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Approximation to x^n by Lower Degree Rational Functions

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Recently it was discovered that effective approximations to x^n by polynomials of degree k were possible *if and only if* k was *much* larger than $n^{1/2}$ (see [1]). In this note we consider this same problem with the word "polynomial" replaced by "rational function." Interestingly there is then no necessary restriction on k! Effective approximation is possible as long as k is large—independent of n. (Score another one for rational approximation!)

Set $S(x) = \sum_{i=0}^{k} \binom{n+i-1}{i} (1-x)^i$ (the kth partial sum of the power series for x^{-n}). Our result is that

$$\frac{1}{S(x)} - x^n \leqslant \frac{2}{k} \qquad \text{for } 0 \leqslant x \leqslant 1, \tag{1}$$

which is the quantitative form of our assertion above (the left-hand side being clearly nonnegative). In fact we shall prove

$$\frac{1}{S(x)} - x^n \leqslant \frac{2}{k}\left(\frac{2n-2}{2n+k}\right)^{n-1} \qquad \text{for} \quad 0 \leqslant x \leqslant 1. \tag{2}$$

Equation (2) indeed shows that the approximation gets better as n gets larger. The quantity $((2n-2)/(2n+k))^{n-1}$ *decreases* with n and so, although for $n = 1$ we obtain an error estimate of $2/k$, for all $n \geqslant 2$ we obtain $4/k(k+4)$ while for $n \geqslant 3$ we get $32/k(k+6)^2$, etc.

We use the explicit formula for the remainder term of a power series expansion. In our case this gives

$$\begin{aligned} S(x) &= x^{-n} - \int_x^1 \frac{(t-x)^k}{k!}\left(\frac{d}{dt}\right)^{k+1} t^{-n}\,dt \\ &= x^{-n}\left(1 - C\int_x^1 \left(1-\frac{x}{t}\right)^k \left(\frac{x}{t}\right)^n \frac{dt}{t}\right), \qquad C \text{ constant.} \end{aligned}$$

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Next we change variables by writing

$$u = \left(\frac{x}{t}\right)^n, \; z = x^n, \qquad \text{and} \qquad \epsilon = \frac{2}{k}\left(\frac{2n-2}{2n+k}\right)^{n-1}, \tag{3}$$

so that our formula for $S(x)$ becomes

$$S(x) = z^{-1}(1 - cI(z)), \qquad I(z) = \int_z^1 (1 - u^{1/n})^k \, du, \tag{4}$$

where c is a constant. By letting $z \to 0$ we obtain $c = 1/I(0)$ and

$$S(x) = \frac{1}{z}\left(1 - \frac{I(z)}{I(0)}\right). \tag{5}$$

Using (5) we find that (2) may be written

$$\frac{z}{1 - I(z)/I(0)} - z \leqslant \epsilon, \qquad \text{or} \qquad (z + \epsilon)\, I(z) \leqslant \epsilon I(0)$$

which is to say

$$\text{on } [0, 1], (z + \epsilon)\, I(z) \text{ takes its maximum at } 0. \tag{6}$$

We show, in fact, by direct differentiation, that $(z + \epsilon)\, I(z)$ is convex on $[0, 1]$. This forces the maximum to be taken at an endpoint which must be 0 as $I(1) = 0$. We have, namely,

$$\begin{aligned}((z + \epsilon)\, I(z))'' &= 2I'(z) + (z + \epsilon)\, I''(z) \\ &= -2(1 - z^{1/n})^k + (z + \epsilon)\, k(1 - z^{1/n})^{k-1} \frac{1}{n} z^{1/n-1} \\ &= \frac{(1 - z^{1/n})^{k-1}}{n} [(k + 2n)\, z^{1/n} + k\epsilon z^{1/n-1} - 2n]\end{aligned}$$

and so we need only prove that

$$(k + 2n)^{1/n} + k\,\epsilon\, z^{1/n-1} \geqslant 2n. \tag{7}$$

If we write $w = (2n-2)/(k+2n)\, z^{-1/n}$ and recall the definition of ϵ in (3), we find that (7) becomes $(2n-2)/w + 2w^{n-1} \geqslant 2n$ or $(w - 1)$ $((w^{n-1} + w^{n-2} + \cdots + 1) - n \geqslant 0$. Both factors are $\geqslant 0$ if $w \geqslant 1$ and $\leqslant 0$ if $w \leqslant 1$, and in either case, our result follows.

REFERENCE

1. D. J. NEWMAN AND T. J. RIVLIN, Approximation of monomials by polynomials of lower degree, *Aequationes Math.* **14** (1976), 451–455.