which will appear in the proof of Theorem 1), we have

$$\frac{1-|v|^2}{1-|u|^2}\left(\frac{|1-u\bar{v}|+|u-v|}{|1-u\bar{v}|-|u-v|}\right)^2 \leqslant \frac{1+|u|}{1-|v|}\left(\frac{1+|v|}{1-|u|}\right)^3 \tag{8}$$

for $u, v \in \Delta$. Here equality occurs if and only if $u\bar{v} \leqslant 0$. Thus, unless $u\bar{v} \leqslant 0$, (6) gives a sharper upper bound for $|f'(u)|/|f'(v)|$ than (2) or its consequence

$$\left|\frac{f'(u)}{f'(v)}\right| \leqslant \left(\frac{1+r}{1-r}\right)^4, \tag{9}$$

where $\max(|u|, |v|) \leqslant r < 1$.

For the special function $f(z) = z(1+z)^{-2}$, every bound given in Theorem 1 is attained for suitable choice of u, v. Indeed, equality occurs on each side of (4) for u, v real, respectively for $u \geqslant v$ and $u \leqslant v$. Equality occurs on each side of (5) respectively for $0 \leqslant u < 1, -1 < v \leqslant 0$ and $-1 < u \leqslant 0, 0 \leqslant v < 1$. Equality holds in (6) for real u, v such that $u \leqslant v$.

Proof of Theorem 1. For $w \in \Delta$, let μ_w denote the Möbius function

$$\mu_w(z) = \frac{z-w}{1-\bar{w}z},$$

which is analytic, univalent on Δ, and $\mu_w(\Delta) = \Delta$. The inverse function of μ_w is μ_{-w}. For a fixed $v \in \Delta$, define g, h on Δ by

$$g = f \circ \mu_{-v}, \qquad h = \frac{g - g(0)}{g'(0)}.$$

Then

$$g'(z) = \frac{1-|v|^2}{(1+\bar{v}z)^2}(f' \circ \mu_{-v})(z),$$

$$g(0) = f(v), \qquad g'(0) = (1-|v|^2)f'(v).$$

As f is univalent, $f'(v) \neq 0$, so $g'(0) \neq 0$. Thus h is analytic, univalent on Δ; $h(0) = 0$, $h'(0) = 1$. If we introduce the substitution

$$u = \mu_{-v}(z), \qquad z = \mu_v(u), \tag{10}$$

then $g(z) = f(u)$ and

$$h(z) = \frac{f(u) - f(v)}{(1 - |v|^2) f'(v)}.$$

With substitution (10), the double inequality (1) for $h(z)$ may be written

$$M_1(u, v) \leq \left| \frac{f(u) - f(v)}{(u - v) f'(v)} \right| \leq M_2(u, v) \tag{11}$$

for $u, v \in \Delta$, where

$$M_1(u, v) = \frac{1 - |v|^2}{|u - v|} \frac{|\mu_v(u)|}{(1 + |\mu_v(u)|)^2}, \tag{12}$$

$$M_2(u, v) = \frac{1 - |v|^2}{|u - v|} \frac{|\mu_v(u)|}{(1 - |\mu_v(u)|)^2}. \tag{13}$$

To simplify these expressions, we use the identity

$$|1 - u\bar{v}|^2 - |u - v|^2 = (1 - |u|^2)(1 - |v|^2)$$

to write (always for u, v in Δ)

$$\frac{|\mu_v(u)|}{1 - |\mu_v(u)|^2} = \frac{|(1 - u\bar{v})(u - v)|}{(1 - |u|^2)(1 - |v|^2)}.$$

Then (12), (13) may be written

$$M_1(u, v) = \frac{|1 - u\bar{v}|}{1 - |u|^2} \frac{1 - |\mu_v(u)|}{1 + |\mu_v(u)|}, \tag{14}$$

$$M_2(u, v) = \frac{|1 - u\bar{v}|}{1 - |u|^2} \frac{1 + |\mu_v(u)|}{1 - |\mu_v(u)|}; \tag{15}$$

so (4) follows at once from (11), (14), and (15).

From

$$\left| \frac{u - v}{1 - u\bar{v}} \right|^2 = 1 - \frac{(1 - |u|^2)(1 - |v|^2)}{|1 - u\bar{v}|^2}$$

$$\leq 1 - \frac{(1 - |u|^2)(1 - |v|^2)}{(1 + |uv|)^2} = \left(\frac{|u| + |v|}{1 + |uv|} \right)^2,$$

we derive

$$\frac{|1-u\bar{v}|+|u-v|}{|1-u\bar{v}|-|u-v|} \leqslant \frac{(1+|u|)(1+|v|)}{(1-|u|)(1-|v|)} \tag{16}$$

for u, v in Δ; here equality occurs if and only if $u\bar{v} \leq 0$. Using (16), we infer that the upper bound in (4) is $\leq$ the upper bound in (5), and the lower bound in (4) is $\geqslant$ the lower bound in (5). Thus (5) is a consequence of (4).

If we interchange u, v in the left-side inequality of (4), the resulting inequality combined with the right-side inequality of (4) gives (6). This completes the proof.

It is easily verified that the bounds $M_1(u, v)$, $M_2(u, v)$ in (4) may also be written

$$M_1(u, v) = \left(\frac{1-|v|^2}{1-|u|^2}\right)^{1/2} e^{-2D(u,v)} \cosh D(u, v), \tag{17}$$

$$M_2(u, v) = \left(\frac{1-|v|^2}{1-|u|^2}\right)^{1/2} e^{2D(u,v)} \cosh D(u, v). \tag{18}$$

2

Let f be analytic and univalent on Δ, with $f(0) = 0$, $f'(0) = 1$. If the image $f(\Delta)$ is a convex set in the complex plane, it is well known [2, p. 225] that inequalities (sharper than (1), (2))

$$\frac{|z|}{1+|z|} \leqslant |f(z)| \leqslant \frac{|z|}{1-|z|}, \tag{19}$$

$$\frac{1}{(1+|z|)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1-|z|)^2} \tag{20}$$

hold for $z \in \Delta$. The following is a generalization of this.

THEOREM 2. *If f is analytic, univalent on Δ, and $f(\Delta)$ is convex, then*

$$\frac{1-|v|^2}{|1-u\bar{v}|+|u-v|} \leqslant \left|\frac{f(u)-f(v)}{(u-v)f'(v)}\right| \leqslant \frac{1-|v|^2}{|1-u\bar{v}|-|u-v|}, \tag{21}$$

$$\frac{1-|v|}{1+|u|} \leqslant \left|\frac{f(u)-f(v)}{(u-v)f'(v)}\right| \leqslant \frac{1+|v|}{1-|u|}, \tag{22}$$

$$\left|\frac{f'(u)}{f'(v)}\right| \leqslant \frac{1-|v|^2}{1-|u|^2} e^{2D(u,v)} \tag{23}$$

hold for u, v in Δ.

For functions f normalized with $f(0)=0$, $f'(0)=1$, the case $v=0$ of (21) or (22) is (19), while the case $u=0$ of (21) or (22) is

$$|v|(1-|v|)\leqslant\left|\frac{f(v)}{f'(v)}\right|\leqslant|v|(1+|v|). \tag{24}$$

Clearly (20) is a special case of (23). In view of (16), we have

$$\frac{1-|v|^2}{1-|u|^2}\,\frac{|1-u\bar{v}|+|u-v|}{|1-u\bar{v}|-|u-v|}\leqslant\left(\frac{1+|v|}{1-|u|}\right)^2$$

with strict inequality unless $u\bar{v}\leqslant 0$. Thus except for the case $u\bar{v}\leqslant 0$, (23) gives a sharper upper bound for $|f'(u)|/|f'(v)|$ than (20). Simple calculations will verify that each of (21)–(23) (which are valid only when $f(\Delta)$ is convex) gives sharper bounds than the corresponding one of (4)–(6).

For the special function $f(z)=z(1-z)^{-1}$, every bound in Theorem 2 is attained for suitable choice of u, v. Indeed, equality occurs on each side of (21) for u, v real, respectively for $u\leqslant v$ and $u\geqslant v$. Equality occurs on each side of (22) respectively for $-1<u\leqslant 0$, $0\leqslant v<1$ and $0\leqslant u<1$, $-1<v\leqslant 0$. Equality holds in (23) for real u, v such that $u\geqslant v$.

Proof of Theorem 2. The proof is similar to that of Theorem 1. Define g, h as in the proof of Theorem 1. Observe that $h(\Delta)$ is convex, so (19) holds for h. With the substitution (10), the double inequality (19) for $h(z)$ becomes (21) after simplification.

Next, in view of the identity

$$(1-|u|^2)(1-|v|^2)=(|1-u\bar{v}|+|u-v|)(|1-u\bar{v}|-|u-v|),$$

we have

$$\frac{1-|v|^2}{|1-u\bar{v}|+|u-v|}=\left(\frac{1-|v|^2}{1-|u|^2}\right)^{1/2}\left(\frac{|1-u\bar{v}|-|u-v|}{|1-u\bar{v}|+|u-v|}\right)^{1/2},$$

$$\frac{1-|v|^2}{|1-u\bar{v}|-|u-v|}=\left(\frac{1-|v|^2}{1-|u|^2}\right)^{1/2}\left(\frac{|1-u\bar{v}|+|u-v|}{|1-u\bar{v}|-|u-v|}\right)^{1/2},$$

which combined with (16) imply

$$\frac{1-|v|^2}{|1-u\bar{v}|+|u-v|}\geqslant\frac{1-|v|}{1+|u|},$$

$$\frac{1-|v|^2}{|1-u\bar{v}|-|u-v|}\leqslant\frac{1+|v|}{1-|u|}.$$

This proves (22) as a consequence of (21). Inequality (23) follows directly from (21).

In the above proof, we have already seen that (21) may be given the form

$$\left(\frac{1-|v|^2}{1-|u|^2}\right)^{1/2} e^{-D(u,v)} \leqslant \left|\frac{f(u)-f(v)}{(u-v)f'(v)}\right| \leqslant \left(\frac{1-|v|^2}{1-|u|^2}\right)^{1/2} e^{D(u,v)}. \tag{25}$$

3

Let f be meromorphic and univalent on Δ with $f(0)=0$, $f'(0)=1$. It is known [1, p. 87] that

$$\frac{|f(z)|^2}{|f'(z)|} \leqslant \frac{|z|^2}{1-|z|^2}, \tag{26}$$

if $z \in \Delta$ is not a pole of f. This is the case $v=0$ of the following.

THEOREM 3. *Let f be meromorphic and univalent on Δ. If u, v are in Δ and neither of them is a pole of f, then*

$$\frac{|f(u)-f(v)|^2}{|f'(u)f'(v)|} \leqslant \frac{|(1-u\bar{v})(u-v)|^2}{(1-|u|^2)(1-|v|^2)}. \tag{27}$$

For $f(z) = z(1+z)^{-2}$ and any real u, v in Δ, equality occurs in (27).

Proof. Let $v \in \Delta$ be not a pole of f, and let g, h be defined as in the proof of Theorem 1. h is meromorphic and univalent on Δ; $h(0)=0$, $h'(0)=1$. With the substitution (10), we have

$$h(z) = \frac{f(u)-f(v)}{(1-|v|^2)f'(v)}, \qquad h'(z) = \frac{f'(u)}{(1+\bar{v}\mu_v(u))^2 f'(v)}.$$

If $u \in \Delta$ is not a pole of f, then $z = \mu_v(u)$ is not a pole of h, so by (26):

$$\left|\frac{f(u)-f(v)}{(1-|v|^2)f'(v)}\right|^2 \cdot \left|\frac{(1+\bar{v}\mu_v(u))^2 f'(v)}{f'(u)}\right| \leqslant \frac{|\mu_v(u)|^2}{1-|\mu_v(u)|^2},$$

which easily reduces to (27).

Finally we mention that the upper bound in (27) may be expressed in terms of the non-Euclidean distance:

$$\frac{|f(u)-f(v)|^2}{|f'(u)f'(v)|} \leqslant \frac{(1-|u|^2)(1-|v|^2)}{4} \sinh^2 2D(u,v). \tag{28}$$

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