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Weierstrass Points and Monomial Curves

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1. INTRODUCTION

Let B_H be a semigroup ring over a fixed algebraically closed field k of characteristic 0, i.e., $B_H = k[t^n \mid n \in H]$ where t is transcendental over k. The following work is devoted to the smoothing question for B_H and related problems. We say B_H can be smoothed if there exists a deformation of B_H over R:

$$\begin{array}{ccc} A & \longrightarrow & B_H \\ {\scriptstyle \text{flat}}\uparrow & & \uparrow \\ R & \longrightarrow & k \end{array}$$

s.t. R is a Noetherian k-algebra without zero divisors and the special fiber $A/\mathbf{m}_R A$ is isomorphic to B_H for some maximal ideal $\mathbf{m}_R$ of R while the generic fiber is smooth over the fraction field of R.

Severi conjectured that every variety is the "limit" of nonsingular varieties. Latter day geometers took this to mean every variety can be obtained as the specialization of a nonsingular variety. Doubt was shed on this conjecture by an anonymous correspondent [18] who provided an example of a five-dimensional projective variety which cannot be smoothed in a fixed embedding. Grauert and Kerner [5] have constructed a series of nonsmoothable varieties in dimension n, provided that $n \geqslant 4$ while Rim [15] constructs a rigid isolated singularity on an irreducible rational surface. At that time the question was still open for curves.

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Mumford [11] has recently given a nonconstructive proof that shows most curves cannot be smoothed. Pinkham [12] has given the example of m lines through the origin in general linear position in d space which is not smoothable provided that $m \gg d$.

The smoothability of monomial curves (i.e., irreducible affine curves with $\mathbb{G}_m$ action) including the semigroup ring subcase remains an open question. In Section 4 of the following work we classify those numerical semigroups H for which B_H is negatively graded (see Section 4.0 for definition and details). In Section 5 we describe a method which allows us to smooth a class of semigroup rings including those which are negatively graded. Thus by the work by Pinkham [12], given a negatively graded semigroup H there exists a smooth projective curve X with base point x s.t. H occurs as the order of poles at x of rational functions on X, regular on $X - \{x\}$. Then if X is nonordinary (i.e., H is not of the form $\{0, g + 1, g + 2, g + 3, \ldots\}$), the point x is a Weierstrass point for X with gap sequence specified by H. In the final section we improve a formula by Rauch [13] on the dimension of a subspace of the coarse moduli space $\mathcal{M}_{g,1}$.

2. Preliminaries and the Standard Basis for H

Let H be a subsemigroup of the additive group $\mathbb{N}$ of nonnegative integers. H is called a *numerical semigroup* if the greatest common divisor of the elements of H is 1, so that only finitely many positive integers are missing from H. Such elements are called the *gaps* of H and the number of *gaps* is called the *genus* of H, denoted by $g(H)$. The least positive integer c such that $c + \mathbb{N} \subset H$ is called the *conductor* of H, denoted by $c(H)$. The least positive integer m in H is called the *multiplicity* (or the *transversal generator*) of H. Throughout this paper H will denote a numerical semigroup, k an algebraically closed field of characteristic 0.

Definition 2.0. Let B_H be the subring of the polynomial ring $k[t]$ generated by the monomials t^h, $h \in H$. B_H is called the *semigroup ring* of H.

Where no possible confusion can arise we write B for B_H. Let $\mathfrak{m}$ denote the maximal ideal of B generated by t^h, $h \in H - \{0\}$. We make the following observations.

Proposition 2.1. *Let H be of multiplicity m.*

(i) $\bar{B} = k[t]$ *where $\bar{A}$ denotes the integral closure of A in its total ring of fractions and* $g(H) = \dim \bar{B}/B$.

(ii) *B is smooth over k if and only if $H = \mathbb{N}$. If not, B has an isolated singularity at $\mathfrak{m}$ and $m = e(B_\mathfrak{m})$ (the multiplicity of the local ring).*

Let H^+ denote the positive integers of H. We construct a generating set called the *standard basis* for H, noted S_H, inductively as follows:

Let $n_0 = m$. If $n_0 < \cdots < n_i$ have been chosen and $i < m - 1$ let $n_{i+1} = \min\{n \in H^+ \mid n \in H - \bigcup_{j \leqslant i} \{n_j + m\mathbb{N}\}\}$, i.e., n_{i+1} is the least integer in H having m-residue distinct from those of $n_0, \ldots, n_i$. Unless otherwise stated the residues throughout are assumed to be modulo m.

PROPOSITION 2.2. *Let $S_H = \{m = n_0 < n_1 < \cdots < n_{m-1}\}$ be the standard basis for H. Then*

(i) $c(H) = n_{m-1} - n_0 + 1.$ (1)

(ii) $g(H) = \sum_{i=1}^{m-1} [n_i/n_0]$ *where* $[x]$ *denotes the greatest integer* $\leqslant x$. (2)

(iii) *Let* $l(H) = [H: c(H) + \mathbb{N}]$. *Then* $l(H) = \sum_{i=0}^{m-1} [n_{m-1} - n_i/n_0] + 1.$ (3)

Proof. (i) Suppose $n \geqslant n_{m-1} - n_0 + 1$. Since the elements of S_H form a complete residue system modulo m, we can write $n = n_j + am$ where $0 \leqslant j \leqslant m - 1$, $a \in \mathbb{Z}$. If $a < 0$, $n \leqslant n_j - m < n_{m-1} - n_0 + 1$, a contradiction. So $a \geqslant 0$ and $n \in H$. Now $n_{m-1} - n_0 \notin H$ since n_{m-1} is the least integer in H having given m-residue. Hence

$$c(H) = n_{m-1} - n_0 + 1.$$

(ii) Since S_H is a complete residue system modulo m,

$$g(H) = \sum_{j=1}^{m-1} \#(n \in \mathbb{N} - H \mid n \equiv n_j)$$
$$= \sum_{j=1}^{m-1} [n_j/n_0].$$

(iii) Similarly, if $l(H)$ denotes the number of elements in $H <$ the conductor of H,

$$l(H) = \sum_j \#\{n \in H - (c + \mathbb{N}) \mid n \equiv n_j\}$$
$$= \#\{a \geqslant 0 \mid am \leqslant c - 1\} + \sum_{j \geqslant 1} \#\{a \geqslant 0 \mid n_j + am \leqslant c - 1\}$$
$$= 1 + \left[\frac{n_{m-1} - n_0}{n_0}\right] + \sum_{j \geqslant 1} 1 + \left[\frac{n_{m-1} - n_0 - n_j}{n_0}\right]$$
$$= 1 + \sum_{j=0}^{m-1} \left[\frac{n_{m-1} - n_j}{n_0}\right].$$

Remark. We have defined the standard basis relative to m, the multiplicity of H. The same results (as in 2.2) hold if we similarly construct a complete residue system modulo p, for any positive integer p of H. In applications, if it is more convenient to consider a standard basis relative to p we shall do so.

(2.3) A semigroup H is called *symmetric* if there is an integer c s.t. $n \in H$ if and only if $c - 1 - n \notin H$, equivalently if in the set $\{0, 1, \ldots, c - 1\}$ there are precisely as many elements of H as gaps so that $c(H) = 2g(H)$. It is well known that H is symmetric if and only if B_H is Gorenstein (e.g., see [7]). We obtain the following interesting characterization of the symmetric semigroup.

PROPOSITION 2.4. *The following statements are equivalent*:

(i) *B_H is Gorenstein*;

(ii) $n_{m-1} = n_i + n_{m-i-1}$ *whenever* $1 \leqslant i \leqslant m - 2$;

(iii) $[\mathrm{End}(H) : H] = 1$ *where* $\mathrm{End}(H) = \{n \in \mathbb{N} \mid n + H^+ \subset H\}$, *i.e., translations of H.*

Proof. (i) $\Rightarrow$ (ii) Assume B_H is Gorenstein so that H is symmetric. The c in the definition of symmetric must necessarily be the conductor of H. Then $n \in H$ if and only if $n_{m-1} - m - n \notin H$. For $1 \leqslant i \leqslant m - 2$, $n_i - m \notin H$ entails $n_{m-1} - n_i \in H$. Since n_{m-1} is the least integer in H of given residue, $n_{m-1} = n_i + n_j$ for some n_j of the standard basis. We see that $n_{j_{m-2}} < n_{j_{m-3}} < \cdots < n_{j_1} < n_{m-1}$ so that $j_i = m - i - 1$, i.e., $n_{m-1} = n_i + n_{m-i-1}$ whenever $1 \leqslant i \leqslant m - 2$.

(ii) $\Rightarrow$ (iii) Assume the equalities of (ii). Since $\mathrm{End}(H)$ is itself a semigroup, it suffices to see that $n_j - m \notin \mathrm{End}(H)$ for $1 \leqslant j \leqslant m - 2$. (Note that $n_{m-1} - m = c - 1 \in \mathrm{End}(H)$ since $(c - 1) + H^+ \subset c + \mathbb{N} \subset H$. Also $(n_{m-1} - 2m) + m \notin H$ entails $n_{m-1} - 2m \notin \mathrm{End}(H)$.) But $(n_j - m) + n_{m-j-1} = n_{m-1} - m \notin H$ for $1 \leqslant j \leqslant m - 2$. Hence $\mathrm{End}(H) = H \cup \{c - 1\}$.

(iii) $\Rightarrow$ (i) Assume $[\mathrm{End}(H) : H] = 1$. So see that $B = B_H$ is Gorenstein it suffices to show that the length of the B-module $\mathbf{m}^{-1}/B$ is one where $\mathbf{m} = (t^h \mid h \in H^+)$. Now B is a graded k-subalgebra of $k[t]$ entails $\mathbf{m}^{-1}$ is generated by monomials t^p s.t. $p + H^+ \subset H$. Hence $l(\mathbf{m}^{-1}/B) = [\mathrm{End}(H) : H] = 1$ and B_H is Gorenstein.

3. MONOMIAL CURVES: THE COHOMOLOGICAL FUNCTOR T^i

(3.1) Let $\mathbb{G}_m$ denote the algebraic group over k where the group law is multiplication. Then an affine scheme $V = \mathrm{Spec}(A)$ has $\mathbb{G}_m$-action if and only if A is a graded k-algebra where the indexing set is $\mathbb{Z}$, i.e., $A = \bigoplus_{-\infty<n<\infty} A_n$.

DEFINITION 3.2. A *monomial curve* is an irreducible affine curve with $\mathbb{G}_m$-action.

If H is a numerical semigroup the associated semigroup ring B_H is clearly a monomial curve and since $B_H = \bigoplus_{n\in H} kt^n$ is indexed by nonnegative integers, B_H is the affine cone over $\mathrm{Proj}(B_H)$. Once we fix a semigroup H we write B for B_H and S for S_H.

Let $S = \{m = n_0 < \cdots < n_{m-1}\}$,

$$f_{ij} = X_i X_j - X_0^{e(i,j)} X_{r(i,j)} \tag{4}$$

for $1 \leqslant i \leqslant j \leqslant m-1$ where

$$n_i + n_j = e(i,j)m + n_{r(i,j)}\,. \tag{5}$$

Set I equal to the ideal of $P = k[X_0, ..., X_{m-1}]$ generated by $\{f_{ij}\}_{1\leqslant i\leqslant j\leqslant m-1}$. We define a k-algebra map φ: $k[X_0, ..., X_{m-1}] \to B$ by $\varphi(X_i) = t^{n_i}$ for $0 \leqslant i \leqslant m-1$.

PROPOSITION 3.3. *The sequence*

$$0 \longrightarrow I \longrightarrow P \xrightarrow{\varphi} B \longrightarrow 0$$

is exact. Furthermore, if we assign the weight n_i to X_i in P, then φ is a (degree 0) homomorphism of graded k-algebras and I is homogeneous.

The proof is obvious since B is free over the principal ideal domain $A = k[t]$ and multiplication of the A-module generators $\{t^{n_i}\}$ is defined by (5).

(3.4) We will not attempt to give a precise definition of T^* here. For definition and details of T^0, T^1 one can consult Lichtenbaum and Schlessinger [8]; for the full cohomological properties of T^* one should consult Rim's article "Formal Deformation Theory" [14] (note that our T^i plays the role of Rim's D^i). We state here several important properties of T^* that we will need in later sections; see [14] for proofs of these assertions.

THEOREM 3.5. (1) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then*

$$\begin{aligned} 0 &\to T^0(A \mid R, M') \to T^0(A \mid R, M) \to T^0(A \mid R, M'') \\ &\to T^1(A \mid R, M') \to T^1(A \mid R, M) \to T^1(A \mid R, M'') \\ &\to \cdots \to T^n(A \mid R, M') \to T^n(A \mid R, M) \to T^n(A \mid R, M'') \to \cdots \end{aligned}$$

is exact.

(2) *Let $S \to R \to A$ be ring homomorphisms. Then for any A-module M we have the long exact sequence*

$$\begin{aligned} 0 &\to T^0(A \mid R, M) \to T^0(A \mid S, M) \to T^0(R \mid S, M) \\ &\to T^1(A \mid R, M) \to T^1(A \mid S, M) \to T^1(R \mid S, M) \\ &\to \cdots \to T^n(A \mid R, M) \to T^n(A \mid S, M) \to T^n(R \mid S, M) \to \cdots. \end{aligned}$$

(3) *Let P be a polynomial algebra over R and let $0 \to I \to P \to A \to 0$ be exact. Then*

$$T^0(A \mid R, M) = \mathrm{Der}_R(A, M), \tag{6}$$

$$\begin{aligned} T^1(A \mid R, M) &= \mathrm{Coker}(\mathrm{Der}_R(P, M) \to \mathrm{Hom}_A(I/I^2, M)) \\ &= \textit{the set of isomorphism classes of} \\ &\quad\ R\textit{-algebra extensions of } A \textit{ by } M. \end{aligned} \tag{7}$$

COROLLARY 3.6. (a) *An R-algebra A is formally smooth over R if and only if $T^1(A \mid R, M) = 0$ for every A-module M.*

(b) *Let R be Noetherian and A a local R-algebra of essentially finite type. We denote by $\hat{A}$ the $\mathfrak{m}$-adic completion of A where $\mathfrak{m}$ = the maximal ideal of A. Then for any A-module E of finite type we have a canonical isomorphism*

$$T^1(\hat{A} \mid R, \hat{A} \otimes_A E) \xrightarrow{\sim} \hat{A} \otimes_A T^1(A \mid R, E).$$

COROLLARY 3.7. *Let R be Noetherian and A an R-algebra of finite type. Then*

(a) Supp $T^i(A \mid R, A) \subset \mathrm{Sing}(A \mid R)$ *for all $i > 0$ where* $\mathrm{Sing}(A \mid R) = \{x \in \mathrm{Spec}(A) \mid A$ *is nonsmooth over R at the point x*$\}$.

(b) *Suppose that A is smooth (over R) everywhere except at one closed point $x \in \mathrm{Spec}(A)$. We then have isomorphisms*

$$T^1(A \mid R, A) \simeq T^1(A_x \mid R, A_x) \simeq T^1(\hat{A}_x \mid R, \hat{A}_x).$$

Remarks 3.8. We see (by 3.5(1)) that $T^*(\ )$: (A-mod) $\to$ (A-mod) defined by $T^i(M) = T^i(A \mid R, M)$ is a cohomological functor; i.e., given a short exact sequence of A-modules we get a long exact sequence on T^i.

Similarly (by 3.5(2)) if we fix a target A and an A-module M, given a triple of rings $S \to R \to A$ we get a long exact sequence on T^i. We will often use these results.

4. A Complete Characterization of the Negatively Graded Semigroups

(4.0) Now suppose that we have a graded k-algebra A (indexed by $\mathbb{Z}$) of finite type where we recall that k is an algebraically closed field of characteristic 0. We can then find an exact sequence

$$0 \longrightarrow I \longrightarrow P \xrightarrow{\varphi} A \longrightarrow 0$$

where $P = k[X_1, \ldots, X_m]$ and weights $n_i \in \mathbb{Z}$ s.t. if we assign $\deg(X_i) = n_i$ then

φ becomes a (degree 0) homomorphism of graded k-algebras. In turn $T^1(A) = T^1(A \mid k, A)$ becomes a graded k-vector space via

$$T^1(A) = \bigoplus_{-\infty<p<\infty} T^1(A)_p$$
$$= \bigoplus_{-\infty<p<\infty} \operatorname{Coker}(\operatorname{Der}_k(P, A)_p \to \operatorname{Hom}_A(I/I^2, A)_p),$$

so that

$$T^1(A)_p \simeq \text{the set of isomorphism classes of degree 0 graded } k\text{-algebra extensions of } A \text{ by } A(p),$$

where $A(p)$ is the graded k-module obtained from A by shifting the degree by p; i.e., $A(p)_n = A_{p+n}$.

We are interested in characterizing those monomial curves B_H for which $T^1(H)_+ = T^1(B_H)_+ = 0$. These are the so called *negatively graded semigroups* of Pinkham [12]. For this purpose we describe another characterization of $T^1(H)$.

PROPOSITION 4.1. *Let k be an algebraically closed field, A a reduced k-algebra of finite type. Then*

$$T^1(A) \cong \operatorname{Coker}(\operatorname{Der}_k(A, K) \to \operatorname{Der}_k(A, K/A)), \tag{8}$$

where K denotes the total ring of fractions for A.

Proof. The exact sequence $0 \to A \to K \to K/A \to 0$ gives us the exact sequence

$$0 \to T^0(A \mid k, A) \to T^0(A \mid k, K) \to T^0(A \mid k, K/A)$$
$$\to T^1(A \mid k, A) \to T^1(A \mid k, K).$$

Since k is algebraically closed and A is reduced, A is generically smooth over k (i.e., for any generic point $\mathfrak{p} \in \operatorname{Spec}(A)$, $A(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is smooth over k). Hence $T^1(A \mid k, K) = 0$. Thus

$$T^1(A \mid k, A) \simeq \operatorname{Coker}(T^0(A \mid k, K) \to T^0(A \mid k, K/A))$$
$$\cong \operatorname{Coker}(\operatorname{Der}_k(A, K) \to \operatorname{Der}_k(A, K/A)).$$

Unless otherwise stated $\mathfrak{m}$ shall denote the maximal ideal of B generated by $\{t^h \mid h \in H^+\}$, $\hat{B}$ the $\mathfrak{m}$-adic completion of B and $\hat{K} = k((t))$ the fraction field of $\hat{B}$.

COROLLARY 4.2. *Let $B = B_H$ and $0 \to I \to P \to B \to 0$ be exact where P is a polynomial algebra over k. Then*

$$T^1(B)_l = \operatorname{Coker}(\operatorname{Der}_k(B, \hat{K})_l \to \operatorname{Der}_k(B, \hat{K}/\hat{B})_l), \qquad l \in \mathbb{Z}. \tag{9}$$

Hence $\dim_k T^1(B)_l = \max\{0, \dim_k(\operatorname{Der}_k(B, \hat{K}/\hat{B})_l - 1)\}$.

The proof is as above as we note that $\hat{P}$ is formally smooth over k and $\mathrm{Der}_k(B, \hat{K})_l \simeq \mathrm{Der}_k(k[t], \hat{K})_l$ is 1-dimensional.

(4.3) Before we state the main theorem of this section we need some notation and definitions. For a numerical semigroup H, let $\lambda(H) = [\mathrm{End}(H) : H]$. We say that H is an *ordinary semigroup of genus* g (denoted by H_g) if $\lambda(H) = g$; equivalently if $H = \{0, g+1, g+2, g+3, \ldots\}$. We say that H is *hyperordinary* if $H = m\mathbb{N} + H_g$ where H_g is ordinary and $0 < m < g$.

Let X be a smooth projective curve of genus g, $x \in X$ and $V = X - \{x\}$. Then we have an ascending chain of finite-dimensional k-vector spaces $k = \Gamma(X, 0\{x\}) \subset \Gamma(X, 1\{x\}) \subset \cdots \subset \Gamma(x, n\{x\}) \subset \cdots$ where $\Gamma(X, n\{x\}) = \{f \in k(X) \mid f$ is regular on V having a pole of order at most n at $x\}$. By Riemann–Roch, we know $\dim_k \Gamma(X, 2g-1\{x\}) = g$ and $\dim_k \Gamma(X, n+1\{x\}) - \dim_k \Gamma(X, n\{x\}) \leqslant 1$. Hence between 0 and $2g-1$ there are precisely g integers $s_1 < \cdots < s_g$ called the *gap sequence* for X at x for which there exists no rational function f, regular on V, having a pole of order precisely s_i at x.

Let $H_{X,x} = \{n \in \mathbb{N} \mid \exists f \in k(X)$ regular on V, having a pole of order n at $x\}$. Thus $n \in H_{X,x}$ if and only if $\Gamma(X, n-1\{x\}) \subsetneqq \Gamma(X, n\{x\})$.

Then x is an *ordinary point* if $H^1(X, g\{x\}) = 0$, i.e., $H_{X,x} = \{0, g+1, g+2, \ldots\}$. So x is an ordinary point of X if and only if $H_{X,x}$ is ordinary. Otherwise x is called a *Weierstrass point* of X.

(4.4) Throughout the rest of this section let $S_H = \{m = n_0 < n_1 < \cdots < n_{m-1}\}$ denote the standard basis for H where $m = m(H)$, $c = c(H)$, and $B = B_H$. Let $\hat{B}$ denote the $\mathbf{m}$-adic completion of B where $\mathbf{m} = (t^h \mid h \in H^+)$ and $\hat{K} = k((t))$. Set $E_l = \mathrm{Der}_k(B, \hat{K}/\hat{B})_l$ for each $l \in \mathbb{Z}$. By $\dim(\cdot)$ we mean dimension as a k-vector space unless otherwise stated.

LEMMA 4.5. *For each* $l \in \mathbb{Z}$, *let* $G_l = \{n \in S_H \mid n + l \notin H\}$ *and* $R_l = \{f_{ij} \in I \mid n_i + n_j + l \notin H\}$. *Associate each element* f_{ij} *of* R_l *with the vector* $V^{(i,j)} = (V_0^{(i,j)}, \ldots, V_{m-1}^{(i,j)}) \in k^m$ *where* $m = m(H)$ *and*

$$
\begin{aligned}
V_k^{(i,j)} &= -e(i,j) && \text{if } k = 0 \text{ and } r(i,j) \neq 0\\
&= -(e(i,j)+1) && \text{if } k = 0 \text{ and } r(i,j) = 0\\
&= 2 && \text{if } k = i = j\\
&= 1 && \text{if } k = i \text{ or } k = j \text{ and } i \neq j\\
&= -1 && \text{if } k = r(i,j) \neq 0\\
&= 0 && \text{otherwise.}
\end{aligned}
$$

Associate R_l *with the vector subspace of* k^m *spanned by* $\{V^{(i,j)}\}$. *Then* $\dim T^1(H)_l = \max\{0, \#G_l - \dim R_l - 1\}$.

Proof. A typical element of $E_l = \mathrm{Der}_k(B, k((t))/\hat{B})_l$ is defined by a vector

$(a_0, \ldots, a_{m-1}) \in k^m$ s.t. $a_i = 0$ whenever $n_i \notin G_l$ and $a_i + a_j = e(i, j)\, a_0 + a_{r(i,j)}$ whenever $f_{ij} \in R_l$. Thus $\dim_k E_l = \# G_l - \dim R_l$ and the statement follows from (4.2).

LEMMA 4.6. (a) *If H is negatively graded, then $c \leqslant n_1 + m$. Consequently, H is negatively graded if and only if $\# G_l \leqslant 1$ for all $l > 0$.*

(b) *If H is negatively graded, then $n_{m-2} < n_1 + m$. For the negatively graded semigroup there is at most one gap between n_1 and $n_1 + m$.*

Proof. (a) Suppose that $c > n_1 + m$ so that $c > n_1 + m + 1$ (since $n_1 + m \in H$). Then setting $p = c - 1 - (n_1 + m)$ we obtain $p > 0$. Now $c - 1 - (n_1 + m) + 2n_1 = c - 1 + (n_1 - m) \geqslant c$ entails $R_p \neq \varnothing$. Since G_p contains m and n_1 we have $T^1(H)_p \neq 0$, a contradiction. Hence $c \leqslant n_1 + m$ so that $R_p = \varnothing$ whenever $p > 0$ (since $n_i + n_j + p > n_1 + m + p > c$). Thus $\dim E_p = \# G_p$ and our assertion follows.

(b) This is clearly the case for $m \leqslant 3$ so assume $m > 3$. Suppose $n_{m-2} \geqslant n_1 + m$ so that $n_{m-2} > n_1 + m$. Set $p = n_{m-2} - (n_1 + m) > 0$. Then $p + n_1 \notin H$ entails $p + m \in H$, i.e., $n_{m-2} = n_1 + n_j$ for some $n_j \in S_H$. Set $q = n_{m-1} - n_1 > m$. If $q \in H$, then $n_{m-1} = n_1 + n_k$ for some $n_k \in S_H$. In that case $G_{n_1 - m}$ contains both n_j and n_k, contradicting (a). Hence $q \notin H$. Then G_{q-m} contains both m and n_1, again a contradiction. Hence $n_{m-2} < n_1 + m$. Therefore $n_2, \ldots, n_{m-2}$ and an m-multiple must occur between n_1 and $n_1 + m$ so that there can be at most one gap for H in this interval.

THEOREM 4.7. *Let H, $g = g(H)$, $\lambda = \lambda(H)$ be as above. H is negatively graded if and only if H is of one of the following types:*

(i) *H is ordinary;*

(ii) *H is hyperordinary;*

(iii) *excluding the ordinary and hyperordinary cases, given g and λ with $2 \leqslant \lambda \leqslant g - 2$ there exists a unique negatively graded semigroup (denoted by $H_{g,\lambda}$) of given g and λ. Namely,*

$$H_{g,\lambda} = \{0, g, \ldots, 2g - \lambda - 1, \widehat{2g - \lambda}, 2g - \lambda + 1, 2g - \lambda + 2, \ldots\}.$$

If $\lambda = 1$ we have two possibilities; by abuse of notation we write:

$$H_{g,1} = \{0, g, g + 1, \ldots, 2g - 2, \widehat{2g - 1}, 2g, 2g + 1, \ldots\}$$

or

$$H_{g,1} = \{0, g - 1, \hat{g}, g + 1, \ldots, 2g - 2, \widehat{2g - 1}, 2g, 2g + 1, \ldots\}.$$

Proof. By 4.6 we have two cases to consider, namely, when there is no gap between n_1 and $n_1 + m$ and when there is one gap.

The former case entails that H is ordinary or hyperordinary and clearly $T^1(H)_+ = 0$.

So assume the latter (so H is neither ordinary nor hyperordinary). Let $n_1 + r$ denote that gap so that $n_1 + r - 1$ is either an element of S_H or an m-multiple. If $n_1 + r - 1 \in S_H$, then $T^1(H)_1 = 0$ entails $m + 1 \in H$. In this case $n_1 = m + 1$, $n_{m-1} = 2m + r + 1$, and

$$\begin{aligned} n_i &= m + i, & 1 \leqslant i \leqslant r \\ &= m + i + 1, & r + 1 \leqslant i \leqslant m - 2. \end{aligned}$$

Here we have $g = g(H) = m$ and $\lambda = \lambda(H) = m - r - 1$ so that $1 \leqslant \lambda \leqslant g - 2$.

If $n_1 + r - 1 = qm$ some $q \geqslant 2$, then $n_1 + r - 2 \in S_H$ entails $m + 2 \in H$, i.e., $n_1 = m + 2, \ldots, n_{m-2} = 2m - 1$ and $n_{m-1} = 3m + 1$. Since $\#G_l \leqslant 1$ whenever $l > 0$, $T^1(H)_+ = 0$. In this case $g = g(H) = m + 1$ and $\lambda = \lambda(H) = 1$.

Remark 4.8. Let $g = g(H)$. Just as $\lambda(H) = g$ if and only if H is ordinary, we can characterize those H for which $\lambda(H) = g - 1$. Indeed, $\lambda(H) = g - 1$ entails $\mathrm{End}(H)$ is elliptic, i.e., $\mathrm{End}(H) = \{0, 2, 3, 4, \ldots\}$ and hence $H = \{0, g, g + 2, g + 3, \ldots\}$. If $\lambda(H) = g - 2$, then $\mathrm{End}(H) = \{0, 3, 4, 5, \ldots\}$ or $\{0, 2, 4, 5, \ldots\}$ and hence $H = \{0, g - 1, g + 2, g + 3, g + 4, \ldots\}$ or $\{0, g - 1, g + 1, g + 3, g + 4, \ldots\}$ or $\{0, g, g + 1, g + 3, g + 4, \ldots\}$.

We will now present the proofs of some well-known results (e.g., see [12]) which will be used repeatedly in Sections 5 and 6.

THEOREM 4.9. *Any configuration of m lines through the origin in general linear position in d-space is negatively graded provided that* $(m - 1)/(d - 1) \leqslant 2$.

Proof. Let B denote the (homogeneous) coordinate ring of the m-lines in d-space and $\bar{B}$ the integral closure of B in its total ring of fractions. If $m \leqslant d$, by suitable homogeneous change of coordinates, we can assume that the m-lines are given by the $X_1, \ldots, X_m$ axes in d-space. Then $B = k[X_1, \ldots, X_d]/\{X_iX_j, X_k \mid 1 \leqslant i < j \leqslant m, m < k\}$, i.e., $B \simeq k[X_1, \ldots, X_m]/\{X_iX_j \mid i \neq j\}$ and $\bar{B} \simeq k[X_1] \oplus \cdots \oplus k[X_m]$. Thus B is a graded k-subalgebra of $\bar{B}$ s.t. $(\bar{B}/B)_l = 0$ whenever $l > 0$ so that $T^1(B)_+ = 0$ by 4.1.

So suppose $d < m \leqslant 2d - 1$. As does Saint-Donat [16] we choose homogeneous coordinates so that $L_1, \ldots, L_d$ represent the $X_1, \ldots, X_d$ axes and $L_j = \{t\mathbf{v} \mid t \in k,\ \mathbf{v} = (a_{1,j}, \ldots, a_{d,j})\}$ for $d + 1 \leqslant j \leqslant m$. Then $L_1, \ldots, L_m$ are in general linear position entails any $k \times k$ minor of $A = (a_{i,j})_{1 \leqslant i \leqslant d, d+1 \leqslant j \leqslant m}$ is nonzero whenever $1 \leqslant k \leqslant m - d$. So if $B = k[X_1, \ldots, X_d]/I$ and $\bar{B} = k[Y_1] \oplus \cdots \oplus k[Y_m]$, then $\varphi: B \to \bar{B}$ is given by

$$X_i \to (0, \ldots, Y_i, \ldots, 0, a_{i,d+1}, a_{i,d+2}, \ldots, a_{i,m}Y_m) \qquad 1 \leqslant i \leqslant d.$$

Then for each $l \geqslant 2$, the images of $X_1{}^l,\ldots, X_d{}^l$, $X_1^{l-1}X_2,\ldots, X_1^{l-1}X_d$ span $\bar{B}_l$ (since the dimension of the subspace spanned by these is given by

$$\begin{aligned}
&\operatorname{rank}\{X_1{}^l,\ldots, X_d{}^l\} + \operatorname{rank}\{X_1^{l-1}X_2,\ldots, X_1^{l-1}X_d\} \\
&\quad = d + \operatorname{rank}\{X_1^{l-1}X_2,\ldots, X_1^{l-1}X_d\} = d + \operatorname{rank}\{X_2,\ldots, X_d\} \\
&\quad = d + m - d = m).
\end{aligned}$$

Hence $(\bar{B}/B)_l = 0$ whenever $l \geqslant 2$ and $T^1(B)_+ = 0$ by 4.1.

(4.10) There is a natural correspondence between k-algebras B with descending filtration and $T^1(\operatorname{gr} B)_+$. Let B be a k-algebra with descending filtration $\cdots B_{-n} \supset B_{-n+1} \supset \cdots \supset B_0 \supset B_1 \supset \cdots$ with $B_iB_j \subseteq B_{i+j}$ and $\bigcup_{n\in\mathbb{Z}} B_n = B$ (i.e., $k \subset B_0$ and each B_i is a k vector space). Set $A = \operatorname{gr} B = \bigoplus_{n\in\mathbb{Z}} B_n/B_{n+1}$. Let $B^{\#}$ be that graded k-algebra whose nth homogeneous part is B_n (and multiplication is defined as in B) so that $B^{\#} = \bigoplus_{n\in\mathbb{Z}} B_n$.

Let ϵ denote the image of 1_B in $B^{\#}_{-1}$. Then

$$\begin{aligned}
B^{\#}/\epsilon B^{\#} &\simeq \bigoplus_{m\in\mathbb{Z}} B_m{}^{\#}/(\epsilon B^{\#})_m \\
&\simeq \bigoplus_{m\in\mathbb{Z}} B_m{}^{\#}/B^{\#}_{m+1} \\
&\simeq \bigoplus_{m\in\mathbb{Z}} B_m/B_{m+1} \\
&= \operatorname{gr} B = A.
\end{aligned}$$

If $A(1)$ denotes the graded k-module obtained from A by shifting the degree by 1 (i.e., $A(1)_m = A_{m+1}$), then $0 \to A(1) \xrightarrow{\epsilon} B^{\#}/\epsilon^2B^{\#} \to B^{\#}/\epsilon B^{\#} \to 0$ defines a graded k-algebra extension of A by $A(1)$, i.e., an element of $T^1(A \mid k, A(1))_0 \simeq T^1(A \mid k, A)_1 = T^1(\operatorname{gr} B)_1$.

The relation between k-algebras B with ascending filtration and $T^1(\operatorname{gr} B)_-$ is analogous.

THEOREM 4.11. *Let A be a graded k-algebra of finite type s.t. $T^1(A)_+ = 0$. If B is a k-algebra with descending filtration $\cdots B_{-n} \supset B_{-n+1} \supset \cdots \supset B_0 \supset B_1 \supset \cdots$ as above s.t. $\operatorname{gr} B \simeq A$, then B is formally isomorphic to A; i.e., if $\hat{B}$ denotes the completion of B with respect to the given filtration and $\hat{A}$ denotes the completion of A w.r.t. the filtration induced by the gradation, then $\hat{B} \simeq \hat{A}$.*

Proof. Let $B^{\#}$ and ϵ be as above. Then since $T^1(A)_1 = 0$, the k-algebra extension

$$0 \longrightarrow A(1) \xrightarrow{\;\tilde{\epsilon}\;} B^{\#}/\epsilon^2B^{\#} \underset{s_2}{\overset{\varphi_2}{\rightleftarrows}} A \longrightarrow 0$$

admits a section $s_2\colon A \to B^{\#}/\epsilon^2B^{\#}$ in the category of graded k-algebras.

Continuing in this fashion, consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(n) & \xrightarrow{\epsilon^n} & B^{\#}/\epsilon^{n+1}B^{\#} & \xrightarrow{\varphi_{n+1}} & B^{\#}/\epsilon^{n}B^{\#} & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle s_n} & & \\
0 & \longrightarrow & A(n) & \longrightarrow & B^{\#}/\epsilon^{n+1}B^{\#} \underset{B^{\#}/\epsilon^{n}B^{\#}}{\times} A & \underset{q_n}{\overset{p_2}{\rightleftarrows}} & A & \longrightarrow & 0.
\end{array}$$

Since $T^1(A)_n = 0$, p_2 admits a section q_n so that if $s_{n+1} = p_1 \circ q_n$, then s_{n+1}: $A \to B^{\#}/\epsilon^{n+1}B^{\#}$ is a degree 0 homomorphism of graded k-algebras

$$\text{s.t. } \varphi_{n+1} \circ s_{n+1} = (\varphi_{n+1} \circ p_1) \circ q_n = s_n \circ p_2 \circ q_n = s_n .$$

Thus we obtain a graded map

$$s = \varprojlim(s_n) \colon A \to \varprojlim_n B^{\#}/\epsilon^n B^{\#} = \widehat{B^{\#}},$$

where $B^{\#}$ denotes the $\epsilon B^{\#}$-adic completion of $B^{\#}$. Now

$$\begin{aligned}
\widehat{B^{\#}} &= \varprojlim_n B^{\#}/\epsilon^n B^{\#} \\
&\simeq \varprojlim_n \bigoplus_{m \in \mathbb{Z}} B_m/B_{m+n} \\
&\simeq \bigoplus_{m \in \mathbb{Z}} \varprojlim_n B_m/B_{m+n} \\
&\simeq \bigoplus_{m \in \mathbb{Z}} \hat{B}_m
\end{aligned}$$

where $\hat{B}_m$ denotes the completion of B_m by the induced filtration $B_m \supset B_{m+1} \supset B_{m+2} \supset \cdots$.

Let $\rho \colon B^{\#} \to B$ be the canonical map,

$$\rho((b_\nu)) = \sum b_\nu \qquad \text{and} \qquad \hat{\rho} \colon \widehat{B^{\#}} \to \hat{B}$$

denote the extension of ρ to $\widehat{B^{\#}}$. Then $\hat{B}$ is filtered by $\{B_m\}_{m \in \mathbb{Z}}$ as above and the composition

$$A \xrightarrow{s} \widehat{B^{\#}} \xrightarrow{\hat{\rho}} \hat{B} \to \operatorname{gr} \hat{B} \cong \operatorname{gr} B = A$$

is the identity on A.

Hence the induced map $\hat{A} \to \hat{B}$ is an isomorphism [1, p. 112].

COROLLARY 4.12. (i) *Let A be a geometric local domain s.t.* gr A *is isomorphic to the coordinate ring of m lines through the origin in d-space in general linear* *position. If* $(m-1)/(d-1) \leqslant 2$, *then* $\hat{A} \simeq$ *the k-subalgebra of* $k[[Y_1]] \oplus \cdots \oplus k[[Y_m]]$ *generated by the tangent vectors where* $\hat{A}$ *denotes the completion of* A *at the origin.*

(ii) *Let* H *be a negatively graded semigroup. The one-dimensional unibranch geometric local domain* A *(over* k*) with value group* H *is unique up to formal isomorphism.*

Proof. (i) It is clear in lieu of 4.9 and 4.11.

(ii) Since $(A, \mathfrak{m})$ is a unibranch geometric local domain, $\hat{\bar{A}} \simeq \bar{\hat{A}}$ is a one-dimensional complete, normal, local domain where $(\bar{A}, \mathfrak{n})$ denotes the normalization of A, $\hat{\bar{A}}$ its $\mathfrak{n}$-adic completion, $\hat{A}$ the $\mathfrak{m}$-adic completion of A and $\bar{\hat{A}}$ its normalization. Hence $\hat{\bar{A}} \simeq k[[t]]$. Now $A \subset \hat{\bar{A}} \simeq k[[t]]$ and $v(A) = H$ (where v is the valuation on A induced by t) entails A is equipped with a natural descending filtration s.t. the associated graded ring $\operatorname{gr} A \simeq k[B_H]$. Since H is negatively graded $\hat{A} \simeq \widehat{\operatorname{gr} A} \simeq k[[B_H]]$ by 4.11; i.e., A is unique up to formal isomorphism.

5. Deforming by the Quadratic Transform of H

In this section we show that a large class of monomial curves can be negatively smoothed. By the work of Pinkham [12] the existence of a negative smoothing for the semigroup ring B_H (H nonordinary) is equivalent to the existence of a Weierstrass point x on a smooth projective curve X s.t. $H_{X,x} = H$ (recall the notation of 4.3). The general idea is to split the unibranch singularity at the origin into a unibranch singularity of the same type but of smaller genus and a multibranch point.

Definitions and remarks (5.0). Let A_0 be a reduced k-algebra of finite type. A deformation (A, R) of A_0 over R is said to be a *smoothing* of A_0 if R is a Noetherian k-algebra without zero divisors and A is *generically smooth* over R (i.e., the generic fiber is smooth over the fraction field of R).

Now let $A_0 = B_H$ be a semigroup ring. Then Pinkham showed that if (B, S) represents the formal versal deformation of A_0, (B, S) can be provided with a compatible $\mathbb{G}_m$-action.

Let

$$S = k[[t_1, \ldots, t_r]]/J,$$

$$B = S[[X_1, \ldots, X_m]]/(F^\infty),$$

where t_i has weight $-e_i$ and X_i has weight n_i. Set S', B' equal to the quotients of S and B, respectively, by the ideals generated by the t_i's s.t. the weight $-e_i$ is negative (so the corresponding element in $T^1(H) \simeq \operatorname{Hom}_k(\mathfrak{m}_S/\mathfrak{m}_{S^2}, k)$ is of positive degree). Since the generators of the defining ideals for S' and B' in $k[[t_i]]$ and $k[[t_i, X_j]]$ are polynomials we may and shall replace S' and B' by the corresponding quotient rings of $k[t_i]$ and $k[t_i, X_j]$ which we continue to call S' and B'.

We say A_0 can be *smoothed negatively* if there exists a point $x \in \operatorname{Spec}(S')$ s.t. the fiber $B'(x)$ is smooth over $S'(x)$. Thus if H is negatively graded, $S' = S$ and $B' = B$ so $A_0 = B_H$ can be smoothed if and only if A_0 can be smoothed negatively.

(5.1) Let H be a numerical semigroup of multiplicity m, and let $S_H = \{m = n_0 < n_1 < \cdots < n_{m-1}\}$ be the standard basis for H. Let H' denote the quadratic transform of H, i.e., H' is the semigroup generated by $\{m, n_1 - m, n_2 - m, \ldots, n_{m-1} - m\}$. Set $n_j' = \min\{n \in H' \mid n \equiv n_j \pmod{m}, 1 \leqslant j \leqslant m-1\}$. Then $n_j' = n_j - a_j m$, for some $a_j \geqslant 1$ and $\{m, n_1', n_2', \ldots, n_{m-1}'\}$ defines the standard basis for H' relative to m.

If A is a reduced one-dimensional algebra over k, we let $\bar{A}$ denote its normalization and $\delta(A) = \dim \bar{A}/A$.

We introduce some notation to use in the following sequence of propositions.

(5.2) Set $f = t^m - \alpha$ and define S to be the $k[\alpha]$-subalgebra of $k[\alpha, t]$ generated by $\{f, t^{n_1'} f^{a_1}, \ldots, t^{n_{m-1}'} f^{a_{m-1}}\}$. Further set $T' = T[\gamma]$ where $T = k[\alpha]_\alpha$ and γ satisfies $\gamma^m = \alpha$ (γ in the algebraic closure of $k(\alpha)$). Finally, let C denote the k-subalgebra of $k[z]$ generated by $\{z^m - 1, z^{n_1'}(z^m - 1)^{a_1}, \ldots, z^{n_{m-1}'}(z^m - 1)^{a_{m-1}}\}$.

Lemma 5.3. (a) $S_\alpha \otimes_T T' \simeq C \otimes_k T'$.

(b) *S defines a deformation of $B = B_H$ over $k[\alpha]$ if and only if one of the following equivalent conditions holds*:

(i) *$S/\alpha S \hookrightarrow k[t]$ is an inclusion.*

(i)′ *$S/\alpha S$ contains no nilpotents.*

(ii) *Let $0 \to S \to k[\alpha, t] \to E \to 0$ be exact. Then E is $k[\alpha]$-projective of constant rank.*

(ii)′ *Let $C_{\alpha=a}$ denote the k-subalgebra of $k[t]$ generated by $t^m - a$, $t^{n_1'}(t^m - a)^{a_1}, \ldots, t^{n_{m-1}'}(t^m - a)^{a_{m-1}}$ where $a \in k$. Then $\delta(C_{\alpha=a}) = \delta(B_H)$ for all $a \in k$ (i.e., it is a δ-constant family).*

Proof. (a) The proof is clear since

$$t^{n_i'}(t^m - \alpha)^{a_i} = t^{n_i'}(t^m - \gamma^m)^{a_i} = \gamma^{n_i}\left(\frac{t}{\gamma}\right)^{n_i'}\left(\left(\frac{t}{\gamma}\right)^m - 1\right)^{a_i}.$$

(b) We show (i)′ ⇒ (i) ⇒ (ii) ⇔ (ii)′ ⇒ (i)′.

(i)′ ⇒ (i) Suppose $S/\alpha S$ contains no nilpotents so that $S/\alpha S$ is the affine coordinate ring of a one-dimensional variety over k. Thus the map $S/\alpha S \to k[t]$ is an injection if and only if $\operatorname{Spec}(k[t]) \to \operatorname{Spec}(S/\alpha S)$ is a surjection.

The latter is clearly the case since $S = k[\alpha, t]$ entails that every maximal ideal of S containing α is of the form $(\alpha, t - b) \cap S$ some $b \in k$.

(i) $\Rightarrow$ (ii) Assume that $S/\alpha S \hookrightarrow k[t]$ so that if $E = k[\alpha, t]/S = \bar{S}/S$, then $\mathrm{Tor}_1^{k[\alpha]}(E, k[\alpha]/(\alpha)) = 0$. Then if $\mathbf{m}_a$ corresponds to the maximal ideal $(\alpha - a)$ of $k[\alpha]$, $E_{\mathbf{m}_0}$ is $k[\alpha]_{\mathbf{m}_0}$-free of rank $\delta(S/\alpha S) = \delta(B_H)$, and hence E is $k[\alpha]$-free in some open set of $\mathrm{Spec}(k[\alpha])$ containing the maximal ideal $\mathbf{m}_0$.

In lieu of the isomorphism $S_\alpha \otimes_T T' \simeq C \otimes_k T'$ of (a) we have $E_\alpha \otimes_T T' \simeq k[z]/C \otimes_k T'$. Hence $E_\alpha \otimes_T T'$ is T'-free of rank $\delta(C)$. But T' is a finite T-free module so that E_α is T-free of rank $\delta(C)$. Hence for any maximal ideal $\mathbf{m}_a$ with $a \neq 0$ we have $[E(\mathbf{m}_a) : k] = \delta(C)$ and $E_{\mathbf{m}_a}$ is $k[\alpha]_{\mathbf{m}_a}$-free so that $S/(\alpha - a)S \hookrightarrow k[t]$ is isomorphic to $C_{\alpha=a}$. Thus E is $k[\alpha]$-projective, hence $k[\alpha]$-free. In particular, $\delta(C) = \delta(B_H)$ and E is $k[\alpha]$-projective of constant rank.

(ii) $\Rightarrow$ (ii)$'$ Assume E is $k[\alpha]$-projective of constant rank. For any $a \in k$, we have the exact commutative diagram:

$$\begin{array}{ccccccccc}
0 \to \mathrm{Tor}_1^{k[\alpha]}(E, k[\alpha]/(\alpha - \alpha)) & \to & S/(\alpha - a)S & \to & k[t] & \to & E/(\alpha - a)E & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{\alpha=a} & \longrightarrow & k[t] & \longrightarrow & k[t]/C_{\alpha=a} & \longrightarrow & 0.
\end{array}$$

Hence $E/(\alpha - a)E \simeq k[t]/C_{\alpha=a}$ so that $[E(\mathbf{m}_a): k] = \delta(C_{\alpha=a})$. Hence E is $k[\alpha]$-projective of constant rank if and only if $[E(\mathbf{m}_a) : k]$ is constant if and only if $\delta(C_{\alpha=a})$ is independent of a.

(ii)$'$ $\Rightarrow$ (i) Assume $\delta(C_{\alpha=a})$ is constant. We saw that $\delta(C_{\alpha=a})$ is independent of a entails E is $k[\alpha]$-projective of constant rank. Hence

$$\mathrm{Tor}_1^{k[\alpha]}(E, k[\alpha]/(\alpha)) = 0$$

and $S/\alpha S \hookrightarrow k[t]$ is an inclusion.

We are now in a position to state the main result of this section.

THEOREM 5.4. *Consider the family of rational curves parameterized by* α:

$$C_\alpha : t \to \begin{bmatrix} t^m - \alpha \\ t^{n_1'}(t^m - \alpha)^{a_1} \\ \vdots \\ t^{n_{m-1}'}(t^m - \alpha)^{a_{m-1}} \end{bmatrix}.$$

C_α defines a deformation of $B = B_H$ over $k[\alpha]$ if and only if the following condition (*) is valid:

$$n_i + n_j \equiv n_k \pmod m \Rightarrow a_i + a_j \geqslant a_k \qquad (*)$$
$$\text{all} \quad i, j, k: \qquad 1 \leqslant i, j, k \leqslant m - 1.$$

Proof. The family of curves C_α defines a deformation of $B = B_H$ over $k[\alpha]$ if and only if the special fiber is B (since S is defined to be a $k[\alpha]$-subalgebra of $k[\alpha, t]$ and is a fortiori $k[\alpha]$-flat). Then by 5.3 C_α defines a deformation if and only if $\delta(B) = \delta(C_{\alpha=1})$.

Now $\delta(B) = g(H) = \sum_{i=1}^{m-1} [n_i/m]$ (see 2.2(ii)). As in the proof of 5.3, if $0 \to S \to k[\alpha, t] \to E \to 0$ is exact then $E_{\mathfrak{m}}$ is $k[\alpha]_{\mathfrak{m}}$-free of rank $\delta(C)$ for all maximal ideals $\mathfrak{m}$ s.t. $\alpha \notin \mathfrak{m}$. In particular $C_{\alpha=1} \simeq S/(\alpha - 1)S \hookrightarrow k[\alpha, t]/(\alpha - 1)k[\alpha, t]$ so that $C_{\alpha=1} \simeq C = k[z^m - 1, z^{n_1'}(z^m - 1)^{a_1}, \ldots, z^{n_{m-1}'}(z^m - 1)^{a_{m-1}}]$.

Now set $f = z^m - 1$. Then $\mathrm{Spec}(C) = \mathrm{Spec}(C_f) \cup \mathrm{Spec}(C_{z^m})$ and $\mathrm{Spec}(C_f) \cap \mathrm{Spec}(C_{z^m}) \simeq \mathrm{Spec}(k[z]_{z^m f})$ is smooth over k so that $\delta(C) = \delta(C_f) + \delta(C_{z^m})$. But $C_{z^m} \simeq D_{z^m}$ where $D = k[f, z^{r_1} f^{a_1}, \ldots, z^{r_{m-1}} f^{a_{m-1}}]$, $r_i \equiv n_i \pmod m$ with $0 < r_i < m$. Set $r_0 = 0$, $a_0 = 1$. Consequently, $\delta(C_{z^m}) = \delta(D_{z^m})$.

We have $\{z^{r_i} f^d\}_{0 \leqslant i \leqslant m-1, 0 \leqslant d}$ forms a k-basis for $k[z]$.

Now $h \in D_{z^m}$ if and only if $z^{md} h \in D$ some $d \geqslant 0$. Hence $z^{r_i} f^e \in D_{z^m}$ if and only if $z^{r_i + dm} f^e \in D$ some $d \geqslant 0$ if and only if $e = \sum_{i=0}^{m-1} c_i a_i$ and $r_i + dm = \sum_{j=1}^{m-1} c_j r_j$. Thus if $b_i = \min(\sum_{j=1}^{m-1} c_j a_j \mid c_j \geqslant 0, \sum c_j n_j \equiv n_i \pmod m)$ we have $z^{r_i} f^{b_i} \in D_{z^m}$ but $z^{r_i} f^{b_i - 1} \notin D_{z^m}$. Now by our choice of k-basis for $k[z]$, $k[z]_{z^m}$ is a graded k-algebra and D_{z^m} is a graded subalgebra. Hence $\delta(D_{z^m}) = \dim(k[z]_{z^m}/D_{z^m}) = \sum_{i=1}^{m-1} b_i$.

On the other hand, consider $C_f \simeq k[z^m, z^{n_1'}, \ldots, z^{n_{m-1}'}]_f$. Since $B' = B_{H'}$ is nonsingular except (possibly) at the origin,

$$\delta(C_f) = \delta(B') = g(H') = \sum_{i=1}^{m-1} \left[\frac{n_i'}{m}\right] = \sum_{i=1}^{m-1} \left[\frac{n_i - a_i m}{m}\right]$$

$$= \sum_{i=1}^{m-1} \left[\frac{n_i}{m}\right] - a_i = g(H) - \sum_{i=1}^{m-1} a_i = \delta(B) - \sum_{i=1}^{m-1} a_i .$$

Thus $\delta(C) = \delta(C_f) + \delta(C_{z^m}) = \delta(B) + \sum_{i=1}^{m-1} (b_i - a_i)$.

Now $1 \cdot n_i \equiv n_i \pmod m$ implies $b_i \leqslant 1 \cdot a_i = a_i$. Thus $b_i \leqslant a_i$ each i and $\delta(C) = \delta(B)$ if and only if $b_i = a_i$ all i if and only if $z^{r_i} f^{a_i - 1} \notin D_{z^m}$ $(1 \leqslant i \leqslant m - 1)$ if and only if (*) whenever $n_j + n_k \equiv n_i \pmod m$ we have $a_j + a_k \geqslant a_i$ $1 \leqslant i, j, k \leqslant m - 1$.

Hence the family C_α of rational curves is a deformation of $B = B_H$ if and only if (*) is valid for H.

Remarks 5.5. (a) Recall that a_i was defined by $a_i m = n_i - n_i'$ so that $a_i = \max\{a \mid n_i - am \in H' = \{m, n_1 - m, n_2 - m, \ldots, n_{m-1} - m\}\}$. So another characterization of a_i is given by

$$a_i = \max\{a \mid n_i - am = c_1(n_1 - m) + \cdots + c_{m-1}(n_{m-1} - m)\}$$

$$= \max \left\{a \mid n_i - am = \sum_{j=1}^{m-1} c_j n_j - \sum_{j=1}^{m-1} c_j m\right\}.$$

Now $\sum_{j=1}^{m-1} c_j n_j = n_i + c_0' m$ some $c_0' \geqslant 0$, and $a = -c_0' + \sum_{j=1}^{m-1} c_j$. Thus

$$a_i = \max\left\{\sum_{j=0}^{m-1} c_j \mid c_0 \leqslant 0,\, c_j \geqslant 0\ (1 \leqslant j \leqslant m-1),\, n_i = \sum_{j=0}^{m-1} c_j n_j\right\}.$$

So (*) is valid for H if and only if

$$\max\left\{\sum_{j=0}^{m-1} c_j \mid c_0 \leqslant 0,\, c_j \geqslant 0\ (1 \leqslant j \leqslant m-1),\, n_i = \sum_{j=0}^{m-1} c_j n_j\right\} =$$

$$\min\left\{\sum_{j=1}^{m-1} c_j \mid c_j \geqslant 0\ (1 \leqslant j \leqslant m-1),\, n_i = \sum_{j=0}^{m-1} c_j n_j \text{ some } c_0 \leqslant 0\right\}.$$

(b) Say (*) is valid for H so that the family C_α defines a deformation of $B = B_H$ over $R = k[\alpha]$. The fibers $C_{\alpha=a} \simeq C$ for $0 \neq a \in k$ have a unibranch singularity at $\mathbf{m} = (z^m, z^{n_1'} f^{a_1}, \ldots, z^{n'_{m-1}} f^{a_{m-1}})$ where $f = z^m - 1$ corresponding to the semigroup H' (unless of course $H' = \mathbb{N}$ in which case C is smooth at $\mathbf{m}$). The only other possible singularity occurs at $\mathbf{n} = (f, z^{n_1'} f^{a_1}, \ldots, z^{n'_{m-1}} f^{a_{m-1}})$ which is an m-fold multibranch point. At $\mathbf{m}$ we have lowered the genus of H by $\sum_{i=1}^{m-1} a_i$.

We wish to examine the properties of the a_i's in order to determine which semigroup rings may be deformed via this quadratic tranform.

Notation 5.6. Let M_H be that subset of S_H, the standard basis for H, which is the *minimal generating set* for H (i.e., if $M_H = \{m = n_{i_0} < n_{i_1} < \cdots < n_{i_l}\}$ then for each $j = 1, \ldots, l$ n_{i_j} is not in the semigroup generated by $n_{i_0}, \ldots, n_{i_{j-1}}$).

PROPOSITION 5.7. *If* $\#(S_H - M_H) \leqslant 1$, *then the condition* (*) *of* 5.4 *is valid for* H.

The proof of this proposition is an immediate consequence of the following lemma.

LEMMA 5.8. *If* $\#(S_H - M_H) = l$, *then* $a_i \leqslant l + 1$ *for* $1 \leqslant i \leqslant m - 1$.

Proof. Using the characterization of a_i given in (5.5) it suffices to see that if

$$n_i + \alpha m = n_{j_1} + \cdots + n_{j_p}, \qquad n_{j_k} \in S_H - \{m\}, \qquad \alpha \geqslant 0,$$

then $p - \alpha \leqslant l + 1$.

We proceed by induction on p noticing that for $p \leqslant l + 1$ the statement is obvious. So assume $p > l + 1$ and the statement holds for all $q < p$. Set $n_i + \alpha m = n_{j_1} + \cdots + n_{j_p}$. If any $n_{j_k} \geqslant n_i$, then

$$\begin{aligned} \alpha m &= n_{j_1} + \cdots + (n_{j_k} - n_i) + \cdots + n_{j_p} \\ &> (p-1)m + (n_{j_k} - n_i) \\ &\Rightarrow \alpha > p - 1 \Rightarrow p - \alpha < 1. \end{aligned}$$

So assume $n_{j_k} < n_i$ all $k = 1,..., p$. Consider the partial sums

$$S(r) = n_{j_1} + \cdots + n_{j_r} \quad \text{for} \quad 2 \leqslant r \leqslant p.$$

Notice that if $S(r) = bm$ for some r (so $b > r$), then

$$n_i + \alpha m = bm + n_{j_{r+1}} + \cdots + n_{j_p}$$
$$\Rightarrow n_i + (\alpha - b)m = n_{j_{r+1}} + \cdots + n_{j_p} \quad \text{and} \quad p - r < p$$

so that

$$p - r - (\alpha - b) \leqslant l + 1$$
$$\Rightarrow p - \alpha < p - \alpha + (b - r) \leqslant l + 1.$$

Similarly if $S(r_1) \equiv S(r_2)$ (mod m) the statement follows by induction. So it suffices to assume all the partial sums represent distinct nonzero residues modulo m. Since $S_H - M_H$ contains l elements and we assumed $p > l + 1$ the list of partial sums must contain at least $p - 1 - l$ residues of elements of M_H.

Say $S(r_{k_i}) = n_{s_i} + b_i m$ with $2 \leqslant r_{k_1} < \cdots < r_{k_{p-1-l}}$, $n_{s_i} \in M_H$. Since $S(r_{k_1})$ involves at least two summands and $n_{s_1} \in M_H$ we have $b_1 > 0$. Since $S(r_{k_{j+1}}) - S(r_{k_j}) = n_{s_{j+1}} - n_{s_j} + (b_{j+1} - b_j)m$ represents an element of H and $n_{s_{j+1}} \in M_H$ we have $b_{j+1} > b_j$. In particular, $b_{p-(l+1)} \geqslant p - (l + 1)$. But

$$n_i + \alpha m = S(r_{p-(l+1)}) + (S(p) - S(r_{p-(l+1)}))$$
$$= n_{s_{p-(l+1)}} + b_{p-(l+1)}m + n.$$

PROPOSITION 5.9. *Let H be ordinary or hyperordinary. Then* (*) *is valid for H and B_H can be negatively smoothed.*

Proof. If H is hyperordinary of ordinary, the standard basis for H is also the minimal generating set. Hence (*) is valid for H (by 5.7) and C_α of 5.4 is a flat family with special fiber $C_{\alpha=0} = B_H$. The fiber $C_{\alpha=a} \simeq C$ for $0 \neq a \in k$ has two possible singularities as in the proof of 5.4. The first occurs at $\mathbf{n} = (f, z^{n_1'}f,..., z^{n_{m-1}'}f)$, where $f = z^m - 1$. Now z^m is a unit in $C_\mathbf{n}$ so that $C_\mathbf{n} \simeq k[f, zf,..., z^{m-1}f]_{\mathbf{n}'}$ where $\mathbf{n}' = (f, zf,..., z^{m-1}f)$.

Then consider the maximal ideals $\mathbf{n}_i = (z - \omega_i)\, k[z]$ in $k[z]$, where ω_i are the mth roots of unity in k. Then

$$k[f, zf,..., z^{m-1}f] = k \oplus \mathbf{n}_1 \cap \mathbf{n}_2 \cap \cdots \cap \mathbf{n}_m \subset k[z]$$

and hence $\widehat{C_\mathbf{n}}$ = the subalgebra of $\widehat{k[z]_{\mathbf{n}_1 \cap \cdots \cap \mathbf{n}_m}} \simeq k[[z_1]] \times \cdots \times k[[z_m]]$ having the same constant term, i.e., $\widehat{C_\mathbf{n}} \simeq k[[z_1]] \times_k \cdots \times_k k[[z_m]] \simeq \widehat{k[z_1, ..., z_m]}/(z_i z_j \mid i \neq j)$. Thus it is smoothable (cf. 5.15). Consequently B_H is smoothable.

The other possible singularity occurs at $\mathfrak{m} = (z^m, z^{n_1'}f,..., z^{n'_{m-1}}f)$ and corresponds to H' which is again hyperordinary or ordinary. Since (*) is again valid for H' and $g(H') < g(H)$, B' may be smoothed inductively. Then since C is a curve there is no obstruction to gluing local deformations of C to give a global deformation, so that C may be smoothed. Thus B_H may be smoothed.

PROPOSITION 5.10. *Suppose that H is negatively graded of the third type listed in* 4.7. *Then* $B = B_H$ *can be (negatively) smoothed.*

Proof. If H is of the third type, then $S_H - M_H = \{n_{m-1}\}$ where m is the multiplicity of H so that (*) is valid for H by 5.7. We note that $a_i = 1$ for $1 \leqslant i \leqslant m - 2$ and $a_{m-1} = 2$ (by 5.7 and 5.8).

Set $n_{m-1} = 2m + r$ where $2 \leqslant r \leqslant m + 1$. Then $C \simeq C_{\alpha=1}$ of 5.4 has two possible singularities occurring at $\mathfrak{m} = (z^m, zf,..., \widehat{z^r f},..., z^{m-1}f)$ and at $\mathfrak{n} = (f, zf,..., \widehat{z^r f},..., z^{m-1}f)$ where $f = z^m - 1$ and with the obvious modification zf is missing in case $r = m + 1$.

Now since f is a unit in $C_\mathfrak{m}$ we have $C_\mathfrak{m}$ isomorphic to a localization of $k[z]$ if $r \neq m + 1$ and $k[z^2, z^3]_{\mathfrak{m}'}$ otherwise, where $\mathfrak{m}' = (z^2, z^3)$. In either case $C_\mathfrak{m}$ is smoothable.

To treat the other case, consider $\mathfrak{n}_i = (z - \omega_i)\, k[z]$ where ω_i are the mth roots of unity in k. Then $C_\mathfrak{n} \subset k[z]_{\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_m}$ and $C_\mathfrak{n}$ is negatively graded. Consequently (by 4.12) $\hat{C}_\mathfrak{n}$ is the k-subalgebra of $k[[z_1]] \times \cdots \times k[[z_m]]$ generated by its tangent vectors T_i $(1 \leqslant i \leqslant m)$ where $T_i = m\omega_i^{m-1}(1, \omega_i ,..., \omega_i^{r-1}, 0, \omega_i^{r+1},..., \omega_i^{m-1})$. Thus $\hat{C}_\mathfrak{n} \subset k[[z_1]] \times \cdots \times k[[z_m]]$ represents m lines in general linear position in $(m - 1)$-space and is smoothable (see [12]). Hence $B = B_H$ is smoothable.

Summarizing the results of 5.9 and 5.10 we have the following.

COROLLARY 5.11. *Every negatively graded semigroup ring* B_H *can be negatively smoothed. In turn, there exists a smooth projective curve* X *with base point* x *s.t.* H *consists of the orders of poles at* x *of rational functions on* X, *regular on* $X - \{x\}$.

COROLLARY 5.12. *There is no rigid semigroup, i.e.,* $T^1(H) = 0$ *if and only if* $H = \mathbb{N}$.

It would be interesting to know in exactly what generality 5.4 can be used to inductively smooth the semigroup ring B_H. In concluding this section we give an example for which we actually obtain a smoothing. We do not know the full implications of the following remark, but will take it up at a later date.

Remark 5.13. If H is any numerical semigroup with standard basis $S_H = \{m = n_0 < n_1 < \cdots < n_{m-1}\}$, m the multiplicity of H, let H^* denote the semigroup generated by $\{n + m \mid n \in H\}$ (i.e., H^* is obtained from H via right translation by m). Then we note that the standard basis for H^* is given by $\{m = n_0^* < n_1^* < \cdots < n_{m-1}^*\}$ where $n_i^* = n_i + m$ and that this is also the minimal generating set for H^*. In particular (*) is valid for H^* and the family C_α defines a deformation of $B^* = B_{H^*}$. We note that in taking the quadratic transform of H^* we again obtain H.

The fiber $C = C_{\alpha=1}$ has two singularities occurring at $\mathfrak{m} = (z^m, z^{n_1}f, \ldots, z^{n_{m-1}}f)$ and at $\mathfrak{n} = (f, z^{n_1}f, \ldots, z^{n_{m-1}}f)$ where $f = z^m - 1$. Now $C_\mathfrak{m} \simeq B_{\mathfrak{m}'}$ where $B = B_H$ and $\mathfrak{m}' = (t^m, t^{n_1}, \ldots, t^{n_{m-1}})$. Also $C_\mathfrak{n} \simeq k[f, zf, \ldots, z^{m-1}f]_{\mathfrak{n}'}$ where $\mathfrak{n}' = (f, zf, \ldots, z^{m-1}f)$ and hence is smoothable. Thus if H can be smoothed so can H^*. This is precisely the situation we encountered in smoothing the ordinary and hyperordinary cases. We draw one immediate conclusion and hope to develop more in the future.

COROLLARY 5.14. *Suppose $H = H_{g,\lambda} + qm$ where $H_{g,\lambda}$ is negatively graded (of the third type) of multiplicity m and $q \geqslant 1$. Then $B = B_H$ can be smoothed.*

PROPOSITION 5.15. *Let S_m be the coordinate ring of m lines in general linear position in m space. Then S_m can be (negatively) smoothed.*

Proof. The result is well known for $m \leqslant 2$; so we assume $m \geqslant 3$ and proceed by induction.

By homogeneous change of coordinates we can assume that S_m is the coordinate ring of the coordinate axes in m space, i.e.,

$$\begin{aligned} S_m &\simeq k[X_1] \times_k \cdots \times_k k[X_m] \\ &\simeq k[X_1, \ldots, X_m]/\{X_i X_j\}_{i \neq j}. \end{aligned}$$

Thus $S_{m+1} \simeq S_m \times_k k[X_{m+1}]$ where the k-algebra map $S_m \to k$ is defined by $X_i \to 0$ $(1 \leqslant i \leqslant m)$ and $k[X_{m+1}] \to k$ is defined by $X_{m+1} \to 0$.

Now $S_3 \simeq k[X, Y, Z]/(XZ, XY, YZ)$. Let $R = k[t]$ and set $B = R[X, Y, Z]/((X - t)Z, XY, YZ)$. Then B is R-*flat* (since if $\bar{B}$ denotes the integral closure of B in its total ring of fractions, $\bar{B} \simeq k[t] \otimes_k (k[X_1] \oplus k[X_2] \oplus k[X_3])$ is $k[t]$-free and $\bar{B}/B$ is a finitely generated $k[t]$-module of constant rank 2, hence $k[t]$ projective. Hence B is R-flat). Hence (B, R) is a deformation of S_3 whose fiber away from $t = 0$ is two ordinary double points, and hence is smoothable. Thus S_3 is smoothable. We define an R-algebra map $g: B \to R$ via $X \to t$, $Y \to 0$, $Z \to 0$ and note that it extends our mapping $S_3 \to k$. Therefore $(B \times_R R[X_4], R)$ is a deformation of $S_4 \simeq S_3 \times_k k[X_4]$ whose fiber away from $t = 0$ is an ordinary triple point and an ordinary double point, and hence is smoothable. Thus S_4 is smoothable. Inductively, we see that S_m is smoothable whenever $m \geqslant 3$.

6. A Dimension Formula for the Coarse Moduli Space $\mathscr{M}_{g,1}$

(6.1) $\mathscr{M}_{g,1}$ will denote the coarse moduli space of smooth projective curves of genus g with a section (i.e., curves of genus g together with a base point). Consult [10] for precise definition and details.

If H ($\neq \mathbb{N}$) is a numerical semigroup of genus g, let $\mathscr{M}_H$ denote the subscheme of $\mathscr{M}_{g,1}$ defined by

$$\mathscr{M}_H = \{(X, x) \in \mathscr{M}_{g,1} \mid H_{X,x} = H\}.$$

(Recall the notation of 4.3.) Then $\mathscr{M}_{g,1} = \bigsqcup_{g(H)=g} \mathscr{M}_H$ where the union indicated is disjoint. If $\lambda(H) = [\mathrm{End}(H) : H]$ (so that $1 \leqslant \lambda \leqslant g$), then we define $\mathscr{M}_{g,1}^{(\lambda)}$ by

$$\mathscr{M}_{g,1}^{(\lambda)} = \bigsqcup_{\substack{g(H)=g \\ \lambda(H)=\lambda}} \mathscr{M}_H ,$$

where the union indicated is again disjoint. Thus

$$\dim \mathscr{M}_{g,1}^{(\lambda)} = \max_{\substack{g(H)=g \\ \lambda(H)=\lambda}} \{\dim \mathscr{M}_H\}.$$

We recall the situation described in Section 5.0. For a fixed semigroup H let (B, S) denote the versal deformation of B_H where

$$S = k[[t_1, ..., t_r]]/J,$$
$$B = S[[X_1, ..., X_m]]/(F^\infty),$$

and t_i has weight $-e_i$ and X_i has weight n_i. Let N denote the ideal of S generated by the images of those t_i s.t. the corresponding weight $-e_i$ is negative. Set $S' = S/NS$ and $B' = B/NB$ so that (B', S') is a deformation of B_H.

Set $U = \{x \in \mathrm{Spec}(S') \mid$ the fiber above x in B' is smooth$\}$. Then Pinkham [12] has shown that U is invariant under the $\mathbb{G}_m$ action on S' and if $\bar{U}$ denotes the quotient of U by the $\mathbb{G}_m$ action, then $\bar{U} \simeq \mathscr{M}_H$. Thus $\dim \mathscr{M}_H = \dim U - 1$.

By the work of Deligne [4] we know $\dim U \leqslant 3\delta + d - c$ where we recall $\delta = \dim(k[t]/B_H) = g(H) = g$; $c = \dim(k[t]/C) = c(H)$; and $C =$ the conductor ideal of B_H, $d = \dim(\mathrm{Coker}(\mathrm{Hom}_{B_H}(\Omega_{k[t]}, C) \to \mathrm{Hom}_{B_H}(\Omega_{B_H}, B_H)))$. Now $c = [\mathbb{N} : c + \mathbb{N}] = [\mathbb{N} : H] + [H : c + \mathbb{N}] = g + l$ where $g = g(H)$, $l = l(H) = [H : c + \mathbb{N}]$.

In our case the formula above can be simplified. Set $B = B_H$, $\bar{B} = \bar{B}_H = k[t]$, and $K =$ the fraction field of $B_H =$ the fraction field of $k[t]$. Since C is torsion free we have $\mathrm{Hom}_B(\Omega_{\bar{B}}, C) \subset \mathrm{Hom}_B(\Omega_{\bar{B}}, C) \otimes_B K \simeq \mathrm{Hom}_K(\Omega_K, K)$. Similarly, $\mathrm{Hom}_{\bar{B}}(\Omega_{\bar{B}}, C) \subset \mathrm{Hom}_K(\Omega_K, K)$. By viewing each as a submodule of $\mathrm{Hom}_K(\Omega_K, K)$ we see that $\mathrm{Hom}_B(\Omega_{\bar{B}}, C) \simeq \mathrm{Hom}_{\bar{B}}(\Omega_{\bar{B}}, C) \simeq \mathrm{Der}_k(\bar{B}, C)$. Setting $D(B) = \mathrm{Der}_k(B, B)$ and $D(\bar{B}) = \mathrm{Der}_k(\bar{B}, \bar{B})$ we have $D(B) \subset D(\bar{B})$

(since k is of characteristic 0) and each contains $\mathrm{Der}_k(\bar{B}, C)$. Now $\dim \mathrm{Der}_k(\bar{B}, C) = c$ and $d = \dim(\mathrm{Coker}(\mathrm{Der}_k(\bar{B}, C) \to D(B)))$ entails $c - d = [D(\bar{B}) : D(B)]$ so that $3\delta + d - c$ can be replaced by $3\delta - [D(\bar{B}) : D(B)] = 3g - [D(\bar{B}) : D(B)]$.

Now $D(\bar{B})_n$ is one dimensional for all $n \geqslant -1$ entails

$$\begin{aligned}[D(\bar{B}) : D(B)] &= \#\{n \geqslant -1 \mid D(B)_n = 0\} \\ &= 1 + \{n \geqslant 0 \mid n + H^+ \not\subset H\} \\ &= 1 + [\mathbb{N} : \mathrm{End}(H)] \\ &= 1 + g - \lambda.\end{aligned}$$

Thus $3g - [D(\bar{B}) : D(B)] = 2g + \lambda - 1$.

COROLLARY 6.3. $\dim \mathscr{M}_{g,1}^{(\lambda)} = 2g + \lambda - 2$.

Proof. We saw (in 6.1) that if H is a semigroup s.t. $g(H) = g$ and $\lambda(H) = \lambda$, then $\dim \mathscr{M}_H = \dim U - 1 \leqslant 2g + \lambda - 2$. By 4.7 and 4.9 given any λ and g with $1 \leqslant \lambda \leqslant g$ we can find a negatively graded semigroup of given g and λ. By 5.11 the corresponding monomial curve B_H can be negatively smoothed. Hence for that H, $\dim U = 3g + d - c = 2g + \lambda - 1$ so that $\dim \mathscr{M}_H = 2g + \lambda - 2$. Thus

$$\dim \mathscr{M}_{g,1}^{(\lambda)} = \max_{\substack{g(H)=g \\ \lambda(H)=\lambda}} \{\dim \mathscr{M}_H\} = 2g + \lambda - 2.$$

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