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Two-sided bounds and perturbation results for regularized determinants of infinite order compact operators ☆

M.I. Gil'

Department of Mathematics, Ben Gurion University of the Negev, PO Box 653, Beer-Sheva 84105, Israel

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ABSTRACT

A compact operator in a separable Hilbert space is of infinite order if it does not belong to any Schatten–von Neumann ideal. In the paper, upper and lower bounds for the regularized determinants of infinite order operators are derived. By these bounds, perturbations results for the regularized determinants are established.

1. Ideals of infinite order operators

Let H be a separable Hilbert space. For a compact linear operator A in H, A^* is the adjoint, $\lambda_j(A)$ are the eigenvalues and $s_k(A) = \sqrt{\lambda_k(A^*A)}$ $(k = 1, 2, \dots)$ are the singular values taken with their multiplicities and ordered in the decreasing way. In addition, S_p $(1 < p < \infty)$ is the Schatten–von Neumann ideal of operators A with the finite norm $N_p(A) := [\mathrm{Trace}(A^*A)^{p/2}]^{1/p}$. We will say that a compact operator in H is of infinite order if it does not belong to any Schatten–von Neumann ideal. Such operators arise in various applications. Many fundamental results on infinite order compact linear operators can be found in the well-known book [9, Section 3.1]. Literature on determinants of compact operators and their applications is very rich, cf. the very interesting recent papers [3,4,10,17,18] and references cited therein, about the classical results see [2,7,8]. At the same time to the best of our knowledge, bounds for the determinants of infinite order operators were not investigated in the available literature. The motivation of this paper is to extend some useful results on determinants of Schatten–von Neumann operators to infinite order operators.

For a non-decreasing sequence of positive integers $\pi = \{p_j\}_{j=1}^{\infty}$ $(p_1 \geqslant 1)$, assume that

$$\sum_{j=1}^{\infty} \big(ts_j(A)\big)^{p_j} < \infty \tag{1.1}$$

for all $t > 0$. *We denote the set of all compact operators A satisfying* (1.1) *by* Γ_π.

Put

$$\gamma_\pi(A) := \sum_{j=1}^{\infty} s_j^{p_j}(A).$$

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E-mail address: gilmi@cs.bgu.ac.il.

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The regularized determinant $\det_\pi(I-A)$ for an operator A with $\gamma_\pi(A)<\infty$ is defined as

$$\det{}_\pi(I-A):=\prod_{j=1}^{\infty}\bigl(1-\lambda_j(A)\bigr)\exp\left[\sum_{m=1}^{p_j-1}\frac{\lambda_j^m(A)}{m}\right].$$

Here we put

$$\sum_{m=1}^{0}\frac{\lambda_j^m(A)}{m}=0.$$

Lemma 1.1. *Let A be a compact operator such that $\gamma_\pi(A)<\infty$. Then*

$$\bigl|\det{}_\pi(I-A)\bigr|\leqslant\exp\left[\sum_{j=1}^{\infty}\bigl|\lambda_j(A)\bigr|^{p_j}\right]\leqslant e^{\gamma_\pi(A)}.$$

Proof. Thanks to Lemma 3.1 from the paper [12], for any integer $p\geqslant 3$ we have

$$\left|(1-w)\exp\left[\sum_{m=1}^{p-1}\frac{w^m}{m}\right]\right|\leqslant\exp\left[\frac{(p-1)|w|^p}{p}\right]\quad(w\in\mathbb{C}).\tag{1.2}$$

But, $|1-w|\leqslant e^{|w|}$. In addition,

$$\bigl|(1-w)e^w\bigr|^2=\bigl(1-2r\cos t+r^2\bigr)e^{2r\cos t}\leqslant e^{r^2}\quad\bigl(r=|w|,\ t=\arg w\bigr),$$

since $1+x\leqslant e^x$, $x\in\mathbb{R}$. Thus

$$\left|(1-w)\exp\left[\sum_{m=1}^{p-1}\frac{w^m}{m}\right]\right|\leqslant\exp\bigl[|w|^p\bigr]\quad(p=1,2,3,\ldots;\ w\in\mathbb{C}).\tag{1.3}$$

Hence,

$$\bigl|\det{}_\pi(I-A)\bigr|\leqslant\prod_{j=1}^{\infty}\exp\bigl[\bigl|\lambda_j(A)\bigr|^{p_j}\bigr]=\exp\left[\sum_{j=1}^{\infty}\bigl|\lambda_j(A)\bigr|^{p_j}\right].$$

Let us check that

$$\sum_{j=1}^{\infty}\bigl|\lambda_j(A)\bigr|^{p_j}\leqslant\sum_{j=1}^{\infty}s_j^{p_j}(A).$$

Indeed, let N be an arbitrary positive integer number and A be an arbitrary compact operator. Weyl's inequality (see e.g. [15, Theorem 1.15]) implies that

$$\sum_{j=1}^{N}\bigl|\lambda_j(A)\bigr|\leqslant\sum_{j=1}^{N}s_j(A).$$

Taking into account that $|\lambda_j(A)|$ and $s_j(A)$ are non-increasing sequences and the function

$$f(x_1,\ldots,x_N)=\sum_{j=1}^{N}|x_j|^{p_j}$$

is a convex function of $(x_1,\ldots,x_N)\in\mathbb{C}^N$, by Markus' theorem (see e.g. [15, Theorem 1.9]),

$$\sum_{j=1}^{N}\bigl|\lambda_j(A)\bigr|^{p_j}\leqslant\sum_{j=1}^{N}s_j^{p_j}(A).$$

Passing to the limits, we get the required inequality, and thus we prove the lemma. □

Lemma 1.2. *Let A and B be compact operators such that $\gamma_\pi(A)<\infty$. Then*

$$\gamma_\pi\bigl((A+B)/2\bigr)\leqslant\frac{1}{2}\bigl(\gamma_\pi(A)+\gamma_\pi(B)\bigr)\quad(A,B\in\Gamma_\pi).$$

Proof. By the Hölder inequality, we have

$$(a+b)^p \leqslant 2^{p/p'}\big(a^p+b^p\big) = 2^{p-1}\big(a^p+b^p\big) \quad (p>1;\ 1/p+1/p'=1;\ a,b>0).$$

So

$$\gamma_\pi\big((A+B)/2\big) = \sum_{k=1}^{\infty} s_k^{p_k}\big((A+B)/2\big) \leqslant \sum_{k=1}^{\infty} 2^{-p_k}\big(s_k(A)+s_k(B)\big)^{p_k}$$
$$\leqslant \sum_{k=1}^{\infty} 2^{-p_k}2^{p_k-1}\big(s_k^{p_k}(A)+s_k^{p_k}(B)\big) = \frac{1}{2}\big(\gamma_\pi(A)+\gamma_\pi(B)\big),$$

as claimed. □

Obviously, $A \in \Gamma_\pi$ if and only if $\gamma_\pi(tA) < \infty$ for all $t \geqslant 0$.

Lemma 1.3. *Γ_π is a linear space.*

Proof. Let $A, B \in \Gamma_\pi$. Then for any $c \in \mathbb{C}$ and all $t > 0$,

$$\gamma_\pi(ctA) := \sum_{j=1}^{\infty} s_j^{p_j}(ctA) = \sum_{j=1}^{\infty} s_j^{p_j}(A)\big(|c|t\big)^{p_j} < \infty.$$

Moreover, by the previous lemma

$$\gamma_\pi(tA+tB) \leqslant \frac{1}{2}\big(\gamma_\pi(2tA)+\gamma_\pi(2tB)\big) < \infty.$$

This proves the lemma. □

Notice that Γ_π can be smaller than the set of all compact operators A satisfying $\gamma_\pi(A) < \infty$. Besides, the set of all A such that $\gamma_\pi(A) < \infty$ is convex. However, this set is not linear in general. Now we are going to introduce a norm in Γ_π. Let $f = \{f_j\}_{j=1}^{\infty}$ be a sequence of complex numbers. Put

$$\rho(f) = \sum_{j=1}^{\infty} |f_j|^{p_j}.$$

The *Nakano space* [14] ℓ^{p_j} is the set of all sequences f such that $\rho(f/\lambda) < \infty$ for some $\lambda > 0$. Consider also the set $\ell_0^{p_j}$ of all sequences f such that $\rho(f) < \infty$ and the set h^{p_j} of all sequences f such that $\rho(f/\lambda) < \infty$ for all $\lambda > 0$. It is clear that $h^{p_j} \subset \ell_0^{p_j} \subset \ell^{p_j}$. It is well known that ℓ^{p_j} is a Banach space under the norm

$$\|f\|_{\ell^{p_j}} := \inf_{\lambda>0} \rho(f/\lambda)$$

and h^{p_j} is its closed subspace. Moreover, $\ell_0^{p_j}$ is not a linear space in general. Nakano spaces are a particular case of Musielak–Orlicz spaces (see Lindenstrauss and Tzafriri [11] and Musielak [13]). From Proposition 4.d.3 of [11] it follows that $\ell^{p_j} = h^{p_j}$ if and only if $\{p_j\}_{j=1}^{\infty}$ is a bounded sequence.

Denote by $\widetilde{\Gamma}_\pi$ the set of all compact operators A on a separable Hilbert space such that $\{s_j(A)\}_{j=1}^{\infty} \in \ell^{p_j}$. Γ_π is its subset, since Γ_π is the set of all compact operators A satisfying $\{s_j(A)\}_{j=1}^{\infty} \in h^{p_j}$. It is now clear that $\widetilde{\Gamma}_\pi$ is a Banach space under the norm

$$\|A\|_\pi := \big\|\{s_j(A)\}_{j=1}^{\infty}\big\|_{\ell^{p_j}}$$

and Γ_π is its closed subspace. It is obvious that if $p_j = p \in [1,\infty)$ for all j, then both spaces $\widetilde{\Gamma}_\pi$ and Γ_π coincide with the Schatten–von Neumann class S_p.

So the following result is true.

Lemma 1.4. *The set $\widetilde{\Gamma}_\pi$ is a closed normed two-sided ideal of the algebra of all bounded linear operators on H. That is, if $A \in \widetilde{\Gamma}_\pi$ and T is a bounded linear operator, then*

$$\|AT\|_\pi \leqslant \|A\|_\pi \|T\|, \qquad \|TA\|_\pi \leqslant \|T\|\|A\|_\pi. \tag{1.4}$$

Proof. It is well known that $s_j(AT) \leqslant s_j(A)\|T\|$ for all j (see e.g. [8, Chap. II, Section 2]). Assume that $\|A\|_\pi > 0$ and $\|T\| > 0$ (otherwise the proof is obvious). Then from the above inequality and the definition of the norm $\|\cdot\|_\pi$ it follows that

$$\sum_{j=1}^{\infty}\left(\frac{s_j(AT)}{\|A\|_\pi\|T\|}\right)^{p_j} \leqslant \sum_{j=1}^{\infty}\left(\frac{s_j(A)}{\|A\|_\pi}\right)^{p_j} \leqslant 1.$$

Then $\|AT\|_\pi \leqslant \|A\|_\pi\|T\|$. The second inequality is proved similarly. $\square$

From Lemmas 1.3 and 1.4 and the fact that Γ_π is a closed subspace of $\widetilde{\Gamma}_\pi$ it follows that Γ_π is also a closed normed two-sided ideal of the Banach algebra of all bounded linear operators and inequalities (1.4) hold.

2. The main result

Suppose that for a number $c_0 > 1$, the condition

$$w(c_0) := \sum_{j=1}^{\infty}\frac{1}{c_0^{p_j}} < \infty \tag{2.1}$$

holds. For example, let $c_0 = e$ and $p_j = 2[\ln j + 1]$ where $[x]$ means the integer part of x. Then

$$w(c_0) \leqslant \sum_{j=1}^{\infty}\frac{1}{j^2} = \zeta(2) < \infty$$

where $\zeta(.)$ is the Riemann Zeta function.

Now we are in a position to formulate the main result of the paper.

Theorem 2.1. *Under condition* (2.1), *let* $A, B \in \Gamma_\pi$. *Then*

$$\left|\det_\pi(I-A) - \det_\pi(I-B)\right| \leqslant 2c_0 e^{w_0(c_0)}\|A-B\| \exp\left[\gamma_\pi\big(2(A+B)\big) + \gamma_\pi\big(2(A-B)\big)\right].$$

To prove this result we need a scalar complex-valued function f defined on Γ_π, such that $f(A+\lambda B)$ $(\lambda \in \mathbb{C})$ is an entire function for all $A, B \in \Gamma_\pi$ and there is a non-decreasing function $G : [0,\infty) \to [0,\infty)$ satisfying the inequality

$$\left|f(A)\right| \leqslant G\big(\gamma_\pi(A)\big) \quad (A \in \Gamma_\pi). \tag{2.2}$$

Lemma 2.2. *Let a function* $f(A+\lambda B)$ $(\lambda \in \mathbb{C})$ *be entire for all* $A, B \in \Gamma_\pi$ *and conditions* (2.1), (2.2) *hold. Then*

$$\left|f(A) - f(B)\right| \leqslant 2c_0\|A-B\| G\big(w(c_0) + \gamma_\pi\big(2(A+B)\big) + \gamma_\pi\big(2(A-B)\big)\big).$$

Proof. For the brevity put $\gamma_\pi(.) = \gamma(.)$. Introduce the function

$$g(\lambda) = f\left(\frac{1}{2}(A+B) + \lambda(A-B)\right).$$

Then $g(\lambda)$ is an entire function and thanks to the Cauchy integral,

$$f(A) - f(B) = g(1/2) - g(-1/2) = \frac{1}{2\pi i}\oint_{|z|=1/2+r}\frac{g(z)\,dz}{(z-1/2)(z+1/2)} \quad (r > 0).$$

So

$$\left|g(1/2) - g(-1/2)\right| \leqslant (1/2+r)\sup_{|z|=1/2+r}\frac{|g(z)|}{|z^2-1/4|}.$$

But

$$\left|z^2 - 1/4\right| = \left|(r+1/2)^2 e^{i2t} - 1/4\right| \geqslant (r+1/2)^2 - 1/4 = r^2 + r \quad \big(z = (1/2+r)e^{it},\ 0 \leqslant t < 2\pi\big). \tag{2.3}$$

In addition, by (2.2),

$$\begin{aligned}\left|g(z)\right| &= \left|f\left(\frac{1}{2}(A+B) + z(A-B)\right)\right| = \left|f\left(\frac{1}{2}(A+B) + \left(r+\frac{1}{2}\right)e^{it}(A-B)\right)\right| \\ &\leqslant G\left[\gamma\left(\frac{1}{2}(A+B) + \left(r+\frac{1}{2}\right)e^{it}(A-B)\right)\right].\end{aligned}$$

Consequently,

$$\left|f(A)-f(B)\right| \leqslant \frac{1}{r}G\left[\gamma\left(\frac{1}{2}(A+B)+\left(r+\frac{1}{2}\right)e^{it}(A-B)\right)\right]. \tag{2.4}$$

Take into account that γ is a convex function. So

$$\gamma\left(\frac{1}{2}(A+B)+\left(\frac{1}{2}+r\right)e^{it}(A-B)\right) \leqslant \gamma\big(2(A+B)\big)+\gamma\big(2(A-B)\big)+\gamma\big(2r(A-B)\big)$$

and

$$\gamma\big(2r(A-B)\big)=\sum_{j=1}^{\infty}(2r)^{p_j}s_j^{p_j}(A-B).$$

Take

$$r=\frac{1}{2c_0\|A-B\|}.$$

Since $s_k(A)\leqslant\|A\|$, we get

$$\gamma\big(2r(A-B)\big)=\sum_{j=1}^{\infty}\frac{s_j^{p_j}(A-B)}{(c_0\|A-B\|)^{p_j}}\leqslant w(c_0).$$

Now (2.4) implies the required result. □

The assertion of Theorem 2.1 directly follows from the previous lemma and Lemma 1.1, since $\det_\pi(I-A-\lambda B)$ is an entire function.

3. Lower bounds

In this section for the brevity we put $\lambda_j(A)=\lambda_j$. Denote by L *a Jordan contour connecting* 0 *and* 1, *lying in the disc* $\{z\in\mathbb{C}\colon |z|\leqslant 1\}$ *and does not containing the points* $1/\lambda_j$ for any eigenvalue λ_j, such that

$$\phi(A):=\inf_{s\in L;\,k=1,2,\ldots}|1-s\lambda_k|>0. \tag{3.1}$$

Theorem 3.1. *Under condition* (3.1), *let* $A\in\Gamma_\pi$ *and* $1\notin\sigma(A)$. *Then*

$$\left|\det_\pi(I-A)\right|\geqslant\exp\left[-\frac{1}{\phi(A)}\sum_{k=1}^{\infty}s_k^{p_k}(A)J_{p_k}\right]$$

where

$$J_p:=\int_L|s|^{p-1}|ds|.$$

Proof. Put

$$D(z)=\det_\pi(I-zA)\quad(z\in L).$$

We have

$$D(z)=\prod_{j=1}^{\infty}E_j(z)\quad\text{where } E_j(z):=(1-z\lambda_j)\exp\left[\sum_{m=1}^{p_j-1}\frac{z^m\lambda_j^m}{m}\right].$$

Clearly,

$$D'(z)=\sum_{k=1}^{\infty}E_k'(z)\prod_{j=1,\,j\neq k}^{\infty}E_j(z)$$

and

$$E_k'(z)=\left[-\lambda_k+(1-z\lambda_k)\sum_{m=0}^{p_k-2}z^m\lambda_k^{m+1}\right]\exp\left[\sum_{s=1}^{p_k-1}\frac{z^s\lambda_k^s}{s}\right].$$

But

$$-\lambda_j + (1 - z\lambda_j) \sum_{m=0}^{p_j-2} z^m \lambda_j^{m+1} = -z^{p_j-1}\lambda_j^{p_j},$$

since

$$\sum_{m=0}^{p_j-2} z^m \lambda_j^m = \frac{1-(z\lambda_j)^{p_j-1}}{1-z\lambda_j}.$$

So

$$E_j'(z) = -z^{p_j-1}\lambda_j^{p_j} \exp\left[\sum_{m=1}^{p_j-1} \frac{z^m\lambda_j^m}{m}\right] = -\frac{z^{p_j-1}\lambda_j^{p_j}}{1-z\lambda_j} E_j(z).$$

Hence, $D'(z) = h(z)D(z)$, where

$$h(z) := -\sum_{k=1}^{\infty} \frac{z^{p_k-1}\lambda_k^{p_k}}{1-z\lambda_k}.$$

Consequently,

$$D(1) = \det_\pi(I-A) = \exp\left[\int_L h(s)\,ds\right].$$

Thus

$$\left|\int_L h(s)\,ds\right| \leqslant \sum_{k=1}^{\infty} |\lambda_k|^{p_k} \int_L \frac{|s|^{p_k-1}\,|ds|}{|1-s\lambda_k|} \leqslant \sum_{k=1}^{\infty} s_k^{p_k}(A)\phi^{-1}(A) J_{p_k}$$

and therefore

$$\left|\det_\pi(I-A)\right| = \left|\exp\left[\int_L h(s)\,ds\right]\right| \geqslant \exp\left[-\left|\int_L h(s)\,ds\right|\right] \geqslant \exp\left[-\phi^{-1}(A)\sum_{k=1}^{\infty} s_k^{p_k}(A) J_{p_k}\right],$$

as claimed. □

The latter theorem generalizes a previous result by the author [6, Theorem 2.1] for Schatten–von Neumann classes S_p. The proof is essentially the same.

Let $l = |L|$ be the length of L. Then $J_{p_k} \leqslant l$ since $|s| \leqslant 1$ for any $s \in L$. Now the previous theorem implies

Corollary 3.2. *Under condition* (3.1) *let* $A \in \Gamma_\pi$ *and* $1 \notin \sigma(A)$. *Then*

$$\left|\det_\pi(I-A)\right| \geqslant \exp\left[-\frac{l\gamma_\pi(A)}{\phi(A)}\right].$$

Furthermore, if A does not have the eigenvalues on $[1,\infty)$, then one can take $L = [0,1]$. In this case $l = 1$ and therefore

$$J_{p_k} \leqslant \frac{1}{p_k}.$$

Let $\sigma(A)$ be the spectrum of A. Now the previous theorem implies

Corollary 3.3. *Under condition* (3.1), *let* $A \in \Gamma_\pi$ *and* $[1,\infty) \cap \sigma(A) = 0$. *Then*

$$\left|\det_\pi(I-A)\right| \geqslant \exp\left[-\frac{1}{\phi(A)}\sum_{k=1}^{\infty} \frac{s_k^{p_k}(A)}{p_k}\right].$$

If, in addition, the spectral radius $r_s(A)$ *of* A *is less than one, then*

$$\left|\det_\pi(I-A)\right| \geqslant \exp\left[-\frac{1}{1-r_s(A)}\sum_{k=1}^{\infty} \frac{s_k^{p_k}(A)}{p_k}\right].$$

4. Schatten–von Neumann ideals

In this section we improve Theorem 2.1 in the case of the Schatten–von Neumann operators S_p, i.e. $p_j \equiv p$ for an integer $p \geqslant 3$, and

$$N_p(A) := \left[\sum_{k=1}^{\infty} s_k^p(A)\right]^{1/p} < \infty, \quad \text{and} \quad \det_p(I-A) := \prod_{j=1}^{\infty}\bigl(1-\lambda_j(A)\bigr)\exp\left[\sum_{m=1}^{p-1}\frac{\lambda_j^m(A)}{m}\right].$$

Let X and Y be complex normed spaces with norms $\|.\|_X$ and $\|.\|_Y$, respectively and F a Y-valued function defined on X. Assume that $F(C+\lambda\tilde{C})$ $(\lambda \in \mathbb{C})$ is an entire function for all $C, \tilde{C} \in X$. That is, for any $\phi \in Y^*$, the functional $\langle \phi, F(C+\lambda\tilde{C})\rangle$ defined on Y is an entire scalar-valued function. Let us prove the following technical lemma.

Lemma 4.1. *Let $F(C+\lambda\tilde{C})$ $(\lambda \in \mathbb{C})$ be an entire function for all $C, \tilde{C} \in X$ and there be a monotone non-decreasing function $G : [0,\infty) \to [0,\infty)$, such that*

$$\bigl\|F(C)\bigr\|_Y \leqslant G\bigl(\|C\|_X\bigr) \quad (C \in X). \tag{4.1}$$

Then

$$\bigl\|F(C)-F(\tilde{C})\bigr\|_Y \leqslant \|C-\tilde{C}\|_X G\left(1+\frac{1}{2}\|C+\tilde{C}\|_X+\frac{1}{2}\|C-\tilde{C}\|_X\right) \quad (C, \tilde{C} \in X).$$

Proof. Put $g_1(\lambda) = F(\frac{1}{2}(C+\tilde{C})+\lambda(C-\tilde{C}))$. Then $g_1(\lambda)$ is an entire function and $F(C)-F(\tilde{C}) = g_1(1/2)-g_1(-1/2)$. Thanks to the Cauchy integral,

$$g_1(1/2)-g_1(-1/2) = \frac{1}{2\pi i}\oint_{|z|=1/2+r}\frac{g_1(z)\,dz}{(z-1/2)(z+1/2)} \quad (r>0).$$

Hence, by (2.3),

$$\bigl\|g_1(1/2)-g_1(-1/2)\bigr\|_Y \leqslant (1/2+r)\sup_{|z|=1/2+r}\frac{\|g_1(z)\|_Y}{|z^2-1/4|} \leqslant \frac{1}{r}\sup_{|z|=1/2+r}\bigl\|g_1(z)\bigr\|_Y.$$

In addition,

$$\begin{aligned}\bigl\|g_1(z)\bigr\|_Y &= \left\|F\left(\frac{1}{2}(C+\tilde{C})+z(C-\tilde{C})\right)\right\|_Y = \left\|F\left(\frac{1}{2}(C+\tilde{C})+\left(r+\frac{1}{2}\right)e^{it}(C-\tilde{C})\right)\right\|_Y \\ &\leqslant G\left(\frac{1}{2}\|C+\tilde{C}\|_X+\left(\frac{1}{2}+r\right)\|C-\tilde{C}\|_X\right) \quad \left(|z|=\frac{1}{2}+r\right).\end{aligned}$$

Therefore according to (4.1),

$$\bigl\|F(C)-F(\tilde{C})\bigr\|_Y = \bigl\|g_1(1/2)-g_1(-1/2)\bigr\|_Y \leqslant \frac{1}{r}G\left(\frac{1}{2}\|C+\tilde{C}\|_X+\left(\frac{1}{2}+r\right)\|C-\tilde{C}\|_X\right).$$

Taking $r = 1/\|C-\tilde{C}\|_X$, we get the required result. □

Corollary 4.2. *Let $A, B \in S_p$ $(p = 2, 3, \ldots)$. Then*

$$\bigl|\det_p(I-A)-\det_p(I-B)\bigr| \leqslant N_p(A-B)\exp\left[\frac{p-1}{p}\left(1+\frac{1}{2}\bigl(N_p(A+B)+N_p(A-B)\bigr)\right)^p\right].$$

Indeed, by (1.2) we easily have

$$\bigl|\det_p(I-A)\bigr| \leqslant \exp\left[\frac{p-1}{p}N_p^p(A)\right] \quad (p \geqslant 2). \tag{4.2}$$

Take in $Y = \mathbb{R}$, $X = S_p$, $\|.\|_X = N_p(.)$. Besides, $\det_p(I-A-B\lambda)$ $(\lambda \in \mathbb{C};\ A, B \in S_p)$ is an entire function of λ. Now the required result follows from the previous lemma.

The latter corollary improves the well-known Theorem 11.2.2 [7].

Clearly, Theorem 3.1 and Corollary 3.2 are true for operators from S_p. Corollary 3.3 takes the form

Corollary 4.3. *Under condition* (3.1)*, let* $A \in S_p$ $(p = 1, 2, \ldots)$ *and* $[1, \infty) \cap \sigma(A) = 0$*. Then*

$$\left|\det_p(I - A)\right| \geqslant \exp\left[-\frac{1}{\phi(A)p} N_p^p(A)\right].$$

If, in addition, the spectral radius $r_s(A)$ *of* A *is less than one, then*

$$\left|\det_p(I - A)\right| \geqslant \exp\left[-\frac{N_p^p(A)}{(1 - r_s(A))p}\right].$$

One can easily see that conditions of Theorem 2.1 hold only for unbounded sequences p_j (because of (2.1)). Let us prove also some unconditional result (without (2.1)) which includes also the classical case of constant p. It is formulated in terms of the norm $\|A - B\|_\pi$ instead of $\|A - B\|$. That result is based on the following.

Lemma 4.4. *Let a function* $f(A + \lambda B)$ *be an entire function of* $\lambda \in \mathbb{C}$ *for all* $A, B \in \Gamma_\pi$ *and* $G : [0, \infty) \to [0, \infty)$ *be a non-decreasing function satisfying* $|f(A)| \leqslant G(\gamma_\pi(A))$ *for all* $A \in \Gamma_\pi$*. Then*

$$\left|f(A) - f(B)\right| \leqslant 2\|A - B\|_\pi G\left(\frac{1}{2} + \frac{1}{4}\gamma_\pi\big(2(A + B)\big) + \frac{1}{4}\gamma_\pi\big(2(A - B)\big)\right). \tag{4.3}$$

Proof. As in the proof of Lemma 2.2 one can get

$$\left|f(A) - f(B)\right| \leqslant \frac{1}{r} G\left(\gamma_\pi\left(\frac{1}{2}(A + B) + \left(\frac{1}{2} + r\right)e^{it}(A - B)\right)\right) \tag{4.4}$$

for every $r > 0$ and $t \in [0, 2\pi)$. Notice that the operator inside $\gamma_\pi(\cdot)$ belongs to Γ_π because Γ_π is a linear space, hence $\gamma_\pi(\cdot)$ is well defined for this operator. Applying Lemma 1.2 two times, one can get

$$\begin{aligned}
\gamma_\pi\left(\frac{1}{2}(A + B) + \left(\frac{1}{2} + r\right)e^{it}(A - B)\right) &\leqslant \frac{1}{2}\gamma_\pi\big(A + B + e^{it}(A - B)\big) + \frac{1}{2}\gamma_\pi\big(2re^{it}(A - B)\big) \\
&\leqslant \frac{1}{4}\gamma_\pi\big(2(A + B)\big) + \frac{1}{4}\gamma_\pi\big(2e^{it}(A - B)\big) + \frac{1}{2}\gamma_\pi\big(2re^{it}(A - B)\big) \\
&= \frac{1}{4}\gamma_\pi\big(2(A + B)\big) + \frac{1}{4}\gamma_\pi\big(2(A - B)\big) + \frac{1}{2}\gamma_\pi\big(2r(A - B)\big).
\end{aligned} \tag{4.5}$$

Note that the last step is justified because $s_j(e^{it}(C)) = |e^{it}|s_j(C) = s_j(C)$ for every compact operator C and every j, hence $\gamma_\pi(e^{it}C) = \gamma_\pi(C)$.

Take

$$r = \big(2\|A - B\|_\pi\big)^{-1}.$$

From the definition of $\|\cdot\|_\pi$ it follows that

$$\gamma_\pi\big(2r(A - B)\big) = \sum_{j=1}^{\infty}\left(\frac{s_j(A - B)}{\|A - B\|_\pi}\right)^{p_j} \leqslant 1. \tag{4.6}$$

Combining (4.4)–(4.6), we arrive at (4.3). □

Thanks to the previous lemma, an analogue of Theorem 2.1 reads as follows.

Theorem 4.5. *If* $A, B \in \Gamma_\pi$*, then*

$$\left|\det_\pi(I - A) - \det_\pi(I - B)\right| \leqslant 2\|A - B\|_\pi \exp\left(\frac{1}{2} + \frac{1}{4}\gamma_\pi\big(2(A + B)\big) + \frac{1}{4}\gamma_\pi\big(2(A - B)\big)\right).$$

The results of this section supplement the recent very interesting investigations of ideals S_p, cf. [1,5,16,19,20].

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References

[1] P. Chaisuriya, S.C. Ong, Schatten's theorems on functionally defined Schur algebras, Int. J. Math. Math. Sci. 2005 (14) (2005) 2175–2193.
[2] N. Dunford, J.T. Schwartz, Linear Operators, Part II. Spectral Theory, Interscience Publishers, New York, London, 1963.
[3] T. Ehrhardt, Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel, Comm. Math. Phys. 262 (2) (2006) 317–341.
[4] J. Fujii, M. Nakamura, Y. Seo, Operator valued determinant and Hadamard product, Sci. Math. Jpn. 60 (1) (2004) 1–8.
[5] L.G. Gheorghe, Hankel operators in Schatten ideals, Ann. Mat. Pura Appl. (4) 180 (2) (2001) 203–210.
[6] M.I. Gil', Upper and lower bounds for regularized determinants, JIPAM. J. Inequal. Pure Appl. Math. 9 (1) (2008), Article 2, 6 pp. http://jipam.vu.edu.au.
[7] I. Gohberg, S. Goldberg, N. Krupnik, Traces and Determinants of Linear Operators, Birkhäuser Verlag, Basel, 2000.
[8] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monogr., vol. 18, Amer. Math. Soc., Providence, RI, 1969.
[9] I.C. Gohberg, M.G. Krein, Theory and Applications of Volterra Operators in Hilbert Space, Transl. Math. Monogr., vol. 24, Amer. Math. Soc., Providence, RI, 1970.
[10] A.Yu. Karlovich, Higher order asymptotics of Toeplitz determinants with symbols in weighted Wiener algebras, J. Math. Anal. Appl. 320 (2006) 944–963.
[11] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I. Sequence Spaces, Springer, Berlin, 1977.
[12] D.H.U. Marchetti, An alternative to Plemelj–Smithies formulas on infinite determinants, J. Funct. Anal. 117 (1993) 360–376.
[13] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin, 1983.
[14] H. Nakano, Modulared Semi-ordered Linear Spaces, Maruzen Co., Ltd., Tokyo, 1950.
[15] B. Simon, Trace Ideals and Their Applications, 2nd edition, Amer. Math. Soc., 2005.
[16] M. Sigg, A Minkowski-type inequality for the Schatten norm, JIPAM. J. Inequal. Pure Appl. Math. 6 (3) (2005), Paper No. 87, 7 pp., electronic only.
[17] Hans Henrik Rugh, Intermittency and regularized Fredholm determinants, Invent. Math. 135 (1) (1999) 1–24.
[18] O. Tsuchiya, Determinant formula for the six-vertex model with reflecting end, J. Math. Phys. 39 (11) (1998) 5946–5951.
[19] M.W. Wong, Schatten–von Neumann norms of localization operators, Arch. Inequal. Appl. 2 (4) (2004) 391–396.
[20] J. Xia, On the Schatten class membership of Hankel operators on the unit ball, Illinois J. Math. 46 (3) (2002) 913–928.