An iterative method for generalized set-valued nonlinear mixed quasi-variational inequalities

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Received 10 March 2003; received in revised form 15 January 2004

Abstract

This paper presents an iterative method for solving the generalized nonlinear set-valued mixed quasi-variational inequality, a problem class that was introduced by Huang et al. (Comp. Math. Appl. 40 (2–3) (2000) 205–215). The method incorporates step size controls that enable application to problems where certain set-valued mappings do not always map to nonempty closed bounded sets.

Keywords: Quasi-variational inequalities; Fixed point methods; Set-valued mappings

1. Introduction

In recent years, a significant number of publications have appeared that define generalizations of the variational and quasi-variational inequality problems; see, for example, [1–3,5,9–15,17], and references therein. One of the most general of these new problem classes is the generalized nonlinear set-valued mixed quasi-variational inequality (GNSVMQVI), which was introduced and studied in [4]. Before the GNSVMQVI can be defined, some definitions are needed. Let $H$ be a real Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$. Let $2^H$ represent the family of all subsets of $H$. A set-valued mapping $F: H \to 2^H$ is said to be monotone if for all $x_1, x_2 \in H$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$,

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0.$$

This material is based on work supported by the National Science Foundation under Grant No. DMS-9973321.

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$F$ is said to be maximal monotone if its graph (i.e., the set \{(x,y)| y \in F(x)\}) is not properly contained in the graph of any other monotone mapping. The effective domain of $F$, denoted $\text{dom}(F)$ is the set \{x|F(x) \neq \emptyset\}.

Let $G, S, T : H \rightarrow 2^H$ be set-valued mappings, and let $p : H \rightarrow H$ and $N : H \times H \rightarrow H$ be single-valued mappings. Suppose that $A : H \times H \rightarrow 2^H$ is a set-valued mapping such that for each fixed $t \in H$, $A(\cdot,t) : H \rightarrow 2^H$ is a maximal monotone mapping and $\text{Range}(p) \cap \text{dom}(A(\cdot,t)) \neq \emptyset$ for each $t \in H$. The GNSVMQVI is to find $u \in H$, $x \in S(u)$, $y \in T(u)$, $z \in G(u)$ such that $p(u) \in \text{dom}A(\cdot,z)$ and

$$0 \in N(x,y) + A(p(u),z).$$

The above definition differs from the one given by Huang, et al. [4] in one important respect: Huang, et al. restricted $2^H$ to be the family of all nonempty subsets of $H$. In other words, they restricted the mappings $G, S, T$ and $A$ to map only to nonempty sets. This restriction is not at all unusual in the literature. In fact, many of the recent generalizations of quasi-variational inequalities have similar restrictions, apparently because the restriction is needed to make the algorithms work. However, this restriction is of considerable negative consequence because it prevents the application of the GNSVMQVI framework to certain problem classes.

As a simple example, let $X$ be a convex subset of $H$ and let $f : H \rightarrow H$ be a single-valued operator. The variational inequality problem (VI) is to

\[
\text{find } x \in X \text{ such that } \langle f(x), z - x \rangle \geq 0 \text{ for all } z \in X.
\]

It is well known (see, for example [16]) that this problem is equivalent to the generalized equation

\[
\text{find } x \in H \text{ such that } 0 \in f(x) + N_X(x),
\]

where $N_X : H \rightarrow 2^H$ is the normal cone operator to the set $X$, defined by

\[
N_X(x) := \begin{cases} 
\{z|\langle z, y - x \rangle \leq 0 \text{ for all } y \in X\}, & x \in X, \\
\emptyset, & x \notin X.
\end{cases}
\]

Note that depending on the choice of $x$, $N_X(x)$ is either the empty set, the singleton \{0\}, or an unbounded cone.

Since the normal cone operator is maximal monotone, the variational inequality problem is a special case of GNSVMQVI formed by choosing $S, T, G, N, A$ and $p$ by the relations $S(x) := x$, $T(x) := 0$, $G(x) := X$, $N(x,y) := f(x)$, $A(x,X) := N_X(x)$ and $p(x) := x$. However, because $N_X$ can map to the empty set, the above formulation of VI as a special case of GNSVMQVI would be excluded from the framework of Huang et al. [4].

It is a simple matter, as we have done in this paper, to change the definition of GNSVMQVI to remove the above difficulty. However, the algorithm proposed in [4] is not capable of solving the unrestricted problem. In fact, the main convergence theorem for that algorithm requires that $S, T,$ and $G$ map everywhere to nonempty, closed, bounded sets. Therefore, this paper proposes a new iterative method for solving GNSVMQVI that does not require the set-valued mappings to map everywhere to nonempty or closed sets. To prove convergence to a solution, we do however, assume that $S, G,$ and $T$ map only to closed sets. This algorithm is an adaptation of Algorithm 1 from [4]. The main change is to introduce step-size controls that enable the algorithm to ensure that the iterates stay in the effective domain of all of the set-valued mappings.
2. Preliminaries

Let \( F : H \rightarrow 2^H \) be a maximal monotone mapping. Given a constant \( \rho > 0 \), the resolvent operator for \( F \) is defined by
\[
J^F_\rho := (I + \rho F)^{-1},
\]
where \( I \) is the identity mapping on \( H \) and the inverse is the set-valued inverse defined by \( F^{-1}(y) = \{ x | y \in F(x) \} \). It is well known that \( J^F_\rho \) is a single-valued, nonexpansive mapping [8].

We define a pseudo-metric \( M : 2^H \times 2^H \rightarrow \mathbb{R} \cup \{ \infty \} \) by
\[
M(\Gamma, \Lambda) := \max \left\{ \sup_{u \in \Gamma} \text{dist}(u|\Lambda), \sup_{v \in \Lambda} \text{dist}(v|\Gamma) \right\},
\]
where \( \text{dist}(u|S) := \inf_{v \in S} \| u - v \| \). Note that if the domain of \( M \) is restricted to closed bounded sets, then \( M \) is the Hausdorff metric.

3. Iterative algorithm

A key to solving (1) is the following lemma, which relates solutions of (1) to the resolvent operator for \( A(\cdot, z) \)

Lemma 3.1 (Huang et al. [4, Lemma 3.1]). \((u, x, y, z)\) is a solution of problem (1) if and only if \((u, x, y, z)\) satisfies the relation
\[
p(u) = J^A_\rho(\cdot, z)(p(u) - \rho N(x, y)),
\]
where \( \rho > 0 \) is a constant and \( J^A_\rho(\cdot, z) \) is the resolvent operator defined by (2).

To develop a fixed point algorithm for (1), we rewrite (4) as follows:
\[
u = u - p(u) + J^A_\rho(\cdot, z)(p(u) - \rho N(x, y)),
\]
where \( \rho > 0 \) is a constant. This fixed point formulation allows us to suggest the following iterative algorithm.

Algorithm 1. Step 0: Let \( \rho > 0 \) be a constant. Choose \( u_0 \in \text{int}(\text{dom}(S) \cap \text{dom}(T) \cap \text{dom}(G)) \) and choose \( x_0 \in S(u_0), y_0 \in T(u_0), \) and \( z_0 \in G(u_0) \). Set \( n = 0 \).

Step 1: Let
\[
u_{n+1} = u_n + \alpha_n( - p(u_n) + J^A_\rho(\cdot, z_n)(p(u_n) - \rho N(x_n, y_n))),
\]
where \( \alpha_n \in (0, 1] \) is chosen sufficiently small to ensure that \( u_{n+1} \in \text{int}(\text{dom}(S) \cap \text{dom}(T) \cap \text{dom}(G)) \).

Step 2: Choose \( \varepsilon_{n+1} \geq 0 \), and choose \( x_{n+1} \in S(u_{n+1}), y_{n+1} \in T(u_{n+1}), z_{n+1} \in G(u_{n+1}) \) satisfying
\[
\| x_{n+1} - x_n \| \leq (1 + \varepsilon_{n+1})M(S(u_{n+1}), S(u_n)),
\]
\[
\| y_{n+1} - y_n \| \leq (1 + \varepsilon_{n+1})M(T(u_{n+1}), T(u_n)),
\]
where \( M(S(u_{n+1}), S(u_n)) \) and \( M(T(u_{n+1}), T(u_n)) \) are the Hausdorff metrics for \( S \) and \( T \), respectively.
\[ \|z_{n+1} - z_n\| \leq (1 + \varepsilon_{n+1})M(G(u_{n+1}), G(u_n)), \]  
(9)

**Step 3:** If \( u_{n+1}, x_{n+1}, y_{n+1}, z_{n+1} \) satisfy (4) to sufficient accuracy, stop; otherwise, set \( n := n + 1 \) and return to Step 1.

**Discussion.** From the definition of \( M \), (3), it is clear that the restrictions (7)–(9) imposed on the points \( x_n, y_n, \) and \( z_n \) can always be satisfied for any \( \varepsilon_n > 0 \). If \( S, T, \) and \( G \) always map to closed bounded sets, then the restrictions can be satisfied with \( \varepsilon_n = 0 \).

Since \( u_n \) is always in the interior of the intersections of the domains of \( S, T, \) and \( G \), it is always possible to choose positive values of \( z_n \) that ensure that \( u_{n+1} \) remains in the interior of the intersections of the domains of \( S, T, \) and \( G \).

In order to ensure convergence, we will need to make the additional assumption that \( \sum_{n=0}^{\infty} \varepsilon_n = \infty \). Note that for \( \varepsilon_n = 1 \), Algorithm 1 collapses to 3.1 of Huang et al. [4].

**4. Existence and convergence theorems**

This section proves that under suitable conditions, the iterates produced by Algorithm 1 converge to a solution of problem (1). Note that, unlike the convergence result presented in [4], our result does not require that \( S, G, T \) and \( A \) map to nonempty, or bounded sets. We also note that the result in [4] assumes that \( N(S(\cdot), z) \) is Lipschitz continuous and strongly monotone for each \( z \in H \). However, Liu and Li pointed out [7, Theorem 3.1] that Lipschitz continuous set-valued operators cannot be monotone unless they are single-valued. Thus, the conditions of that theorem imply that \( N(S(\cdot), z) \) is single-valued for each \( z \in H \). We therefore use a different set of assumptions that are similar to those used in [7, Theorem 3.2] to establish our convergence result.

**Definition 4.1.** A mapping \( g : H \rightarrow H \) is said to be **strongly monotone** if there exists some \( \delta > 0 \) such that
\[
\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \geq \delta \| u_1 - u_2 \|^2,
\]
for all \( u_1, u_2 \in H \). \( g \) is **Lipschitz continuous** if there exists some \( \sigma > 0 \) such that
\[
\| g(u_1) - g(u_2) \| \leq \sigma \| u_1 - u_2 \|,
\]
for all \( u_1, u_2 \in H \).

**Definition 4.2.** A set-valued mapping \( S : H \rightarrow 2^H \) is said to be \( M \)-**Lipschitz continuous** if there exists a constant \( \eta > 0 \) such that
\[
M(S(u_1), S(u_2)) \leq \eta \| u_1 - u_2 \|,
\]
for all \( u_1, u_2 \in H \).

**Definition 4.3.** The operator \( N : H \times H \rightarrow H \) is said to be **Lipschitz continuous with respect to the first argument** if there exists a constant \( \beta > 0 \) such that
\[
\| N(u_1, \cdot) - N(u_2, \cdot) \| \leq \beta \| u_1 - u_2 \|.
\]
for all \(u_1, u_2 \in H\). Similarly, \(N\) is Lipschitz continuous with respect to the second argument if there exists \(\xi > 0\) such that
\[
\|N(\cdot, v_1) - N(\cdot, v_2)\| \leq \xi \|v_1 - v_2\|
\]
for all \(v_1, v_2 \in H\).

Lipschitz continuity of \(N\) with respect to the second argument is defined similarly.

The following two technical lemmas will be needed in the proof of our main theorem. The first Lemma is from [6].

**Lemma 4.4.** (Li and Feng [6]) Let \(F : H \to 2^H\) be maximal strongly monotone with constant \(\nu > 0\). Then, for any constant \(\rho > 0\), the resolvent operator \(J_F := (I + \rho F)^{-1}\) is Lipschitz continuous with constant \(1/(1 + \nu \rho)\).

**Lemma 4.5.** Let \(\delta\) and \(\sigma\) be positive scalars with \(\delta \leq \sigma\). Then for all \(x \in [0, 1]\),
\[
1 - 2\delta x + \sigma^2 x^2 \leq \left(1 - x + x\sqrt{1 - 2\delta + \sigma^2}\right)^2.
\]

**Proof.** Since \(\sigma \geq \delta > 0\), we have
\[
1 - 2\delta + \sigma^2 = (1 - \delta)^2 + \sigma^2 - \delta^2 \\
\geq (1 - \delta)^2 \geq 0.
\]
Thus,
\[
1 - \delta \leq \sqrt{1 - 2\delta + \sigma^2}
\]
so,
\[
2x(1 - x)(1 - \delta) \leq 2x(1 - x)\sqrt{1 - 2\delta + \sigma^2}.
\]
Adding \((1 - x)^2 + x^2(1 - 2\delta + \sigma^2)\) to both sides and simplifying yields the desired result. \(\square\)

For the following theorem, define \(C(H)\) to be the collection of all closed subsets of \(H\).

**Theorem 4.6.** Let \(N\) be Lipschitz continuous with respect to the first and second arguments with constants \(\beta\) and \(\xi\), respectively. Let \(S, T, G : H \to C(H)\) be \(M\)-Lipschitz with constants \(\eta, \gamma\) and \(\mu\), respectively; and suppose that \(\text{int}(\text{dom}(S) \cap \text{dom}(T) \cap \text{dom}(G)) \neq \emptyset\). Suppose that for each fixed \(z \in H\), \(A(\cdot, z)\) is a maximal strongly monotone mapping with constant \(v(z) \geq v > 0\). Let \(p : H \to H\) be strongly monotone and Lipschitz continuous with constants \(\delta\) and \(\sigma\), respectively. Suppose that there exist constants \(\lambda > 0\) and \(\rho > 0\) such that, for each \(x, y, z \in H\),
\[
\|J_p^{A(\cdot, x)}(z) - J_p^{A(\cdot, y)}(z)\| \leq \lambda \|x - y\| \tag{10}
\]
and
\[
\theta := 1 - \sqrt{1 - 2\delta + \sigma^2} - \lambda \mu - \frac{\rho(\xi \gamma + \beta \eta) + \sigma}{1 + \rho v} > 0. \tag{11}
\]
Proof. For \( n = 0, 1, \ldots \), define

\[
\Gamma_n := -p(u_n) + J^A(\cdot; z_n)(p(u_n) - \rho N(x_n, y_n))
\]  

(12)

and note that

\[
u_{n+1} = u_n + z_n \Gamma_n.
\]  

(13)

We will first establish a bound on \( \|\Gamma_n\| \). From (12) and (13), we have

\[
\|\Gamma_n\| = \|\Gamma_{n-1} + \Gamma_n - \Gamma_{n-1}\| = \|(u_n - u_{n-1})/z_{n-1} + \Gamma_n - \Gamma_{n-1}\|
\]

\[
\leq \| (u_n - u_{n-1})/z_{n-1} - (p(u_n) - p(u_{n-1})) \|
\]

\[
+ \| J^A(\cdot; z_n)(p(u_n) - \rho N(x_n, y_n)) - J^A(\cdot; z_{n-1})(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \|
\]

\[
\leq \| (u_n - u_{n-1})/z_{n-1} - (p(u_n) - p(u_{n-1})) \|
\]

\[
+ \| J^A(\cdot; z_n)(p(u_n) - \rho N(x_n, y_n)) - J^A(\cdot; z_{n-1})(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \|
\]

\[
- J^A(\cdot; z_{n-1})(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})).
\]  

(14)

By (9) and (10), and the M-Lipschitz continuity of \( G \),

\[
\| J^A(\cdot; z_n)(p(u_n) - \rho N(x_n, y_n)) - J^A(\cdot; z_{n-1})(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \|
\]

\[
\leq \lambda \| z_n - z_{n-1} \|
\]

\[
\leq \lambda (1 + \varepsilon_n) M(G(u_n), G(u_{n-1}))
\]

\[
\leq \lambda \mu (1 + \varepsilon_n) \| u_n - u_{n-1} \|.
\]  

(15)

By Lemma 4.4, the last term in (14) is bounded by

\[
\| J^A(\cdot; z_{n-1})(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})) \|
\]

\[
\leq \frac{1}{1 + \rho v} \| p(u_n) - p(u_{n-1}) - \rho N(x_n, y_n) + \rho N(x_{n-1}, y_{n-1}) \|
\]

\[
\leq \frac{1}{1 + \rho v} (\| p(u_n) - p(u_{n-1}) \| + \rho \| N(x_n, y_n) - N(x_{n-1}, y_{n-1}) \|)
\]

\[
+ \rho \| N(x_{n-1}, y_{n-1}) - N(x_{n-1}, y_{n-1}) \|.
\]  

(16)

Since \( p \) is Lipschitz continuous, we have

\[
\| p(u_n) - p(u_{n-1}) \| \leq \sigma \| u_n - u_{n-1} \|.
\]  

(17)
Using the Lipschitz continuity of the operator $N(\cdot, \cdot)$ with respect to the second argument and $M$-Lipschitz continuity of $T$, we have
\[
\|N(x_n, y_n) - N(x_n, y_{n-1})\| \leq \xi \|y_n - y_{n-1}\|
\leq \xi(1 + \varepsilon_n)M(T(u_n), T(u_{n-1}))
\leq \xi\gamma(1 + \varepsilon_n)\|u_n - u_{n-1}\|.
\]
Similarly, using the Lipschitz continuity of the operator $N(\cdot, \cdot)$ with respect to the first argument and $M$-Lipschitz continuity of $S$ for all $x \in S(u)$, we have
\[
\|N(x_n, y_{n-1}) - N(x_{n-1}, y_{n-1})\| \leq \beta \|x_n - x_{n-1}\|
\leq \beta(1 + \varepsilon_n)M(S(u_n), S(u_{n-1}))
\leq \beta\eta(1 + \varepsilon_n)\|u_n - u_{n-1}\|.
\]
Finally, since $p$ is strongly monotone and Lipschitz continuous, we have
\[
\|(u_n - u_{n-1})/x_{n-1} - (p(u_n) - p(u_{n-1}))\|^2
= \|u_n - u_{n-1}\|^2 - \frac{2}{x_{n-1}}(u_n - u_{n-1}, p(u_n) - p(u_{n-1})) + \|p(u_n) - p(u_{n-1})\|^2
\leq \frac{1}{x_{n-1}^2}(1 - 2\delta x_{n-1} + \sigma^2 x_{n-1})\|u_n - u_{n-1}\|^2
\leq \frac{1}{x_{n-1}^2} \left(1 - x_{n-1} + x_{n-1}\sqrt{1 - 2\delta + \sigma^2}\right)^2 \|u_n - u_{n-1}\|^2,
\]
where the last inequality follows from Lemma 4.5 and the fact that the Lipschitz constant $\sigma$ of $p$ must be at least as large as the constant of monotonicity $\delta$.

Combining (14)–(20) yields
\[
\|I_n\| \leq (1 - x_{n-1}^2\theta_n)\|u_n - u_{n-1}\|/x_{n-1} = (1 - x_{n-1}^2\theta_n)\|I_{n-1}\|
\]
with $\theta_n$ defined by
\[
\theta_n := 1 - \left\{ \sqrt{1 - 2\delta + \sigma^2} + (1 + \varepsilon_n)(\xi\gamma + \beta\eta + \sigma) \right\}
\]
\[
\leq 1 + (1 + \varepsilon_n)(\xi\gamma + \beta\eta + \sigma)
\]
\[
\frac{1}{1 + \rho\mu}
\]
Since $\varepsilon_n \to 0$, then $\theta_n \to \theta$. By Assumption (11), $\theta > 0$. Thus, for all $n$ sufficiently large, $\theta_n \geq \theta/2 > 0$. Define $\Phi := \theta/2$. Without loss of generality, we can assume $\theta_n \geq \Phi > 0$ for all $n$. It follows that
\[
\|I_n\| \leq \|I_0\| \prod_{i=0}^{n-1} (1 - z_i\Phi).
\]
Since $\sum_{n=0}^{\infty} z_n = \infty$, we conclude that $\lim_{n \to \infty} \|I_n\| = 0$ and therefore
\[
\lim_{n \to \infty} \|u_n - u_{n-1}\| = 0.
\]
Next, we show that \( \{u_n\} \) converges. Let \( m \) be an arbitrary index. Since \( \sum_{i=0}^{\infty} \alpha_i = \infty \) and \( \alpha_i \leq 1 \), there exists a sequence \( \{k_j\} \) of indices, with \( k_0 = m \) such that
\[
1 \leq \sum_{i=k_j}^{k_{j+1}-1} \alpha_i < 2.
\]
(22)

Define
\[
\kappa_j := \left( \prod_{i=k_j}^{k_{j+1}-1} (1 - \alpha_i \Phi) \right)^{1/(k_{j+1} - k_j)}
\]
and
\[
\tau_j := \left( \frac{\sum_{i=k_j}^{k_{j+1}-1} (1 - \alpha_i \Phi)}{k_{j+1} - k_j} \right).
\]

Note that \( \kappa_j \) and \( \tau_j \) are the geometric and arithmetic means, respectively, of \((1 - \alpha_{k_j} \Phi), (1 - \alpha_{k_{j+1}} \Phi), \ldots, (1 - \alpha_{k_{j+1} - 1})\); so \( \kappa_j \leq \tau_j \). Thus,
\[
\prod_{i=k_j}^{k_{j+1}-1} (1 - \alpha_i \Phi) = \kappa_j^{(k_{j+1} - k_j)}
\]
\[
\leq \tau_j^{(k_{j+1} - k_j)}
\]
\[
= \left( \frac{\sum_{i=k_j}^{k_{j+1}-1} (1 - \alpha_i \Phi)}{k_{j+1} - k_j} \right)^{(k_{j+1} - k_j)}
\]
\[
= \left( 1 - \frac{\Phi \sum_{i=k_j}^{k_{j+1}-1} \alpha_i}{k_{j+1} - k_j} \right)^{(k_{j+1} - k_j)}
\]
\[
\leq \left( 1 - \frac{\Phi}{k_{j+1} - k_j} \right)^{(k_{j+1} - k_j)} \quad \text{(using (22))}
\]
\[
\leq e^{-\Phi}.
\]
(23)

It follows that
\[
\| \Gamma_{k_{j+1}} \| \leq e^{-\Phi} \| \Gamma_k \| \leq (e^{-\Phi})^{j+1} \| \Gamma_m \|.
\]
(24)

Thus,
\[
\lim_{n \to \infty} \| u_n - u_m \| \leq \sum_{i=0}^{\infty} \alpha_i \| \Gamma_i \|
\]
\[
= \sum_{j=0}^{\infty} \sum_{i=k_j}^{k_{j+1}-1} \alpha_i \| \Gamma_i \|
\]
\[ \sum_{j=0}^{\infty} 2\|\Gamma_j\| \quad \text{(by (22))} \]

\[ \leq 2\|\Gamma_m\| \sum_{j=0}^{\infty} (e^{-\phi})^j \quad \text{(by (24))} \]

\[ = 2\|\Gamma_m\|/(1 - e^{-\phi}). \]

Since \( \lim_{m \to \infty} \|\Gamma_m\| = 0 \), it follows that \( \lim_{n,m \to \infty} \|u_n - u_m\| = 0 \), so \( \{u_n\} \) converges strongly to some fixed \( u \in H \).

Now we prove that \( x_n \to x \in S(u) \). From (7), we have
\[
\|x_n - x_{n-1}\| \leq (1 + \varepsilon_n)M(S(u_n), S(u_{n-1})) \leq 2\eta\|u_n - u_{n-1}\|
\]
which implies that \( \{x_n\} \) is a Cauchy sequence in \( H \), so there exists \( x \in H \) such that \( x_n \to x \). Further,
\[
d(x, S(u)) = \inf \{\|x - t\| : t \in S(u)\} \leq \|x - x_n\| + d(x_n, S(u))
\]
\[
\leq \|x - x_n\| + M(S(u_n), S(u)) \leq \|x - x_n\| + \eta\|u_n - u\| \to 0.
\]
Hence, since \( S(u) \) is closed, we have \( x \in S(u) \). Similarly, \( \{y_n\} \) converges to some fixed \( y \in T(u) \) and \( \{z_n\} \) converges to some fixed \( z \in G(u) \). By continuity, \( u, x, y, z \) satisfy (4) and therefore solve (1).

5. Summary

Algorithm 1 is based on the algorithm presented in [4]. The only difference is the addition of the stepsize \( \varepsilon_n \) in Step 1 of the algorithm. This stepsize is crucial because it allows us to prove convergence of the algorithm without assuming that the set-valued mappings map only to nonempty or bounded sets. This advance is important because it enables some well-known problems to be solved as instances of GNSVMQVI. Another important improvement over [4] is that our assumptions do not force the mapping \( N(S(\cdot), z) \) to be single-valued. The ideas behind this algorithm and the convergence proof are applicable to many other generalizations of quasi-variational inequalities. We developed them for the GNSVMQVI because it is among the most general such problem classes studied to date.

Acknowledgements

We are grateful to an anonymous referee who called our attention to the result given in [7, Theorem 3.1].

References