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Triangular Matrix Representations

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In this paper we develop the theory of generalized triangular matrix representation in an abstract setting. This is accomplished by introducing the concept of a *set of left triangulating idempotents*. These idempotents determine a generalized triangular matrix representation for an algebra. The existence of a set of left triangulating idempotents does not depend on any specific conditions on the algebras; however, if the algebra satisfies a mild finiteness condition, then such a set can be refined to a "complete" set of left triangulating idempotents in which each "diagonal" subalgebra has no nontrivial generalized triangular matrix representation. We then apply our theory to obtain new results on generalized triangular matrix representations, including extensions of several well known results. © 2000 Academic Press

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INTRODUCTION

Throughout this paper K will denote a commutative ring with unity and R an associative K-algebra with unity, unless indicated otherwise. All modules are unital. We say R has a *generalized triangular matrix representation* if there exists a K-algebra isomorphism

$$\theta : R \to \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix}, \qquad (*)$$

where each R_i is a K-algebra with unity and R_{ij} is a left R_i-right R_j-bimodule for $i < j$.

Generalized triangular matrix representations provide an important tool in the investigation of the structure of a wide range of algebras. Some of the diverse applications associated with generalized triangular matrix representations appear in the study of operator theory [HKL, PS], quasitriangular Hopf algebras [CP], and various Lie algebras (Kac–Moody, Virasoro, and Heisenberg) [MP]. Previous authors (e.g., [CH1, GS, Ha, LZ]) have used a variety of conditions (e.g., CS, hereditary, semiprimary, Artinian) to obtain a generalized triangular matrix representation for an algebra.

In this paper we develop the theory of generalized triangular matrix representation in an abstract setting. This is accomplished by introducing the concept of a *set of left triangulating idempotents*. These idempotents determine a generalized triangular matrix representation for an algebra. The existence of a set of left triangulating idempotents does not depend on any specific conditions on the algebra (e.g., $\{1\}$ is a set of left triangulating idempotents); however, if the algebra satisfies a mild finiteness condition, then such a set can be refined to a "complete" set of left triangulating idempotents in which each "diagonal" subalgebra has no nontrivial generalized triangular matrix representation. When this occurs we say the generalized triangular matrix representation is *complete*. We then apply our theorem to obtain new results on generalized triangular matrix representations, including extensions of several well known results.

An idempotent $e \in R$ is *left* (respectively, *right*) *semicentral* in R if $Re = eRe$ (respectively, $eR = eRe$) [Bi4, p. 569]. As is well known [Ch], a left semicentral idempotent induces a 2-by-2 generalized triangular matrix representation on R. We use $\mathcal{S}_\ell(R)$ and $\mathcal{S}_r(R)$ for the sets of all left and right semicentral idempotents, respectively. Again taking e to be an idempotent of R, observe that $\mathcal{S}_\ell(eRe) = \{0, e\}$ if and only if $\mathcal{S}_r(eRe) = \{0, e\}$; when this occurs we say e is *semicentral reduced*. If 1 is semicentral reduced, then we say R is *semicentral reduced*. We say that R has a *complete generalized triangular matrix representation* if each R_i in $(*)$ is semicentral reduced (i.e., each R_i has no nontrivial generalized triangular matrix representation). In Section 1, we develop further properties of left (right) semicentral idempotents.

An ordered set $\{b_1, \ldots, b_n\}$ of nonzero distinct idempotents in R is called a set of *left triangulating idempotents* of R if all the following hold:

(i) $1 = b_1 + \cdots + b_n$;

(ii) $b_1 \in \mathcal{S}_\ell(R)$; and

(iii) $b_{k+1} \in \mathcal{S}_\ell(c_k R c_k)$, where $c_k = 1 - (b_1 + \cdots + b_k)$, for $1 \leq k \leq n - 1$.

Similarly we define a *set of right triangulating idempotents* of R using (i), $b_1 \in \mathcal{S}_r(R)$, and $b_{k+1} \in \mathcal{S}_r(c_k R c_k)$. From part (iii) of the above definition, a set of left (right) triangulating idempotents is a set of pairwise orthogonal idempotents.

In [KMW], the authors state that there is a "search for a new internal characterization of upper triangular matrix rings." The concept of a set of left triangulating idempotents can be used to obtain such an internal characterization (see Proposition 1.9). This characterization is different than that obtained in [SW].

A set $\{b_1, \ldots, b_n\}$ of left (right) triangulating idempotents of R is said to be *complete* if each b_i is also semicentral reduced. Proposition 1.3 shows that R has a (respectively, complete) set of left triangulating idempotents if and only if R has a (respectively, complete) generalized triangular matrix representation. In the sequel, the behavior of a complete set of left triangulating idempotents is shown to be "strictly between" that of a complete set of primitive idempotents and a complete set of centrally primitive idempotents (see Proposition 2.14(i), Example 2.15, Proposition 2.18, Proposition 2.20, and Example 2.21). We show in Corollary 1.7 that R has a (complete) set of left triangulating idempotents if and only if R has a corresponding (complete) set of right triangulating idempotents.

The condition that R has a complete set of left triangulating idempotents is—among other things—a type of finiteness condition. In Theorem 2.9 we show that having a complete set of left triangulating idempotents is equivalent to six other finiteness conditions, including $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ is finite, and $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ satisfies both ACC and DCC. Theorem 2.10 establishes the "uniqueness" of a complete set of left triangulating idempotents. From Proposition 2.14, if R satisfies almost any of the well known finiteness conditions, then R has a generalized triangular matrix representation with semicentral reduced subalgebras on the "main diagonal" which satisfy the same finiteness condition as R. This result and Proposition 2.16 show that the study of many well known classes of algebras can be reduced to the investigation of the algebras with no nontrivial generalized triangular matrix representation (i.e., semicentral reduced algebras) from the respective class.

In Section 3 we derive a canonical form for the generalized triangular matrix representation of R and use it to determine the global dimension of R.

We apply our theory to the class of quasi-Baer rings in Section 4. Theorem 4.4 and 4.11 describe quasi-Baer rings with a complete set of left triangulating idempotents. As corollaries of these results we obtain Michler's splitting theorem for right hereditary right Noetherian rings [Mi], Levy's decomposition of right Goldie semiprime right hereditary rings [Le], the characterization of a piecewise domain given in [GS], and the characterization of a semiprime right FPF ring with no infinite set of central orthogonal idempotents given in [Fa1].

Standard terminology and notation are adhered to as much as possible. Where there is conflict or confusion in the literature we define the term or notation as we plan to use it herein. We use $\mathbf{I}(R)$, $\mathbf{B}(R)$, and $\mathbf{N}(R)$ for the sets of idempotents, central idempotents, and nilpotents of R, respectively. Observe that $\mathcal{S}_r(R) \cap \mathcal{S}_\ell(R) = \mathbf{B}(R)$. If $\mathbf{N}(R) = 0$, then R is called *reduced*. We use $\mathbf{P}(R)$ for the prime radical of R (in the category of K-algebras), and for any nonempty subset X of R, write $r_R(X)$ for $\{c \in R \mid Xc = 0\}$ and $l_R(X)$ for $\{c \in R \mid cX = 0\}$, which are called the right annihilator of X in R and the left annihilator of X in R, respectively. The subscript R might be omitted when the context is clear. By prime ideal we mean a *proper* prime ideal of R. We denote by $T_n(R)$ and $M_n(R)$ the K-algebras of all n-by-n upper triangular matrices and n-by-n matrices, respectively, with entries from R.

1. TRIANGULATING IDEMPOTENTS

In this section we develop some basic properties of semicentral idempotents and triangulating idempotents. Among other things we establish a connection between triangulating idempotents and upper triangular matrix algebras. We use the notation $[a_{ij}]$ for the square matrix whose (i, j)th position is a_{ij}. Some of these results are known or are part of the "folklore"; however, we include them to make the paper self-contained. We begin with a technical lemma, the proof of which is routine and will be omitted.

LEMMA 1.1. *Let* $e \in \mathbf{I}(R)$. *Then the following conditions are equivalent*:

(i) $e \in \mathcal{S}_\ell(R)$;

(ii) $1 - e \in \mathcal{S}_r(R)$;

(iii) $xe = exe$, *for each* $x \in R$;

(iv) $(1 - e)Re = 0$;

(v) $(1 - e)x = (1 - e)x(1 - e)$, *for each* $x \in R$;

(vi) eR *is an ideal of* R;

(vii) $R(1 - e)$ *is an ideal of* R.

LEMMA 1.2. *Let* $\{b_1, \ldots, b_n\}$ *be a set of nonzero idempotents of* R. *Then the following are equivalent*:

(i) $\{b_1, \ldots, b_n\}$ *is a set of left triangulating idempotents*;

(ii) $\{b_1, \ldots, b_n\}$ *is an ordered set such that* $1 = b_1 + \cdots + b_n$ *and* $b_j R b_i = 0$, *for all* $i < j \leq n$.

Proof. (i) $\Rightarrow$ (ii). Since $b_2 \in (1 - b_1)R(1 - b_1)$, we have $b_2 b_1 = 0$ and hence $b_2 R b_1 = b_2 b_1 R b_1 = 0$, because $b_1 \in \mathcal{S}_\ell(R)$. Proceeding similarly one obtains $b_j R b_1 = 0$, for all $j > 1$. By hypothesis $b_2 \in \mathcal{S}_\ell((1 - b_1)R(1 - b_1))$. Let $R_1 = (1 - b_1)R(1 - b_1)$ and $c_1 = 1 - b_1$, the unity of R_1. Then $b_3 \in \mathcal{S}_\ell((c_1 - b_2)R(c_1 - b_2))$. Analogous to the calculation above, using R_1 in place of R and b_2 in place of b_1, we get $b_j b_2 = 0$ and $b_j R_1 b_2 = 0$, for all $j > 2$.

Thus for any $r \in R$, we have $0 = b_j[(1 - b_1)r(1 - b_1)]b_2 = [b_j(1 - b_1)r - b_j(1 - b_1)rb_1]b_2$. Since $b_j(1 - b_1)rb_1 = 0$, we have $0 = b_j(1 - b_1)rb_2 = b_j r b_2 - b_j b_1 r b_2 = b_j r b_2$. Thus $b_j R b_2 = 0$. Continue the process, using $R_2 = (1 - b_1 - b_2)R(1 - b_1 - b_2)$ and $c_2 = 1 - b_1 - b_2$ in the next step, and so on, to get $b_j R b_i = 0$ for all $j > i \geq 1$.

(ii) $\Rightarrow$ (i). We will make use of the equivalence of conditions (i) and (iv) in Lemma 1.1 repeatedly in this argument. Observe that $(1 - b_1)Rb_1 = (b_2 + \cdots + b_n)Rb_1 = 0$. So $b_1 \in \mathcal{S}_\ell(R)$. To see that $b_2 \in \mathcal{S}_\ell((1 - b_1)R(1 - b_1))$, observe that $(1 - b_1)b_2(1 - b_1) = (b_2 + \cdots + b_n)b_2(1 - b_1) = b_2(1 - b_1) = b_2$. Next, since $(1 - b_2)(1 - b_1) = (b_1 + b_3 + \cdots + b_n)(1 - b_1) = b_3 + b_4 + \cdots + b_n$, we have $(1 - b_2)[(1 - b_1)R(1 - b_1)]b_2 = \sum_{i=3}^n b_i R(1 - b_1)b_2 = \sum_{i=3}^n b_i R b_2 = 0$. Continuing this process yields the desired result.

PROPOSITION 1.3. *R has a* (*respectively*, *complete*) *set of left triangulating idempotents if and only if R has a* (*respectively*, *complete*) *generalized triangular matrix representation.*

Proof. Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of R. Using Lemma 1.2 and a routine argument shows that the mapping

$$\phi : R \to \begin{pmatrix} b_1Rb_1 & b_1Rb_2 & \cdots & b_1Rb_n \\ 0 & b_2Rb_2 & \cdots & b_2Rb_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_nRb_n \end{pmatrix}$$

defined by $\phi(r) = [r_{ij}]$, where $r_{ij} = b_irb_j$, is a K-algebra isomorphism.

Conversely, assume

$$\theta : R \to \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix}$$

is a K-algebra isomorphism. Then by routine calculations $\{\theta^{-1}(e_1), \ldots, \theta^{-1}(e_n)\}$ is a set of left triangulating idempotents of R, where e_i is the n-by-n matrix with the unity of R_i in the (i,i)-position and 0 elsewhere.

Thus if the set of left triangulating idempotents is complete, then the "diagonal" algebras are semicentral reduced. Also note that R is semicentral reduced if and only if R has no nontrivial generalized triangular matrix representation.

LEMMA 1.4. *Let $e \in \mathcal{S}_\ell(R)$ and $f \in \mathbf{I}(R)$. Then:*

(i) $\mathcal{S}_\ell(eRe) \subseteq \mathcal{S}_\ell(R)$;

(ii) $f\mathcal{S}_\ell(R)f \subseteq \mathcal{S}_\ell(fRf)$;

(iii) *if f is a primitive idempotent of R such that $fe \neq 0$, then $fef = f$ and efe is a primitive idempotent in eRe*;

(iv) *if $f \in \mathcal{S}_r(R)$ and X is an ideal of R, then eXf is an ideal of R.*

Proof. (i) Let $g \in \mathcal{S}_\ell(eRe)$. Then $gRg = geReg = eReg = Rg$. So $g \in \mathcal{S}_\ell(R)$.

(ii) Let $g \in \mathcal{S}_\ell(R)$ and $r \in R$. Then by Lemma 1.1, $(fgf)(frf)(fgf) = (ff)(frf)(fgf) = (frf)(fgf)$. So $fgf \in \mathcal{S}_\ell(fRf)$ by Lemma 1.1.

(iii) First observe that $0 \neq fe = fefe$; so $fef \neq 0$. Then primitivity of f implies the idempotent fef must be f. Let u be a nonzero idempotent in $(efe)(eRe)(efe)$. Routine calculation, making use of $e \in \mathcal{S}_\ell(R)$ and f an idempotent, yields that $ue = u$, $fu = u$, $uf = fuf$, and $(uf)(uf) = uf$. Since $uf = 0$ implies $u = ufe = 0$, we have that uf is a nonzero idempotent in

fRf. Primitivity of f in R then yields $uf = f$. Then $u = ufe = fe = efe$. So efe is the only nonzero idempotent in eRe.

(iv) Observe that $R(eXf)R = eReXfRf \subseteq eXf$.

LEMMA 1.5. (i) *If h is a K-algebra homomorphism from R into a K-algebra A, then $h(\mathcal{S}_\ell(R)) \subseteq \mathcal{S}_\ell(h(R))$.*

(ii) *Let $e \in \mathcal{S}_\ell(R) \cup \mathcal{S}_r(R)$ and $f \in \mathcal{S}_\ell(eRe) \cup \mathcal{S}_r(eRe)$. The function $h : R \to fRf$, defined by $h(r) = frf$, for each $r \in R$, is a K-algebra homomorphism.*

Proof. The proof of part (i) is straightforward. For (ii), observe $f \in eRe$ implies $ef = f = fe$. So for each $x, y \in R$, $fxyf = fexyef$. Using $e \in \mathcal{S}_\ell(R) \cup \mathcal{S}_r(R)$ and $f \in \mathcal{S}_\ell(eRe) \cup \mathcal{S}_r(eRe)$, we have that $fxyf = fexeyef = fexefeyef = fxfyf$ by Lemma 1.1. So $h(xy) = fxf^2yf = h(x)h(y)$.

PROPOSITION 1.6. *Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of R and let $c_k = 1 - (b_1 + \cdots + b_k)$, $k = 1, \ldots, n-1$. Then we have the following*:

(i) $c_k \in \mathcal{S}_r(R)$, $k = 1, \ldots, n-1$;

(ii) $b_1 + \cdots + b_k \in \mathcal{S}_\ell(R)$, $k = 1, \ldots, n$;

(iii) *the function $h_j : R \to b_jRb_j$, defined by $h_j(r) = b_jrb_j$, for all $r \in R$, is a K-algebra homomorphism*;

(iv) *let h be a K-algebra homomorphism from R onto A, then some ordered subset H formed from the list $h(b_1), \ldots, h(b_n)$ is a set of left triangulating idempotents of A*;

(v) *$b_i \in \mathcal{S}_\ell(R)$ if and only if $b_jRb_i = 0$ for all $j < i$*;

(vi) *$b_i \in \mathcal{S}_r(R)$ if and only if $b_iRb_j = 0$ for all $j > i$.*

Proof. (i) Recall that $b_1 \in \mathcal{S}_\ell(R)$ implies $c_1 = 1 - b_1 \in \mathcal{S}_r(R)$ by Lemma 1.1. Since $b_2 \in \mathcal{S}_\ell(c_1Rc_1)$ and $c_2 = 1 - b_1 - b_2$ is in $\mathcal{S}_r(c_1Rc_1)$, using the right-sided analog to Lemma 1.4(i) we get that $c_2 \in \mathcal{S}_r(R)$. Using this procedure an induction proof completes the argument.

(ii) This part follows immediately from part (i) and Lemma 1.1.

(iii) Refer to Lemma 1.5(ii). Use $e = c_k$ and $b = b_{k+1}$. Then from part (i) above we have $e \in \mathcal{S}_r(R)$ and by definition we have $b \in \mathcal{S}_\ell(eRe)$. So the mapping given by $r \to brb$ is a K-algebra homomorphism.

(iv) This part follows from a routine application of Lemma 1.5(i).

(v) Assume $b_i \in \mathcal{S}_\ell(R)$. Since $\{b_1, \ldots, b_n\}$ is a set of pairwise orthogonal idempotents, Lemma 1.1(iii) yields $b_jRb_i = b_jb_iRb_i = 0$ for $j < i$. Conversely, assume $b_jRb_i = 0$, for all $j < i$. Use the triangulating properties of b_1 and b_i, together with Lemma 1.2, to obtain $b_kRb_i = 0$ for all

$k > i$. Therefore $(1 - b_i)Rb_i = (\sum_{k \neq i} b_k)Rb_i = 0$. Lemma 1.1 then yields $b_i \in \mathcal{S}_\ell(R)$.

(vi) Proceed analogously as in the proof of part (v), using the right-sided versions of Lemmas 1.1 and 1.2 where necessary.

COROLLARY 1.7. *The ordered set* $\{b_1, \dots, b_n\}$ *is a (complete) set of left triangulating idempotents of* R *if and only if* $\{b_n, \dots, b_1\}$ *is a (complete) ordered set of right triangulating idempotents.*

Proof. Let $b_1, \dots, b_n$ be a set of left triangulating idempotents of R. Then by Proposition 1.6(i), $1 - (b_1 + \cdots + b_{n-1}) = b_n \in \mathcal{S}_r(R)$. We next show that $b_{n-1} \in \mathcal{S}_r((1 - b_n)R(1 - b_n))$. Let $R' = (1 - b_n)R(1 - b_n)$ and note that $1 - b_n$ is the unity in this algebra. Let $d = b_1 + \cdots + b_{n-2}$. Proposition 1.6(ii) yields that $d \in \mathcal{S}_\ell(R)$. Using the orthogonality of the b_j we obtain $d = (1 - b_n)d(1 - b_n)$, and Lemma 1.4(ii) gives $(1 - b_n)\mathcal{S}_\ell(R)(1 - b_n) \subseteq \mathcal{S}_\ell((1 - b_n)R(1 - b_n))$; so $d \in \mathcal{S}_\ell(R')$. Consequently $b_{n-1} = (1 - b_n) - d$ is in $\mathcal{S}_r(R')$. Repeat the argument, using $R'' = (1 - b_n - b_{n-1})R(1 - b_n - b_{n-1})$ and $d' = b_1 + b_2 + \cdots + b_{n-3}$, to get that $d' \in \mathcal{S}_\ell(R'')$ and consequently $b_{n-2} \in \mathcal{S}_r(R'')$. This argument is then repeated until the set $b_n, b_{n-1}, \dots, b_1$ is exhausted.

The right−implies−left converse is proved similarly. Since $\mathcal{S}_\ell(b_iRb_i) = \{0, b_i\}$ if and only if $\mathcal{S}_r(b_iRb_i) = \{0, b_i\}$, "completeness" is left-right symmetric.

COROLLARY 1.8. *Let* R *be a subdirectly irreducible* K*-algebra. If* $\{b_1, \dots, b_n\}$ *is a set of left triangulating idempotents, then*:

(i) *for each* $i \neq 1$ *there exists* $j < i$ *such that* $b_jRb_i \neq 0$;

(ii) *for each* $i \neq n$ *there exists* $j > i$ *such that* $b_iRb_j \neq 0$;

(iii) *the heart is contained in* b_1Rb_n.

Proof. By definition, $b_1 \in \mathcal{S}_\ell(R)$. If $b_k \in \mathcal{S}_\ell(R)$, for $k > 1$, then Lemma 1.1(vi) shows that b_kR is an ideal. But $b_1b_k = 0$, so $b_1R \cap b_kR = 0$, a contradiction. From Proposition 1.6(v), there is some $j < i$ such that $b_jRb_i \neq 0$. This shows part (i). An analogous argument yields part (ii). Part (iii) is a consequence of Lemma 1.4(iv).

From Corollary 1.8, one can see that if a subdirectly irreducible K-algebra has a generalized triangular matrix representation then in each column except for the first there are nonzero entries off the diagonal, and for each row except the last there are nonzero entries off the diagonal. Our final result of this section shows that a set of left triangulating idempotents can be used to provide an internal characterization of an upper triangular matrix algebra.

PROPOSITION 1.9. *$R \cong T_n(A)$ for some K-algebra A if and only if there exists a set of left triangulating idempotents $\{b_1, \ldots, b_n\}$ of R such that*:

(i) *there exist K-algebra isomorphisms $\phi_j : b_j R b_j \to b_1 R b_1$ for all j, $1 \le j \le n$; and*

(ii) *there exist group isomorphisms $\theta_{ij} : b_i R b_j \to b_1 R b_1$ such that*

(α) *$b_1 r b_1 \cdot (\theta_{ij}(b_i s b_j)) = \theta_{ij}((\phi_i^{-1}(b_1 r b_1) \cdot b_i s b_j)$, and*

(β) *$(\theta_{ij}(b_i s b_j)) \cdot b_1 r b_1 = \theta_{ij}(b_i s b_j \cdot (\phi_j^{-1}(b_1 r b_1)))$ for all i, j, $1 \le i, j \le n$ and $r, s \in R$.*

Proof. Let e_{ij} be the n-by-n matrix with the unity of A in the (i, j)-position and zero elsewhere. Assume $\psi : T_n(A) \cong R$ is a K-algebra isomorphism. Let $b_i = \psi(e_{ii})$. Then $\{b_1, \ldots, b_n\}$ is a set of left triangulating idempotents. Define $\phi_j : b_j R b_j \to b_1 R b_1$ by $\phi_j(b_j r b_j) = \psi(e_{1j}\psi^{-1}(r)e_{j1})$ for all $r \in R$. Define $\theta_{ij} : b_i R b_j \to b_1 R b_1$ by $\theta_{ij}(b_i r b_j) = \psi(e_{1i}\psi^{-1}(r)e_{j1})$ for all $r \in R$. Routine, but technical arguments, show that the ϕ_j and θ_{ij} have the desired properties.

Conversely, define $\Phi : R \to T_n(b_1 R b_1)$ by

$$\Phi(R) = \begin{pmatrix} r_1 & r_{12} & \cdots & r_{1n} \\ 0 & r_2 & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix},$$

where $r_j = \phi_j(b_j r b_j)$ and $r_{ij} = \theta_{ij}(b_i r b_j)$. A routine, but technical argument, shows that Φ is a K-algebra isomorphism.

2. GENERALIZED TRIANGULAR MATRIX REPRESENTATIONS

Since the concept of "semicentral reduced" plays a key role in the main results of this section, we begin by deriving several fundamental properties of semicentral reduced algebras. Our main theorem provides a characterization of a K-algebra with a complete set of left triangulating idempotents in terms of finiteness conditions on ideals generated by left semicentral idempotents. We then consider the "uniqueness" of a complete set of triangulating idempotents. Next we show that if R satisfies almost any of the well known finiteness conditions, then R has a generalized triangular matrix representation with semicentral reduced subalgebras on the "main diagonal" which satisfy the same finiteness condition as R. Also we show that the condition of having a complete set of left triangulating idempotents is "strictly between" that of having a complete set of primitive idempotents and that of having a complete set of centrally primitive idempotents.

LEMMA 2.1. *The following are equivalent*:

(i) *R is semicentral reduced*;

(ii) *$(1-e)Re$ and $eR(1-e)$ are both nonzero for each nontrivial idempotent e*;

(iii) *if e is a nontrivial idempotent of R and A and B are subsets of R that contain e and $1-e$, respectively, then neither ARB nor BRA can be zero*;

(iv) *if X is a right ideal and $e \in \mathbf{I}(R)$ such that $1-e \in X$, then $XeR = 0$ implies $e = 0$ or $e = 1$.*

Proof. Assume (i). Then (ii) follows from Lemma 1.1 and its right-sided analog. Assume (ii). Since $ARB = 0$ implies $eR(1-e) = 0$, and $BRA = 0$ implies $(1-e)Re = 0$, when e is nontrivial, we have neither ARB nor BRA can be zero. Assume (iii). If X is a right ideal of R and $e \in \mathbf{I}(R)$ such that $1-e \in X$ and $XeR = 0$, then $(XR)eR \subseteq XeR = 0$ and hence $e = 0$ or $e = 1$. Finally, assume (iv). Consider $e \in \mathcal{S}_\ell(R)$. Then $((1-e)R)eR = (1-e)(eRe)R = 0$. So using $X = (1-e)R$ in (iv), we get e is 0 or 1.

COROLLARY 2.2. *Let R be semicentral reduced. Then exclusively, either*:

(i) *R is reduced and $\mathbf{I}(R) = \{0,1\}$; or*

(ii) *for each nonzero idempotent $e \in R$, eR and Re each contain nonzero nilpotent elements.*

Proof. If R is reduced, then all idempotents are central. So $\mathbf{I}(R) = \mathcal{S}_\ell(R) = \{0,1\}$. Part (ii) follows from the equivalence of parts (i) and (ii) in Lemma 2.1.

Observe that if R is a prime K-algebra, then R is semicentral reduced. Thus one may think of Corollary 2.2 as generalizing the result that a prime ring is either a domain or every nonzero ideal contains a nonzero nilpotent element [Bi1, Proposition 3.1].

We use $\mathrm{Soc}(R_R)$ for the sum of the minimal right ideals of R.

PROPOSITION 2.3. *Let R be semicentral reduced with $\mathrm{Soc}(R_R) \neq 0$. Then either R is a skewfield or $\mathrm{Soc}(R_R) = U \oplus V \oplus X$, as a direct sum of right ideals, where*:

(i) *U is an ideal of R and $U^2 = 0$;*

(ii) *V is a direct sum of minimal ideals V_j (of R), where $V_j = Re_jR$, $e_j \in \mathbf{I}(R)$, and each e_jR is a minimal right ideal of R with $e_jR \cap \mathbf{N}(R) \neq 0$;*

(iii) *X is a direct sum of the minimal right ideals X_i (of R), where each X_i is isomorphic as a right R-module to a nilpotent right ideal of R and each X_i is generated by an idempotent.*

Proof. First consider R reduced. Since $\mathrm{Soc}(R_R) \neq 0$, there is a minimal right ideal of R of the form eR for some $e \in \mathbf{I}(R)$. Since R is reduced, $e \in \mathbf{B}(R)$. Hence $R = eR$. By [J, Proposition 1, p. 65], R is a skewfield. Next consider $\mathbf{N}(R) \neq 0$. Then using a K-algebra version of Theorem 4.6 of [BHL], we get $\mathrm{Soc}(R_R) = U \oplus V \oplus X$, where V and X are as desired and $U = D \oplus B$, where D is a direct sum of minimal right ideals of R which are skewfields and B is a square zero ideal. However, Lemma 2.1 forces D to be zero, thus completing the decomposition as described.

Note that V_j is a homogeneous component of $e_j R$ and $e_j R$ cannot be isomorphic (as an R-module) to a square zero right ideal of R (otherwise V_j would contain a nonzero nilpotent ideal, contrary to V_j being a minimal ideal).

It is worth noting that in the above proposition if R is also assumed to be semiprime, then either R is a skewfield or $\mathrm{Soc}(R_R) = V$.

In the next corollary we use the notation introduced in Proposition 2.3 and its proof.

COROLLARY 2.4. *The following are equivalent*:

(i) *R is semicentral reduced with a complete set of primitive idempotents and $V \neq 0$*;

(ii) *R is isomorphic to a full n-by-n matrix algebra over a skewfield*, *for some n*.

Proof. Assume R is semicentral reduced, $V \neq 0$, and $\{e_1, \ldots, e_n\}$ is a complete set of primitive idempotents of R. First observe either $e_i R \cap V = 0$ or $e_i R \subseteq V$. To see this suppose $e_i R \cap V \neq 0$. Then there exists a minimal right ideal cR such that $c = c^2$ and $cR \subseteq e_i R \cap V$. By the primitivity of e_i, then $e_i R = cR \subseteq V$. Let $b = \sum e_i$, where $e_i \in V$. So $V = V \cap (bR \oplus (1-b)R) = bR \oplus (V \cap (1-b)R)$. Observe that $1 - b = \sum e_j$, where $e_j V = 0$. Thus $V \cap (1-b)R = 0$ and hence $V = bR$. By Lemma 1.1, $b \in \mathcal{S}_\ell(R)$; so $R = bR = V$. Hence $e_i R$ is a minimal right ideal for all $i = 1, \ldots, n$. Therefore R is isomorphic to a full n-by-n matrix algebra over a skewfield. The converse is immediate.

As a direct consequence of Corollary 2.4 we have the well known result that a prime ring with nonzero socle and a complete set of primitive idempotents is isomorphic to a full matrix ring over a skewfield.

We begin now a buildup to one of our main results, Theorem 2.9, and its several useful corollaries.

LEMMA 2.5. *Let* $0 \neq f \in \mathbf{I}(R)$. *Assume* $eR = fR$ *for each* $0 \neq e \in \mathcal{S}_{\ell}(fRf)$. *Then* f *is semicentral reduced.*

Proof. Let $e \in \mathcal{S}_{\ell}(fRf)$. Then since $eR = fR$, it follows that $f = ex$ for some $x \in R$. So $e = ef = eex = ex = f$; so f is semicentral reduced.

LEMMA 2.6. (i) *R has DCC on* $\{bR \mid b \in \mathcal{S}_{\ell}(R)\}$ *if and only if R has ACC on* $\{Rc \mid c \in \mathcal{S}_r(R)\}$.

(ii) *R has ACC on* $\{br \mid b \in \mathcal{S}_{\ell}(R)\}$ *if and only if R has DCC on* $\{Rc \mid c \in \mathcal{S}_r(R)\}$.

(iii) *If R has DCC on* $\{Rc \mid c \in \mathcal{S}_r(R)\}$, *then R has DCC on* $\{cR \mid c \in \mathcal{S}_r(R)\}$.

Proof. (i) Assume R has DCC on $\{bR \mid b \in \mathcal{S}_{\ell}(R)\}$. Consider a chain $Rc_1 \subseteq Rc_2 \subseteq \dots$, where $c_i \in \mathcal{S}_r(R)$. Then $(1 - c_1)R \supseteq (1 - c_2)R \supseteq \dots$, with $1 - c_i \in \mathcal{S}_{\ell}(R)$. This descending chain becomes stationary, say with $(1 - c_n)R = (1 - c_{n+j})R$, for each $j \geq 1$. Then $c_{n+j}(1 - c_n)R = 0$ and hence $c_{n+j} = c_{n+j}c_n$. So $Rc_{n+j} = Rc_n$. The converse is proved similarly.

(ii) The proof of this part is analogous to that of part (i).

(iii) Assume R has DCC on $\{Rc \mid c \in \mathcal{S}_r(R)\}$. Let $c_1R \supseteq c_2R \supseteq \cdots$ be a descending chain with $c_i \in \mathcal{S}_r(R)$. Observe $c_{i+1} = c_ic_{i+1} = c_ic_{i+1}c_i$. But multiplying $c_{i+1} = c_ic_{i+1}$ on the right yields $c_{i+1}c_i = c_ic_{i+1}c_i$. Hence $c_{i+1} = c_ic_{i+1} = c_{i+1}c_i$, so $Rc_i \supseteq Rc_{i+1}$. Thus we have the descending chain $Rc_1 \supseteq Rc_2 \supseteq \cdots$. Then there exists n such that $Rc_n = Rc_{n+1}$. A routine argument yields $(1 - c_n)R = (1 - c_{n+1})R$. Hence $(1 - c_n)Rc_n = (1 - c_{n+1})Rc_{n+1}$. Observe that $Rc_n = c_nRc_n + (1 - c_n)Rc_n = c_nR + (1 - c_n)Rc_n$, and $Rc_{n+1} = c_{n+1}Rc_{n+1} + (1 - c_{n+1})Rc_{n+1} = c_{n+1}R + (1 - c_{n+1})Rc_{n+1}$. Since $c_nR \supseteq c_{n+1}R$ and $c_nR + (1 - c_n)Rc_n = c_{n+1}R + (1 - c_{n+1})Rc_{n+1}$, then $c_nR = c_{n+1}R$.

LEMMA 2.7. *Let* $e \in \mathcal{S}_r(R)$. *If R has DCC on* $\{bR \mid b \in \mathcal{S}_{\ell}(R)\}$, *then eRe has DCC on* $\{d(eRe) \mid d \in \mathcal{S}_{\ell}(eRe)\}$.

Proof. Let $R_1 = eRe$ and $c_i \in \mathcal{S}_r(R_1)$ such that $R_1c_i \subseteq R_1c_{i+1}$, $i = 1, 2, \dots$, is an ascending chain. By the right-sided analog of Lemma 1.4(i), $c_i \in \mathcal{S}_r(R)$. Since $(1 - e)Rec_i = (1 - e)Rec_ie = (1 - e)Rc_i \subseteq (1 - e)Rc_{i+1}$, it follows that $Rc_i \subseteq Rc_{i+1}$. By Lemma 2.6, there exists a positive integer n such that $Rc_n = Rc_{n+1}$. But $Rec_n = Rc_n = Rc_{n+1} = Rec_{n+1}$. So $R_1c_n = R_1c_{n+1}$. From Lemma 2.6, R_1 has DCC on $\{dR_1 \mid d \in \mathcal{S}_{\ell}(R_1)\}$.

LEMMA 2.8. *Let* $\{b_1, \dots, b_n\}$ *be a complete set of left triangulating idempotents for R.*

(i) *If* $e \in \mathcal{S}_{\ell}(R)$, *then* $eR = \oplus_i b_iR$, *as a direct sum of right ideals of R, where the sum runs over a subset of* $\{1, \dots, n\}$.

(ii) *There are at most* 2^n *distinct right ideals of the form* eR *with* $e \in \mathcal{S}_\ell(R)$.

Proof. Let $0 \neq e \in \mathcal{S}_\ell(R)$. Consider i such that $b_i e \neq 0$. We show that $b_i eR = b_i R$. Observe that by Lemma 1.1, $b_i e b_i e = b_i e \neq 0$; so $b_i e b_i \neq 0$. Recall from Lemma 1.4(ii) that $b_i \mathcal{S}_\ell(R) b_i \subseteq \mathcal{S}_\ell(b_i R b_i)$. So $b_i e b_i \in \mathcal{S}_\ell(b_i R b_i)$, but by hypothesis this latter set is $\{0, b_i\}$. So $b_i e b_i = b_i$ and $b_i R = b_i e b_i R \subseteq b_i eR \subseteq b_i R$; hence $b_i eR = b_i R$. Recall that the b_i are pairwise orthogonal. This and $b_i e b_j e = b_i b_j e$ yields $b_1 e, \ldots, b_n e$ are pairwise orthogonal idempotents. Let $I = \{i \mid 1 \leq i \leq n \text{ and } b_i e \neq 0\}$. Then $eR = \oplus_{i \in I} b_i eR = \oplus_{i \in I} b_i R$. Observe that there can be no more than 2^n such direct sums.

Note that if in the above the b_i are all central, then we achieve the upper bound of 2^n.

The following result, which is one of our main theorems, fully characterizes rings which have a complete generalized triangular matrix representation.

THEOREM 2.9. *The following conditions are equivalent*:

(i) *R has a complete set of left triangulating idempotents*;

(ii) $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ *is a finite set*;

(iii) $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ *satisfies ACC and DCC*;

(iv) $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ *and* $\{Rc \mid c \in \mathcal{S}_r(R)\}$ *satisfy ACC*;

(v) $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ *and* $\{Rc \mid c \in \mathcal{S}_r(R)\}$ *satisfy DCC*;

(vi) $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ *and* $\{cR \mid c \in \mathcal{S}_r(R)\}$ *satisfy DCC*;

(vii) *R has a complete set of right triangulating idempotents.*

(viii) *R has a complete generalized triangular matrix representation.*

Proof. Lemma 2.8 yields (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) is trivial. Then (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) follow immediately from Lemma 2.6. The equivalence of (vii) and (i) follows from Corollary 1.7. Finally, assume (vi). We will show (vi) $\Rightarrow$ (i). If $\mathcal{S}_\ell(R) = \{0, 1\}$, then we are finished. Otherwise take e_1 to be a nontrivial element of $\mathcal{S}_\ell(R)$. If e_1 is not semicentral reduced, then there exists a nontrivial element e_2 of $\mathcal{S}_\ell(e_1 R e_1)$; so $e_1 R \supseteq e_2 R$. By Lemma 1.4(i), $e_2 \in \mathcal{S}_\ell(R)$. Next, if e_2 is not semicentral reduced, then there exists a nontrivial element e_3 of $\mathcal{S}_\ell(e_2 R e_2)$ and so $e_2 R \supseteq e_3 R$. Again by Lemma 1.4(i), $e_3 \in \mathcal{S}_\ell(R)$. Continue this procedure to get a descending chain $e_1 R \supseteq e_2 R \supseteq e_3 R \supseteq \cdots$. The DCC on $\{bR \mid b \in \mathcal{S}_\ell(R)\}$ guarantees this chain must become stationary, and Lemma 2.5 yields that it does so with an $e_n \in \mathcal{S}_\ell(R)$ such that e_n is semicentral reduced. Starting a new process, let $b_1 = e_n$ and observe that $1 - b_1 \in \mathcal{S}_r(R)$. If $1 - b_1$ is semicentral reduced, then $\{b_1, 1 - b_1\}$ is a complete set of left triangulating idempotents. Otherwise, consider the algebra $R_1 = (1 - b_1)R(1 - b_1)$, observe that by Lemma 2.7, R_1 has DCC on $\{dR_1 \mid d \in \mathcal{S}_\ell(R_1)\}$, and using a strictly analogous argument to that used to get b_1, obtain $b_2 \in \mathcal{S}_\ell(R_1)$ such that $\mathcal{S}_\ell(b_2 R_1 b_2) = \{0, b_2\}$. Since $1 - b_1$ is the unity for R_1, and $b_2 \in R_1$, we have $b_2 R_1 b_2 = b_2 R b_2$; so $\mathcal{S}_\ell(b_2 R_1 b_2) = \{0, b_2\}$. Also, $(1 - b_1) - b_2 \in \mathcal{S}_r(R_1)$. The right-sided analog of Lemma 1.4(i) yields $\mathcal{S}_r(R_1) \subseteq \mathcal{S}_r(R)$. If $1 - b_1 - b_2$ is semicentral reduced in R, then $\{b_1, b_2, 1 - b_1 - b_2\}$ is a complete set of left triangulating idempotents. Otherwise we can continue the process, generating a descending chain of terms from $\{cR \mid c \in \mathcal{S}_r(R)\}$. By the DCC hypothesized this chain must become stationary after a finite number of steps, yielding a complete set of left triangulating idempotents. The equivalence of (i) and (viii) follows from Proposition 1.3.

It is worth noting that if $\{b_1, \dots, b_n\}$ is a complete set of left triangulating idempotents of an algebra R and each of the rings $b_i R b_i$ satisfies $\mathbf{I}(b_i R b_i) = \mathbf{B}(b_i R b_i)$, then $\{b_1, \dots, b_n\}$ is a complete set of primitive idempotents. This occurs, for example, if each $b_i R b_i$ is commutative or duo.

The next theorem shows the uniqueness of a complete generalized triangular matrix representation. In the proof of this theorem we make use of the following result due to Azumaya: if I is a quasi-regular ideal of R and $\{e_1, \dots, e_n\}$, $\{f_1, \dots, f_n\}$ are two sets of pairwise orthogonal idempotent elements of R such that $\bar{e}_i = \bar{f}_i$ for every i with images $\bar{e}_i$ and $\bar{f}_i$ in $\overline{R} = R/I$, then there exists an invertible element $\alpha \in R$ such that $f_i = \alpha^{-1} e_i \alpha$ for each i [Az, Theorem 3]. Recall for $0 \neq e \in \mathbf{B}(R)$, e is said to be *centrally primitive* if 0 and e are the only central idempotents in eR. Also, R is said to have a *complete set of centrally primitive idempotents* if there exists a finite set of centrally primitive pairwise orthogonal idempotents whose sum is the unity of R [Lam 1, Sects. 21 and 22].

THEOREM 2.10 (Uniqueness). *Let $\{b_1, \dots, b_n\}$ and $\{c_1, \dots, c_k\}$ each be a complete set of left triangulating idempotents for R. Then $n = k$ and there exists an invertible element $\alpha \in R$ and a permutation σ on $\{1, \dots, n\}$ such that $b_{\sigma(i)} = \alpha^{-1} c_i \alpha$ for each i. Thus for each i, $c_i R \cong b_{\sigma(i)} R$, as R-modules, and $c_i R c_i \cong b_{\sigma(i)} R b_{\sigma(i)}$, as K-algebras.*

Proof. Let $U = \sum_{i<j} b_i R b_j$. Thus U corresponds to the strictly upper triangular matrix subalgebra of $T_n(R)$ induced by $\{b_1, \dots, b_n\}$ as in the proof of Proposition 1.3. Recall U is an ideal of R and $U^n = 0$. Let $\overline{R} = R/U$ and denote by $\bar{x}$ the image of $x \in R$ under the natural homomorphism $R \to \overline{R}$. Since $(b_i R b_i) \cap U = 0$, for $i = 1, \dots, n$, we have that each $b_i R b_i$ is isomorphic as a K-algebra to $\bar{b}_i \overline{R} \bar{b}_i$. So $\overline{R}$ is a direct sum of the $\bar{b}_i \overline{R} \bar{b}_i$, and consequently $\{\bar{b}_1, \dots, \bar{b}_n\}$ is a complete set of centrally primitive idempotents for $\overline{R}$.

Now by Lemma 1.5(i), $\bar{c}_1 \in \mathcal{S}_\ell(\bar{R})$. Hence $\bar{c}_1 = \sum_{i=1}^n \bar{b}_i \bar{c}_1$. Since $\bar{b}_i$ is semicentral reduced, then $\bar{b}_i \bar{c}_1 \in \{\bar{0}, \bar{b}_i\}$. Thus $\bar{c}_1 \in \mathbf{B}(\bar{R})$. Then $\bar{c}_2 \in \mathcal{S}_\ell((\bar{1} - \bar{c}_1)\bar{R}(\bar{1} - \bar{c}_1))$. Because $\bar{1} - \bar{c}_1 \in \mathbf{B}(\bar{R})$, Lemma 1.4(i) yields $\bar{c}_2 \in \mathcal{S}_\ell(\bar{R})$. Using the above argument, with $\bar{c}_2$ in place of $\bar{c}_1$, we obtain $\bar{c}_2 \in \mathbf{B}(\bar{R})$. Continuing this procedure yields that $\{\bar{c}_1, \ldots, \bar{c}_k\}$ is a set of pairwise orthogonal nonzero central idempotents in $\bar{R}$. Hence $\bar{c}_i \bar{R} \bar{c}_j = \bar{0}$, for $i < j$. Thus $c_i R c_j \subseteq U$, for all $1 \leq i < j \leq k$.

Let $V = \sum_{i<j} c_i R c_j$. Thus V corresponds to the strictly upper triangular matrix subalgebra of $T_k(R)$ induced by $\{c_1, \ldots, c_k\}$ as in the proof of Proposition 1.3. Using an argument similar to the above one, we obtain $b_i R b_j \subseteq V$, for all $1 \leq i < j \leq n$. Hence $U = V$. Therefore $\{\bar{b}_1, \ldots, \bar{b}_n\}$ and $\{\bar{c}_1, \ldots, \bar{c}_k\}$ are both complete sets of centrally primitive idempotents for $\bar{R}$. It is well known that for such sets of centrally primitive idempotents $n = k$ and there exists a permutation σ on $\{1, \ldots, n\}$ such that $\bar{c}_i = \bar{b}_{\sigma(i)}$, [Lam1, (22.1) Proposition]. Use Azumaya's result to get an invertible element α in R such that $b_{\sigma(i)} = \alpha^{-1} c_i \alpha$ for every i. Thus $c_i R \cong b_{\sigma(i)} R$ as R-modules. Since $\mathrm{End}_R(c_i R) \cong c_i R c_i$ and $\mathrm{End}_R(b_j R) \cong b_j R b_j$, we have $c_i R c_i \cong b_{\sigma(i)} R b_{\sigma(i)}$, as K-algebras.

With a different but much longer proof of Theorem 2.10, one can show that the permutation σ can be chosen so that $c_1 R = b_{\sigma(1)} R$ and $(1 - b_{\sigma(1)} - \cdots - b_{\sigma(k-1)}) c_k R = b_{\sigma(k)} R$, $k = 2, \ldots, n$. Then the function $f : c_k R \to b_{\sigma(k)} R$ defined by $f(c_k r) = d_k c_k r$, where $d_1 = 1$ and $d_k = 1 - b_{\sigma(1)} - \cdots - b_{\sigma(k-1)}$, for $2 \leq k \leq n$, is an R-module isomorphism. Note that $b_{\sigma(1)} \in \mathcal{S}_\ell(R)$.

The next example shows that the isomorphism $c_i R \cong b_{\sigma(i)} R$, given in Theorem 2.10, cannot be sharpened to equality. In fact, there can be infinitely many different complete sets of triangulating idempotents for certain R. This is in stark contrast to the result for complete sets of centrally primitive idempotents [Lam1, Sect. 22].

EXAMPLE 2.11. Let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where F is a field of characteristic not 2. Let

$$e_x = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \qquad f_x = \begin{pmatrix} 0 & -x \\ 0 & 1 \end{pmatrix}, \qquad x \in F,\ x \neq 0;$$

and let

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\{e_x, f_x\}$, for each x, and $\{b_1, b_2\}$ are complete sets of left triangulating idempotents for R, giving infinitely many such sets when F is infinite.

Moreover, $b_1 R = e_x R$, $b_2 R \neq f_x R$, but $b_2 R \cong f_x R$ and $(1 - b_1) f_x R = b_2 R$.

According to [Lam3, p. 35] I. Kaplansky raised the following question: Let A and B be two rings. If $M_n(A) \cong M_n(B)$ as rings, does it follow that $A \cong B$ as rings? It is known that there are nonisomorphic semicentral reduced rings (e.g., simple Noetherian domains) which have isomorphic full matrix rings [Lam2, Lam4, Smi]. Surprisingly, our next result shows this situation *cannot* hold for n-by-n $(n > 1)$ triangular matrix rings over semicentral reduced rings.

COROLLARY 2.12. *Let A and B be semicentral reduced K-algebras. If $T_m(A) \cong T_n(B)$ as K-algebras for some m and n, then $m = n$ and $A \cong B$ as K-algebras.*

Proof. Because A is semicentral reduced, the matrix units E_{ii} form a complete set of left triangulating idempotents for $T_m(A)$. A similar result holds for $T_n(B)$. Since the property of being a complete set of left triangulating idempotents is preserved under isomorphisms, the hypothesis $T_m(A) \cong T_n(B)$ together with Theorem 2.10 yields $m = n$ and also the isomorphism of the algebras in the first row, first column (i.e., $A \cong B$).

We note that in [Wh] it has been shown that if F and K are fields then $F \cong K$ if and only if their strictly upper triangular matrix rings are isomorphic.

LEMMA 2.13. *Let $\{e_1, \dots, e_n\}$ be a complete set of primitive idempotents of R. If b is a nontrivial idempotent in $\mathcal{S}_\ell(R) \cup \mathcal{S}_r(R)$, then there exists a subset P of $\{e_1, \dots, e_n\}$ such that $\{be_jb \mid e_j \in P\}$ is a complete set of primitive idempotents of bRb and this set has less than n elements.*

Proof. Consider $b \in \mathcal{S}_\ell(R)$. Let P be the set of all e_j such that the elements be_jb are distinct and nonzero. Without loss of generality let $P = \{e_1, \dots, e_n\}$. By Lemma 1.4(iii), the be_jb, $j = 1, \dots, m$, are primitive idempotents in bRb, and $e_jbe_j = e_j$. From $b = (e_1 + \cdots + e_n)b = e_1 b + \cdots + e_n b = be_1 b + \cdots + be_n b = be_1 b + \cdots + be_m b$, we have that $\{be_jb \mid j = 1, \dots, m\}$ is a complete set of primitive idempotents for bRb. Suppose $n = m$. Then $1 = e_1 + \cdots + e_n = e_1 b e_1 + \cdots + e_n b e_n = be_1 be_1 + \cdots + be_n be_n = b(e_1 be_1 + \cdots + e_n be_n) = b(e_1 + \cdots + e_n) = b$, a contradiction. So $m < n$.

The proof for $b \in \mathcal{S}_r(R)$ is a right-sided dual of the proof given above.

From Proposition 1.3, Theorem 2.9, and Theorem 2.10 we have that if R satisfies certain mild finiteness conditions then R has a complete generalized triangular matrix representation. Moreover, by Proposition 1.6, the diagonal subalgebras of R (which are of the form eRe for $e = e^2$) are also homomorphic images of R. In the next two propositions we show that the study of many well known classes of algebras and rings can be reduced to the investigation of semicentral reduced algebras and rings from the same respective class.

PROPOSITION 2.14. *If R satisfies any of the following conditions, then*

$$R \cong \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix},$$

where each R_i is semicentral reduced and satisfies the same condition as R, R_{ij} is a left R_i-right R_j-bimodule, and the K-algebras (rings) $R_1, \ldots, R_n$ are uniquely determined by R up to isomorphism (induced by an inner automorphism of R) and permutation:

(i) *R has a complete set of primitive idempotents;*

(ii) *R has no infinite set of orthogonal idempotents;*

(iii) *R_R has Krull dimension;*

(iv) *R has DCC on (idempotent generated, principal, or finitely generated) ideals;*

(v) *R has DCC on (idempotent generated, principal, or finitely generated) right ideals;*

(vi) *R has ACC on (idempotent generated, principal, or finitely generated) ideals;*

(vii) *R has ACC on (idempotent generated, principal, or finitely generated) right ideals;*

(viii) *R has either ACC or DCC on right annihilators;*

(ix) *R is a semilocal ring;*

(x) *R is a semiperfect ring;*

(xi) *R is a semiprimary ring.*

Proof. (i) Let $\{e_1, \ldots, e_k\}$ be a complete set of primitive idempotents of R. Then for any $b \in \mathcal{S}_\ell(R)$, $b = e_1 b + \cdots + e_k b$, and by Lemma 1.1(iii) each $e_j b$ is an idempotent. Without loss of generality, assume the full set of all $e_j b$ which are not zero is given by $j = 1, \ldots, m$. Then $bR \subseteq e_1 bR + \cdots + e_m bR = be_1 bR + \cdots + be_m bR \subseteq bR$, or $bR = e_1 R + \cdots + e_m bR$. Primitivity of e_j implies $e_j bR = e_j R$, whenever $e_j b \neq 0$. So $bR = e_1 bR + \cdots + e_m R$ and consequently the total number of right ideals of the form bR, $b \in \mathcal{S}_\ell(R)$ cannot exceed 2^k. Then Theorem 2.9 yields that R has a complete set of left triangulating idempotents.

Let $\{b_1, \dots, b_n\}$ be a complete set of left triangulating idempotents for R. Using Proposition 1.3, take $R_i = b_i R b_i$ and $R_{ij} = b_i R b_j$, for $i < j$. Then R_{ij} is a left R_i-right R_j-bimodule, for $i < j$. Lemma 2.13 and Proposition 1.6(i) ensure that each R_i has a complete set of primitive idempotents. The uniqueness of the R_i follows from Theorem 2.10.

(ii) Since this condition implies that R has a complete set of primitive idempotents, we have the unique generalized triangular matrix representation by (i). Each subalgebra of R, in particular R_i, has no infinite set of orthogonal idempotents.

(iii) By [Kr, Proposition 4] R_R has finite uniform dimension. So (ii) is satisfied. Hence R has the unique generalized triangular matrix representation. From [GR, Proposition 1.4] and Proposition 1.6(iii), each R_i has Krull dimension.

(iv)–(viii) By Lemma 1.1, if $b \in \mathcal{S}_\ell(R)$ and $c \in \mathcal{S}_r(R)$, then bR and Rc are ideals of R. For each chain condition under consideration, Theorem 2.9 yields that R has a complete set of left triangulating idempotents. As in the proof of (i) we have the unique generalized triangular matrix representation. Each chain condition is either inherited by K-algebra homomorphic images or by subalgebras of the form eRe, where e is an idempotent [Lam1, p. 322]. Using Proposition 1.6(iii), each R_i satisfies the same chain condition as R.

(ix)–(xi) For each of these conditions, R has a complete set of primitive idempotents. By (i), R has a complete set of left triangulating idempotents. As in the proof of (i), we have the unique generalized triangular matrix representation. From [AF, p. 305; Lam1, p. 312] the class of semilocal and the class of semiperfect rings are closed under homomorphic images. By Proposition 1.6(iii), if R is semilocal (respectively, semiperfect), then so is each R_i. By [AF, p. 319], if R is semiprimary then so is each R_i.

Observe that the class of all left perfect rings is included in the class of rings described in condition (v) of Proposition 2.14. Also Proposition 2.14(i) shows that if R has a complete set of primitive idempotents, then R has a complete set of triangulating idempotents. The next example shows that the converse to this statement is false.

Example 2.15. Let V be an infinite dimensional vector space over a field F and let $R = \operatorname{End}_F(V)$. Then R is a prime ring; hence $\{1\}$ forms a complete set of left triangulating idempotents. Since R is a von Neumann regular ring and is not semisimple Artinian, R cannot have a complete set of primitive idempotents. If $\dim_F V = \aleph_m$, m any finite ordinal, then the F-algebra R has ACC and DCC on ideals. Next if $\dim_F V = \aleph_\omega$, then R has DCC on ideals, but R does not have ACC on idempotent generated ideals. To see the latter, recall that $R_n = \{f \in R \mid \operatorname{rank}(f) < \aleph_n\}$, where n is any ordinal less than ω, is an ideal of R. Observe that if $g \in R$ with $\operatorname{rank}(g) = \aleph_n$, then $RgR = R_{n+1}$.

Some of our motivating ideas for defining triangulating idempotents originated with [Bi2, Theorem 5]. This result decomposed a ring with a complete set of primitive idempotents in terms of iterated generalized triangular matrix representations involving reduced rings and MDSN rings. Recall that R is MDSN if $0 \neq e \in \mathbf{I}(R)$ implies eR contains a nonzero nilpotent element. Now from Corollary 2.2 and Proposition 2.14(i), we obtain that if R has a complete set of primitive idempotents, then R has a complete set of left triangulating idempotents $\{b_1, \ldots, b_n\}$ such that each $b_i R b_i$ is either an indecomposable reduced ring or an MDSN ring.

PROPOSITION 2.16. *Let R be a ring. If R has a complete set of left triangulating idempotents and satisfies any of the following conditions, then*

$$R \cong \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix},$$

where each R_i is semicentral reduced and satisfies the same condition as R, R_{ij} is a left R_i-right R_j-bimodule, and the rings $R_1, \ldots, R_m$ are uniquely determined by R up to isomorphism (induced by an inner automorphism of R) and permutation:

(i) *R_R has Gabriel dimension;*

(ii) *R is a Baer ring;*

(iii) *R is a right semihereditary ring;*

(iv) *R is a right hereditary ring;*

(v) *R is an I-ring (i.e., every non-nil right ideal contains a nonzero idempotent element);*

(vi) *R is a π-regular ring;*

(vii) *R is a right semiartinian ring;*

(viii) *R is a PI-ring;*

(ix) *R is a right PP-ring;*

(x) *R is a semiregular ring;*

(xi) *R has bounded index of nilpotency;*

(xii) *R is right self-injective.*

Proof. Since R has a complete set of left triangulating idempotents, as in the proof of Proposition 2.14(i), R has the indicated unique generalized triangular matrix representation. To see that each R_i satisfies the same condition as R observe that if R satisfies any of the conditions (i)–(vi), then subrings of the form eRe, where $e = e^2$, also satisfy the same condition as R [NO, p. 147; Ka, p. 6; Wi, pp. 336–339; J, p. 211]. Conditions (v) and (vi) are preserved under homomorphism [St, p. 193]. Thus by Proposition 1.6(iii), if R is right semiartinian (respectively, PI) then so is each R_i. For (ix), recall that R is right PP if and only if for any $x \in R$ there exists an idempotent $e \in R$ such that $r(x) = eR$. Hence the proof of this part is similar to [Ka, Theorem 4]. Condition (x) is a consequence of [Ni, Corollary 2.3]. The remaining conditions are satisfied by subrings of the form eRe, where $e = e^2$.

Note that each of the following classes of semiprime rings is closed relative to subrings of the form eRe, where $e = e^2$: (i) von Neumann regular, (ii) biregular, (iii) (right) fully idempotent, (iv) right V-ring. Also if R is semiprime, then R is semicentral reduced if and only if R is indecomposable. Thus Proposition 1.3 and Theorem 2.9 yield that if R has a complete set of left triangulating idempotents and is from one of the above classes, then $R = \oplus R_i$, where each R_i is indecomposable and from the same class as R.

EXAMPLE 2.17. For the ring $\mathbb{Z}$ of integers and a prime p, let $A_n = \mathbb{Z}/p\mathbb{Z}$, $n = 1, 2, \ldots$, and let A be the ring $\prod_{n=1}^{\infty} A_n$. Consider the subring R of A generated by $\oplus_{n=1}^{\infty} A_n$ together with $1 \in A$. Then R is a semiartinian ring [St, Exercise 12, p. 193]. Thus R has Gabriel dimension 1. But since R is a commutative von Neumann regular ring which is not Artinian, R cannot have a complete set of left triangulating idempotents.

Observe from Theorem 2.10 and Corollary 1.7 that the number of elements in a complete set of left triangulating idempotents is unique for a given algebra R (which has such a set) and this is also the number of elements in any complete set of right triangulating idempotents of R. This motivates the following definition: R has *triangulating dimension* n, written $\mathrm{Tdim}(R) = n$, if R has a complete set of left triangulating idempotents with n elements. Note that R is semicentral reduced if and only if $\mathrm{Tdim}(R) = 1$.

PROPOSITION 2.18. *If $\{e_1, \ldots, e_n\}$ is a complete set of primitive idempotents for R, then* $\mathrm{Tdim}(R) \leq n$.

Proof. By Proposition 2.14(i), R has a complete set of left triangulating idempotents, say $\{b_1, \ldots, b_m\}$. Let $c_k = 1 - (b_1 + \cdots + b_k)$, for $1 \leq k < m$.

Then each $c_k \in \mathcal{S}_r(R)$, by Proposition 1.6(i). Lemma 2.13 guarantees that c_1Rc_1 has a complete set of primitive idempotents, and this set has n_1 elements, with $n_1 < n$. Then $c_2 \in c_1Rc_1$ and c_2Rc_2 has a complete set of primitive idempotents, and this set has n_2 elements, with $n_2 < n_1$. If $m > n$, then this process can be continued until one arrives at a contradiction. So $m \le n$.

LEMMA 2.19. (i) *Let* $c = b + e$, *where* $c \in \mathbf{B}(R)$, $b \in \mathcal{S}_\ell(R)$, *and* $e \in \mathbf{I}(R)$. *If* $eb = 0$, *then* $e \in \mathcal{S}_r(R)$.

(ii) *Let* $\{e_1, \dots, e_n\}$ *be a set of nonzero, pairwise orthogonal, left semicentral idempotents in* R. *If* $e_1 + \cdots + e_n$ *is in* $\mathbf{B}(R)$, *then* $\{e_1, \dots, e_n\} \subseteq \mathbf{B}(R)$.

Proof. (i) For any $x \in R$, $xc = xb + xe$, and $exc = exb + exe = ebxb + exe = exe$. Since $ec = e$, we have $ex = ecx = exc$; so $exe = ex$ and hence by Lemma 1.1, $e \in \mathcal{S}_r(R)$.

(ii) The case $n = 1$ being trivial, consider $n > 1$. Assume one of the e_j is not in $\mathbf{B}(R)$; without loss of generality let this be e_1. Let $e = e_2 + \cdots + e_n$. Observe that e is an idempotent orthogonal to e_1. It can be easily checked that $e \in \mathcal{S}_\ell(R)$. Note that $c = e_1 + e$ is in $\mathbf{B}(R)$ and use part (i) to obtain $e_1 \in \mathcal{S}_r(R)$. Since it is given that $e_1 \in \mathcal{S}_\ell(R)$, we get $e_1 \in \mathbf{B}(R)$.

Recall that R has a "block decomposition" if and only if R has a complete set of centrally primitive idempotents [Lam1, Sects. 21, 22]. Our next result shows that if R has a complete set of left triangulating idempotents, then R has a "block decomposition" and $\mathrm{Tdim}(R)$ is greater than or equal to the cardinality of a complete set of centrally primitive idempotents.

PROPOSITION 2.20. *Let* $\{b_1, \dots, b_n\}$ *be a complete set of left triangulating idempotents for* R.

(i) $c \in \mathbf{B}(R)\backslash\{0, 1\}$ *if and only if there exists* $I \subsetneq \{1, \dots, n\}$ *such that* $c = \sum_{i \in I} b_i$ *and* $b_iRb_j = b_jRb_i = 0$, *for each* $i \in I$ *and* $j \notin I$.

(ii) *R has a complete set of centrally primitive idempotents.*

(iii) $\{b_1, \dots, b_n\} \subseteq \mathcal{S}_\ell(R)$ *if and only if* $\{b_1, \dots, b_n\}$ *is a complete set of centrally primitive idempotents.*

Proof. (i) Let $c \in \mathbf{B}(R)\backslash\{0, 1\}$. Then $c = c(b_1 + \cdots + b_n) = cb_1 + \cdots + cb_n$. However, $cb_i \in \mathcal{S}_\ell(b_iRb_i)$ and $\mathcal{S}_\ell(b_iRb_i) = \{0, b_i\}$, for each i. Hence there exists $I \subsetneq \{1, \dots, n\}$ such that $c = \sum_{i \in I} b_i$. Let $i \in I$ and $j \notin I$. Then $b_iRb_j = cb_iRb_j = b_iRb_j = 0$. Similarly, $b_jRb_i = 0$. Conversely, let $r \in R$.

Then $r \in \sum_{j=1}^{n} \sum_{i=1}^{n} b_i R b_j$. A routine argument shows $cr = rc$. Observe that for the converse we need only assume that $\{b_1, \dots, b_n\}$ is a set of left triangulating idempotents.

(ii) By part (i), $\mathbf{B}(R)$ is a finite set. Now a standard argument yields a complete set of centrally primitive idempotents.

(iii) Let $\{b_1, \dots, b_n\} \subseteq \mathcal{S}_\ell(R)$. Observe that the b_i are pairwise orthogonal. Then, recalling that $b_1 + \cdots + b_n = 1$, Lemma 2.19 yields $\{b_1, \dots, b_n\} \subseteq \mathbf{B}(R)$. The converse is immediate.

Note that from Proposition 2.20(i), if R has $\mathrm{Tdim}(R) < \infty$, then one can determine $\mathbf{B}(R)$ (and hence the indecomposability of R) by observing certain patterns of zeros in its generalized triangular matrix representation. Moreover, from Propositions 2.18 and 2.20, if R has a complete set of primitive idempotents of cardinality n, then R has a complete set of centrally primitive idempotents of cardinality m and $m \leq \mathrm{Tdim}(R) \leq n$.

By Proposition 2.20(ii), if R has a complete set of left triangulating idempotents, then R has a complete set of centrally primitive idempotents. But in the following example, the converse does not hold.

EXAMPLE 2.21. Let R be the $\aleph_0$-by-$\aleph_0$ upper triangular row-finite matrix ring over a field. Then obviously $\{1\}$ is a complete set of centrally primitive idempotents of R. Let e_n be the matrix in R with 1 in the (n, n)-position and 0 elsewhere. Then $e_1 + \cdots + e_n$ is a left semicentral idempotent of R for any positive integer n. Since $(e_1 + \cdots + e_n)R \subsetneq (e_1 + \cdots + e_n + e_{n+1})R$, for each n, Theorem 2.9 implies that R cannot have a complete set of left triangulating idempotents.

3. CANONICAL FORM AND GLOBAL DIMENSION

In this section we show that if R has a set of left triangulating idempotents, then it has a "canonical" generalized triangular matrix representation, where the "diagonal" subalgebras are organized into "blocks" of square diagonal matrix algebras. This canonical representation is then used to obtain a result on the left global dimension of rings with a set of left triangulating idempotents.

For the remainder of this section $\{b_1, \dots, b_n\}$ is a set of left triangulating idempotents for R. If J is a subset of $\{1, \dots, n\}$, for notational convenience we denote the sum of all the b_i, $i \in J$, by σ_J, and use $\sigma'_J = 1 - \sigma_J$. If $J = \emptyset$, then take $\sigma_J = 0$ and $\sigma'_J = 1$.

LEMMA 3.1. *Let J be a subset of $\{1, \dots, n\}$ and let $m \in \{1, \dots, n\} \setminus J$. Then $b_m \in \mathcal{S}_\ell(\sigma'_J R \sigma'_J)$ if and only if $b_i R b_m = 0$ for each $i \notin J \cup \{m\}$.*

Proof. First observe that using orthogonality of the idempotents we have that if $i \notin J$, then $b_i = \sigma_J' b_i \sigma_J' = \sigma_J' b_i = b_i \sigma_J'$. Now assume $b_m \in \mathcal{S}_\ell(\sigma_J' R \sigma_J')$ and let $i \notin J \cup \{m\}$. Then $b_i R b_m = b_i(\sigma_J' R \sigma_J') b_m = b_i b_m \sigma_J' R \sigma_J' b_m = 0$.

Conversely, assume $b_i R b_m = 0$, for each $i \notin J \cup \{m\}$. Let $J^* = \{1, \dots, n\} \setminus J$. Then $(\sigma_J' R \sigma_J') b_m = \sigma_{J^*} R b_m = \sum_{i \in J^*} b_i R b_m$. So $\sum_{i \in J^*} b_i R b_m = b_m R b_m = b_m(\sigma_J' R \sigma_J') b_m$. Hence $b_m \in \mathcal{S}_\ell(\sigma_J' R \sigma_J')$.

PROPOSITION 3.2. *Let j and m be in $\{1, \dots, n\}$ with $j < m \leq n$. If $b_i R b_m = 0$ for each i such that $j \leq i < m$, then $\{b_1, \dots, b_{j-1}, b_m, b_j, b_{j+1}, \dots, b_{m-1}, b_{m+1}, \dots, b_n\}$ is a set of left triangulating idempotents of R.*

Proof. Let $J = \{1, \dots, j-1\}$ and $I = J \cup \{m\}$. Since $\{b_1, \dots, b_n\}$ is a set of left triangulating idempotents of R, a moments reflection will reveal that the desired result will be established if we show that $b_m \in \mathcal{S}_\ell(\sigma_J' R \sigma_J')$, $b_j \in \mathcal{S}_\ell(\sigma_I' R \sigma_I')$, and $b_k \in \mathcal{S}_\ell(\sigma_L' R \sigma_L')$, where $L = \{1, \dots, k-1, m\}$, for $j < k < m$. Lemma 1.2 yields that $b_i R b_m = 0$, for all $i > m$. By hypothesis, $b_i R b_m = 0$, for $j \leq i < m$. Use Lemma 3.1 to obtain $b_m \in \mathcal{S}_\ell(\sigma_J' R \sigma_J')$. Next, use Lemma 1.2 to obtain $b_i R b_j = 0$ for $i \geq j + 1$. So $b_i R b_j = 0$ for $i \notin I \cup \{j\}$ and hence by Lemma 3.1 we have $b_j \in \mathcal{S}_\ell(\sigma_I' R \sigma_I')$. In a similar fashion we obtain $b_k \in \mathcal{S}_\ell(\sigma_L' R \sigma_L')$.

We make use of the procedure given in Proposition 3.2 to obtain a canonical form for the generalized triangular matrix representation for R. In this the following notation will be helpful. Let $S = \{b_1, \dots, b_n\}$. Recursively define the sets I_k and $J(k)$ as follows: $I_1 = \{i \mid b_i \in \mathcal{S}_\ell(R)\}$, $J(1) = I_1$; whenever I_k and $J(k)$ are defined, then let $I_{k+1} = \{i \mid i \notin J(k),\ b_i \in \mathcal{S}_\ell(\sigma_{J(k)}' R \sigma_{J(k)'}\}$, and $J(k+1) = I_{k+1} \cup J(k)$. This process terminates after no more than n steps. Consider the situation where $I_1, \dots, I_q$ is a partition for $\{1, \dots, n\}$. Then $S_1, \dots, S_q$ is a partition for $\{b_1, \dots, b_n\}$, where $S_j = \{b_i \mid i \in I_j\}$. (We will show in the proof of Theorem 3.3 that this always occurs.) Then reorder $\{1, \dots, n\}$ so that each I_j has any (fixed) ordering and so that elements of I_j always precede elements in I_{j+1}. This can be thought of in terms of a permutation ψ on $\{1, \dots, n\}$. Then the ordered set $\{b_{\psi(1)}, \dots, b_{\psi(n)}\}$ is called a *canonical form* for $\{b_1, \dots, b_n\}$.

THEOREM 3.3. *Let $\{b_1, \dots, b_n\}$ be a set of left triangulating idempotents. Then a canonical form for $\{b_1, \dots, b_n\}$ exists, and any such canonical form is a set of left triangulating idempotents of R.*

Proof. The proof involves repeated use of Lemma 3.1 and Proposition 3.2, as indicated in the following discussion. Observe that $b_1 \in S_1$. If $b_m \in S_1$ and $m \neq 1$, then Lemma 3.1 yields $b_i R b_m = 0$, for all $i \neq m$. Use Proposition 3.2 to get that $\{b_m, b_1, \dots, b_{m-1}, b_{m+1}, \dots, b_n\}$ is a set of left triangulating idempotents of R. Continue this process using elements of S_1

until they are exhausted. Following the procedure given in Proposition 3.2 this results in a permutation α on $\{1,\ldots,n\}$ such that $b_{\alpha(1)},\ldots,b_{\alpha(n_1)}$ are in S_1, where $n_1 = |S_1|$, and $\alpha(n_1+1)$ is the smallest positive integer i such that $b_i \notin S_1$. Also, the ordered set $\{b_{\alpha(1)}, b_{\alpha(2)},\ldots\}$ is a set of left triangulating idempotents of R.

If $n_1 = n$, then we are finished. So consider $n_1 < n$ and let $q = \alpha(n_1 + 1)$. Observe that b_q is the first element in this new ordering which is not in S_1. We show $b_q \in S_2$. Let y be the sum of all elements in S_1. Note that $y = c + d$, where c is the sum of all elements $b_i \in S_1$ such that $i < q$ and d is the sum of all elements $b_j \in S_1$ such that $q < j$. (If the latter summation is over the empty set, then $d = 0$.)

Observe that inherent in the rearrangement process, we obtain $b_q \in \mathcal{S}_\ell((1-c)R(1-c))$. Orthogonality of the idempotents $b_1,\ldots,b_n$ yields $b_q = (1-y)b_q(1-y)$ and $1-y = (1-c)(1-y)(1-c)$. So b_q is in $(1-y)(\mathcal{S}_\ell(1-c)R(1-c)))(1-y)$. Now by using Lemma 1.4(ii), we have that $(1-y)(\mathcal{S}_\ell((1-c)R(1-c)))(1-y) \subseteq \mathcal{S}_\ell((1-y)(1-c)R(1-c)(1-y))$. Since $(1-y)(1-c) = 1-y$, we have $b_q \in \mathcal{S}_\ell((1-y)R(1-y))$. Consequently, $b_q \in S_2$. Either this exhausts the elements in S_2 or (in the ordering given by α) there is an element b_p in S_2 beyond b_q (i.e., $p = \alpha(k)$ where $k > n_1 + 1$). Use Lemma 3.1 and Proposition 3.2 as before to obtain a set of left triangulating idempotents of R of the form $\{b_{\alpha(1)},\ldots,b_{\alpha(n_1)}, b_p, b_q, b_{\alpha(n_1+2)},\ldots,b_{\alpha(n)}\}$. As before, repeat this process using elements of S_2 until they are exhausted. This results in a permutation γ on $\{1,\ldots,n\}$ such that $\{b_{\gamma(1)},\ldots,b_{\gamma(n_1)}, b_{\gamma(n_1+1)},\ldots,b_{\gamma(n_2)},\ldots,b_{\gamma(n)}\}$ is a set of left triangulating idempotents of R, with $\gamma(i) = \alpha(i)$, $1 \le i \le n_1$, $b_{\gamma(n_2)} = b_q$, and $\{b_{\gamma(n_1+1)},\ldots,b_{\gamma(n_2)}\} = S_2$. Either $S_1 \cup S_2 = \{b_1,\ldots,b_n\}$ or we can continue the process on S_3, and so on. After k steps, $k \le n$, the process terminates in a set of left triangulating idempotents of R in canonical form. In particular $S_1,\ldots,S_k$ is a partition of $\{b_1,\ldots,b_n\}$. Finally, note that for any rearrangement of the elements within a given S_j we obtain another ordered set which serves as a left triangulating set of idempotents of R.

The results of the above theorem together with those of Proposition 1.3 provide the vehicle for a generalized triangular matrix representation for R in a special "canonical" form, which we give below:

Corollary 3.4 (Canonical Representation). *Let $\{b_1,\ldots,b_n\}$, $S_1,\ldots,S_k$, and ψ be as above. Then using $0 = n_0 < n_1 < \cdots < n_k$, we have $S_{j+1} = \{b_{\psi(n_j+1)},\ldots,b_{\psi(n_{j+1})}\}$, $j = 0,1,\ldots,k-1$, and R is isomorphic to the* *n-by-n matrix* $[A(i,j)]$, *where the* $A(i,j)$ *are the* n_i*-by-*n_j *block matrices*

$$A(i+1,i+1) = \begin{pmatrix} b_{\psi(n_i+1)}Rb_{\psi(n_i+1)} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & b_{\psi(n_{i+1})}Rb_{\psi(n_{i+1})} \end{pmatrix};$$

$$A(i+1,j+1) = \begin{pmatrix} b_{\psi(n_i+1)}Rb_{\psi(n_j+1)} & \cdots & b_{\psi(n_i+1)}Rb_{\psi(n_{j+1})} \\ \vdots & \ddots & \vdots \\ b_{\psi(n_{i+1})}Rb_{\psi(n_j+1)} & \cdots & b_{\psi(n_{i+1})}Rb_{\psi(n_{j+1})} \end{pmatrix},$$

for $i < j$; *and* $A(i,j) = 0$, *for* $j < i$, *where* $i, j = 0, 1, \ldots, k-1$.

COROLLARY 3.5. *If* b_i *and* b_j *are distinct elements in some* S_h, *then* $\mathrm{Hom}_R(b_iR, b_jR) = 0$.

Proof. Observe that $\mathrm{Hom}_R(b_iR, b_jR)$ is group-isomorphic to b_jRb_i. From the canonical generalized triangular matrix representation for R we see that since $b_i, b_j \in S_h$ and $i \neq j$, we have $b_jRb_i = 0$.

The final result of this section uses our canonical representation to address a problem implicitly posed by Chase [Ch, p. 19]: "In general it seems to be difficult to express the global dimension of $R = \mathscr{T}(R', S, A)$ in terms of the homological invariants of R', S, and A." Here $\mathscr{T}(R', S, A)$ denotes the formal triangular matrix ring $\left(\begin{smallmatrix} R' & A \\ 0 & S \end{smallmatrix}\right)$.

THEOREM 3.6. *Let* $\{b_1, \ldots, b_n\}$ *and* $S_1, \ldots, S_k$ *be as above. Then R has finite left global dimension if and only if* $D = b_1Rb_1 + \cdots + b_nRb_n$ *has finite left global dimension. In this case*, $\text{l.gl.dim } D \leq \text{l.gl.dim } R \leq k \cdot (\text{l.gl.dim } D) + k - 1$.

Proof. The proof is by induction on k. If $k = 1$, then $R = D$ by Theorem 3.3 or Corollary 3.4 and we are done.

Assume $k \geq 2$. Let $A = \sum_{b_i \in S_1} b_iRb_i$, $M = \sum_{b_i \in S_1, b_j \in S_2 \cup \cdots \cup S_k} b_iRb_j$, and

$$B = \Bigl(1 - \sum_{b_i \in S_1} b_i\Bigr) R \Bigl(1 - \sum_{b_i \in S_1} b_i\Bigr) = \Bigl(\sum_{b_j \in S_2 \cup \cdots \cup S_k} b_j\Bigr) R \Bigl(\sum_{b_j \in S_2 \cup \cdots \cup S_k} b_j\Bigr).$$

Then $S_2 \cup \cdots \cup S_k$ is a complete set of left triangulating idempotents of B and $S_2, \ldots, S_k$ is a partition which establishes a canonical generalized triangular matrix representation for B. Let $D_1 = \sum_{b_j \in S_2 \cup \cdots \cup S_k} b_jRb_j$.

Then by the induction hypothesis, $\text{l.gl.dim } D_1 \leq \text{l.gl.dim } B \leq (k-1) \cdot \text{l.gl.dim } D_1 + k - 2$. Note that $D = A \oplus D_1$ by Theorem 3.3 or Corollary 3.4, so $\max(\text{l.gl.dim } A, \text{l.gl.dim } D_1) = \text{l.gl.dim } D$. Also note that

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

and M is a left A-right B-bimodule. By [Fi, Corollary 5], $\max(\text{l.gl.dim } A, \text{l.gl.dim } B) \leq \text{l.gl.dim } R \leq \max(\text{l.gl.dim } A, \text{l.gl.dim } B + \text{l.pd}_A\, M + 1)$, where l.pd denotes the left projective dimension. Since $\text{l.pd}_A\, M \leq \text{l.gl.dim } A \leq \text{l.gl.dim } D$, it follows that $\text{l.gl.dim } D \leq \text{l.gl.dim } R \leq \text{l.gl.dim } B + \text{l.gl.dim } A + 1 \leq (k-1) \cdot \text{l.gl.dim } D_1 + k - 2 + \text{l.gl.dim } D \leq (k-1) \cdot \text{l.gl.dim } D + k - 2 + \text{l.gl.dim } D + 1 = k \cdot (\text{l.gl.dim } D) + k - 1$.

Consequently, R has finite left global dimension if and only if D has finite left global dimension.

4. QUASI-BAER RINGS

Throughout this section R denotes a ring (i.e., K is the ring of integers). In [Cl] Clark called a ring *quasi-Baer* if the right annihilator of every right ideal is generated by an idempotent as a right ideal. He used quasi-Baer rings to characterize a finite dimensional algebra over an algebraically closed field as a twisted semigroup algebra of a matrix units semigroup.

Examples are provided and various properties of quasi-Baer rings are extensively studied in [Cl, PZ, Bi4]. In particular, Baer rings [Ka], prime rings, semiprime right FPF rings [Fa2, p. 168], and piecewise domains [GS] (see Corollary 4.13) are examples of quasi-Baer rings.

One advantage of the class of quasi-Baer rings is that it is closed under the formation of n-by-n full matrix rings and n-by-n upper triangular matrix rings [PZ]. This is not true of the class of Baer rings. For example, if D is a commutative domain which is not Prüfer, then the ring $M_2(D)$ is a prime ring (hence quasi-Baer) which is not Baer [Ka, p. 17]. Also if R is the 2-by-2 upper triangular matrix ring over a prime ring which is not a Baer ring, then R is a quasi-Baer ring which is not a Baer ring. Observe that prime rings with nonzero left or right singular ideal [Law] are quasi-Baer but not Baer.

In this section we describe quasi-Baer rings which have a complete set of left triangulating idempotents. As corollaries we obtain several well known results, such as Michler's splitting theorem for right hereditary right Noetherian rings [Mi, Theorem 2.2], Faith's characterization of semiprime right FPF rings with no infinite set of central orthogonal idempotents [Fa1], and Gordon and Small's characterization of piecewise domains [GS, Main Theorem]. We extend the structure theorem of Chatters and Hajarnavis for semiprimary right nonsingular right CS-rings [CH1, Theorem 3.1]. Furthermore, the global dimension of quasi-Baer rings with a complete set of left triangulating idempotents is investigated. Also, we provide an intrinsic criterion for a triangular ring to be quasi-Baer, thereby answering a question of Pollingher and Zaks [PZ].

The following result is a generalization of Theorem 2 in [Cl] and Remark 3 in [PZ]. We obtain it immediately from Theorem 2.9.

PROPOSITION 4.1. *Assume that R is a quasi-Baer ring with a complete generalized triangular matrix representation. Then $\{r(I) \mid I$ is a right ideal of $R\}$ and $\{\ell(I) \mid I$ is a left ideal of $R\}$ are finite distribute sublattices of the lattice of all ideals of R.*

We next prove two lemmas which lead to one of the main theorems of this section.

LEMMA 4.2. *R is a prime ring if and only if R is quasi-Baer and semicentral reduced.*

Proof. Clearly a prime ring is quasi-Baer and semicentral reduced. Conversely, assume R is quasi-Baer and semicentral reduced. Let X and Y be ideals of R such that $XY = 0$. Then there exists an idempotent $e \in R$ such that $r(X) = eR$. By Lemma 1.1(i) and (vi), $e \in \mathcal{S}_\ell(R)$. Hence $e \in \{0, 1\}$. If $e = 0$ then $Y = 0$, and if $e = 1$ then $X = 0$.

For a ring R with a generalized triangular matrix representation, the following lemma illustrates the interplay between the structure of R and its "diagonal" rings.

LEMMA 4.3. *Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of R.*

(i) *P is a prime ideal of R if and only if there exists a prime ideal P_m of the ring $b_m R b_m$ such that $P = P_m + \Sigma b_i R b_j$, where the sum is over all $i, j = 1, \ldots, n$, such that $(i, j) \neq (m, m)$. Furthermore, P is a minimal prime ideal of R if and only if P_m is a minimal prime ideal of $b_m R b_m$.*

(ii) *If I is a minimal ideal of R, then there exist, $i, j \in \{1, \ldots, n\}$ such that $I = b_i I b_j$. Moreover if $I^2 \neq 0$, then $i = j$.*

Proof. (i) Referring to the generalized triangular matrix representation for R given in the proof of Proposition 1.3, observe that the set of all elements whose main diagonal is zero is a nilpotent ideal. Corresponding to this in R we have that $S = \Sigma b_i R b_j$, where $i, j = 1, \ldots, n$, $i \neq j$, is a nilpotent ideal of R. Also observe that $D_m = \Sigma b_i R b_j$, where $i, j = 1, \ldots, n$, with $(i, j) \neq (m, m)$, is an ideal of R, for each m.

Let P be a prime ideal of R. Then $S \subseteq P$. Since $b_1R + \cdots + b_nR = R$, there is some m such that $b_mR \nsubseteq P$. If $k \neq m$ and $b_kR \nsubseteq P$, then neither $(b_mR)(b_kR)$ nor $(b_kR)(b_mR)$ is contained in P, contrary to one of these products being zero. So $b_iR \subseteq P$, for all $i \neq m$, and hence $b_iRb_i \subseteq P$, for each $i \neq m$. So $b_iRb_j \subseteq P$, whenever $(i, j) \neq (m, m)$. From the generalized triangular matrix representation for R we see that P is the sum of all b_iRb_j, and hence $P = b_mPb_m + D_m$. Since P is a prime ideal of R, one immediately has b_mPb_m is a prime ideal of the ring b_mRb_m.

Conversely, assume P_m is a prime ideal of the ring b_mRb_m. Observe that $P_m + D_m$ is an ideal of R. Since b_mRb_m/P_m is a prime ring, and $R/(P_m + D_m) \cong b_mRb_m/P_m$, we have that $P_m + D_m$ is a prime ideal of R.

(ii) Let k be the largest element of $\{1, \dots, n\}$ such that $Ib_k \neq 0$. For $t \in \{1, \dots, n\}$ define $d_t = \sum_{i=1}^{t} b_i$. By Proposition 1.6(ii), $d_k \in \mathcal{S}_\ell(R)$. Hence $I = d_kI$. Also by Proposition 1.6(i), $b_k \in \mathcal{S}_r(d_kRd_k)$. Let $r \in R$. Then $r = rd_k + r\sum_{i=k+1}^{n} b_i$. So $Ib_kr = Ib_krd_k = Ib_krd_kb_k \subseteq Ib_k$. Hence $I = d_kIb_k$. If $d_{k-1}Ib_k = 0$, then $I = b_kIb_k$ and we are done. Suppose $d_{k-1}Ib_k \neq 0$. As above, $d_{k-1} \in \mathcal{S}_\ell(R)$, so $I = d_{k-1}Ib_k$, $I^2 = 0$ and $b_kIb_k = 0$. If $d_{k-2}Ib_k = 0$, then $b_{k-1}Ib_k = I$ and we are finished. Otherwise $d_{k-2}Ib_k \neq 0$ and hence $I = d_{k-2}Ib_k$. This process will terminate in a finite number of steps to yield the result.

Using the notation and hypotheses of Lemma 4.3, we see immediately that the mapping defined by $\phi(P) = P_m$, for each prime ideal P of R, is a bijection from Spec(R) onto Spec(V), where $V = b_1Rb_1 + \cdots + b_nRb_n$. Recall [R, p. 418] that the *little Krull dimension* of R, denoted by kdim(R) is the length of a maximal chain of prime ideals of R. Then Lemma 4.3 immediately yields kdim(V) = kdim(R) and $\mathbf{P}(R) = \mathbf{P}(V) + \sum_{i<j} b_iRb_j = \mathbf{P}(b_1Rb_1) + \cdots + \mathbf{P}(b_nRb_n) + \sum_{i<j} b_iRb_j$.

By applying Proposition 1.3, Theorems 2.9 and 2.10 to the case of quasi-Baer rings, we obtain the following structure theorem.

THEOREM 4.4. *Let R be a quasi-Baer ring with* Tdim$(R) = n$. *Then* $R = A \oplus B$ (*ring direct sum*) *such that*:

(i) $A = \oplus_{i=1}^{k} A_i$ *is a direct sum of prime rings*;

(ii) *there exists a ring isomorphism*

$$\phi : B \to \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1m} \\ 0 & B_2 & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{pmatrix},$$

where each B_i is a prime ring, B_{ij} is a left B_i-right B_j-bimodule, and $k + m = n$;

(iii) *for each $i \in \{1, \ldots, m\}$ there exists $j \in \{1, \ldots, m\}$ such that either $B_{ij} \neq 0$ or $B_{ji} \neq 0$;*

(iv) *the rings $B_1, \ldots, B_m$ are uniquely determined by B up to isomorphism (induced by an inner automorphism of R) and permutation;*

(v) *B has exactly m minimal prime ideals $P_1, \ldots, P_m$, R has exactly n minimal prime ideals of the form $A \oplus P_i$ or $C_i \oplus B$ where $C_i = \oplus_{j \neq i} A_j$ and these are mutually comaximal, $\mathbf{P}(R) = \mathbf{P}(B)$, and $(\mathbf{P}(R))^m = 0$;*

(vi) *if I is a minimal ideal of R, then either $I^2 \neq 0$ and $I \subseteq A_i$ for some $1 \leq i \leq k$, or $I^2 = 0$ and $\phi(I) \subseteq (B_{ij})$ for some $1 \leq i < m$ and $1 < j \leq m$, where (B_{ij}) is the set of m-by-m matrices with entries from B_{ij} in the (i, j)th position and zero elsewhere.*

Proof. Let $E = \{b_1, \ldots, b_n\}$ be a complete set of left triangulating idempotents of R.

(i) Let $\{e_1, \ldots, e_k\} = E \cap \mathbf{B}(R)$. Take $A_i = e_i R$. By Lemma 4.2, each A_i is a prime ring.

(ii) Let $\{f_1, \ldots, f_m\} = E \backslash \{e_1, \ldots, e_k\}$, where the f_i are maintained in the same relative order as they were in E. Then $\{f_1, \ldots, f_m\}$ is a complete set of left triangulating idempotents of B. Define ϕ as in the proof of Proposition 1.3, and let $B_i = f_i B f_i$ and $B_{ij} = f_i B f_j$. By [Cl, Lemma 2] and Lemma 4.2, each B_i is a prime ring.

(iii) This part is a consequence of Proposition 2.20(i).

(iv) This part follows from Theorem 2.10.

(v) This follows from a routine argument using Lemma 4.3(i).

(vi) Since $\mathbf{P}(R) = \mathbf{P}(B)$, if $I^2 = 0$, then $I \subseteq f_i B f_j$ for some $i, j \in \{1, \ldots, m\}$. So assume that $I^2 \neq 0$. By Lemma 4.3(ii) there exists $b_v \in E$ such that $I = b_v I b_v$. Consider the following cases:

Case 1. Assume $v = 1$. Then there exists $e \in \mathcal{S}_\ell(R)$ such that $r_R(I) = eR$. Then $(1 - b_1)R \subseteq eR$. Since $eb_1 \in r_{b_1 R b_1}(I)$ and $b_1 R b_1$ is a prime ring, then $eb_1 = 0$. Thus $e = eb_1 + e(1 - b_1) = e(1 - b_1) = 1 - b_1$. By Proposition 1.6(i), $1 - b_1 \in \mathcal{S}_r(R)$. So $1 - b_1 \in \mathcal{S}_\ell(R) \cap \mathcal{S}_r(R)$. Hence $1 - b_1$ is central. Therefore b_1 is also central.

Case 2. Assume $v > 1$. Let $g = \sum_{i=1}^{v} b_i$ and $\Gamma = gRg$. By Proposition 1.6(ii), $g \in \mathcal{S}_\ell(R)$. From [Cl, Lemma 2], Γ is a quasi-Baer ring with a complete set of left triangulating idempotents $\{b_1, \ldots, b_v\}$. Thus there exists $c \in \mathcal{S}_r(\Gamma)$ such that $\ell_\Gamma(I) = \Gamma c$. Hence $R(g - b_v) \subseteq \Gamma c$. Since $b_v c \in \ell_\Gamma(I)$ and $b_v R b_v = b_v \Gamma b_v$ is a prime ring, it follows that $b_v c = 0$. Therefore $c = b_v c + (g - b_v)c = (g - b_v)c = g - b_v$. By Proposition

1.6(ii), $g - b_v \in \mathcal{S}_\ell(\Gamma)$. So $g - b_v \in \mathcal{S}_\ell(\Gamma) \cap \mathcal{S}_r(\Gamma)$. Hence b_v is central in Γ. Thus $b_j \Gamma b_v = b_j R b_v = 0$ for all $j < n$. By Proposition 1.6(v), $b_v \in \mathcal{S}_\ell(R)$.

Now let $h = \sum_{i=v}^{n} b_i$ and $\Lambda = hRh$. By Proposition 1.6(i), $h \in \mathcal{S}_r(R)$. From [Cl, Lemma 2], Λ is a quasi-Baer ring with a complete set of left triangulating idempotents $\{b_v, \dots, b_n\}$. Thus there exists $d \in \mathcal{S}_\ell(\Lambda)$ such that $r_\Lambda(I) = d\Lambda$. Then $(h - b_v)\Lambda \subseteq d\Lambda$. Since $db_v \in r_{b_v \Lambda b_v}(I)$ and $b_v \Lambda b_v = b_v R b_v$ is a prime ring, it follows that $db_v = 0$. Thus $d = db_v + d(h - b_v) = d(h - b_v) = h - b_v$. From Proposition 1.6(i), $h - b_v \in \mathcal{S}_r(\Lambda)$. So $h - b_v \in \mathcal{S}_\ell(\Lambda) \cap \mathcal{S}_r(\Lambda)$. Hence b_v is central in Λ. Thus $b_v \Lambda b_j = b_v R b_j = 0$ for all $j > v$. By Proposition 1.6(vi), $b_v \in \mathcal{S}_r(R)$. Therefore $b_v \in \mathcal{S}_\ell(R) \cap \mathcal{S}_r(R)$. Consequently, b_v is central in R.

Observe that in Theorem 4.4(vi), if $I \subseteq A_i$, then A_i is a subdirectly irreducible ring. Moreover if X is a minimal right ideal of R such that the ideal generated by X is a minimal ideal of R (equivalently, X is not R-isomorphic to a nilpotent right ideal) then $X \subseteq A_i$, for some i, and A_i is a right primitive subdirectly irreducible ring.

COROLLARY 4.5. *For each of the following conditions R has the generalized triangular matrix representation of Theorem* 4.4, *with each R_i a prime Baer ring*, *where $R_i \in \{A_1, \dots, A_k\} \cup \{B_1, \dots, B_m\}$*:

(i) *R is a quasi-Baer left perfect ring. In this case each R_i is a simple Artinian ring.*

(ii) *R is a right nonsingular right CS left perfect ring. In this case each R_i is a simple Artinian ring.*

(iii) *R is a right* (*semi*-) *hereditary ring with no infinite set of orthogonal idempotents. In this case each R_i is a* (*semi*-) *hereditary prime ring.*

(iv) *R is a right hereditary right Noetherian ring. In this case each R_i is a right hereditary right Noetherian prime ring.*

(v) *R is a right Goldie semiprime right PP ring. In this case $R = \oplus R_i$ where each R_i is a prime right Goldie ring.*

Proof. (i) This part is a consequence of Proposition 2.14(v) and Theorem 4.4.

(ii) From [CK, Theorem 2.1] a right nonsingular right CS ring is a Baer ring. Now (i) can be used to obtain the result.

(iii)–(v) From [Sma, Theorem 1], R is a Baer ring. The remainder of the proof follows from Proposition 2.14, Proposition 2.16, and Theorem 4.4.

Since every semiprimary ring is left perfect, Corollary 4.5(i) extends the structure theorem of Chatters and Hajarnavis for semiprimary right nonsingular right CS-rings [CH1, Theorem 3.1]. Recently, Barthwal, Jain, Kanwar, and Lopez-Permouth have obtained a generalization of the Chatters and Hajarnavis result [BJKL-P]. Note that there are semiprimary right nonsingular quasi-Baer rings which are not right CS. For example, take

$$R = \begin{pmatrix} F & F \oplus F \\ 0 & F \end{pmatrix},$$

where F is a field. Thus in Corollary 4.5, condition (i) is weaker than condition (ii). Moreover, Corollary 4.5(iv) and (v) indicate that Theorem 4.4 is a generalization of Michler's splitting theorem for right hereditary right Noetherian rings [Mi, Theorem 2.2] and Levy's decomposition of right Goldie semiprime right hereditary rings [Le, Theorem 4.3], respectively.

The following result of Faith [Fa1, Theorem I.4] is also a corollary of Theorem 4.4.

COROLLARY 4.6. *A ring R is semiprime right FPF with no infinite set of central orthogonal idempotents if and only if R is a finite direct sum of prime right FPF rings.*

Proof. From Lemma 1.1, it follows that in a semiprime ring R, $\mathbf{B}(R) = \mathcal{S}_\ell(R) = \mathcal{S}_r(R)$. Assume R is a right FPF with no infinite set of central orthogonal idempotents. It is known that a semiprime right FPF ring is quasi-Baer [Fa2, p. 168]. Furthermore R has a complete set of centrally primitive idempotents which is also a complete set of left triangulating idempotents. By Theorem 4.4, R is a finite direct sum of prime rings. Since ring direct summands of right FPF rings are right FPF, these prime rings are right FPF. The converse is immediate.

COROLLARY 4.7. *Assume that R is a quasi-Baer ring with a complete generalized triangular matrix representation, then the following are equivalent:*

(i) l.gl.dim R *is finite*;

(ii) l.gl.dim $R/\mathbf{P}(R)$ *is finite*;

(iii) l.gl.dim$(R_1 + \cdots + R_n)$ *is finite, where the R_i are the diagonal rings in the complete generalized triangular matrix representation of R.*

Proof. By Theorem 4.4, if $R/\mathbf{P}(R)$ is prime, then so is R. The proof (i) $\Leftrightarrow$ (ii) then follows as in the proof of [GS, Corollary 5]. As a direct consequence of Theorem 3.6, we have (i) $\Leftrightarrow$ (iii).

If a ring R is semiprimary, then l.gl.dim R = r.gl.dim R by [Au, Corollary 9]. When R is semiprimary, we use gl.dim R instead of l.gl.dim R or r.gl.dim R.

COROLLARY 4.8. *Every semiprimary quasi-Baer ring has finite global dimension. Therefore, every quasi-Frobenius quasi-Baer ring is semisimple Artinian. In particular, if the group algebra $F[G]$ of a finite group G over a field F is quasi-Baer, then $F[G]$ is semisimple Artinian.*

Proof. This follows immediately from Corollary 4.7.

In the following, we use Theorem 3.6 to provide an upper bound for the global dimension of a semiprimary quasi-Baer ring.

COROLLARY 4.9. *Assume that R is a semiprimary quasi-Baer ring. Let $S_1, \dots, S_k$ be the partition (as in Theorem 3.6) of a complete set of left triangulating idempotents $\{b_1, \dots, b_n\}$. Then* gl.dim $R \leq k - 1$.

Proof. Since R is semiprimary quasi-Baer, each $b_i R b_i$ is a simple Artinian ring. Hence $D = b_1 R b_1 + \cdots + b_n R b_n$ is a semisimple Artinian ring. Thus Theorem 3.6 yields that gl.dim $R \leq k - 1$.

There is a semiprimary quasi-Baer ring R whose global dimension is $k - 1$. For example, let R be the 2-by-2 upper triangular matrix ring over a field. Then $k = 2$ and gl.dim $R = 1$. Also for a semiprimary quasi-Baer ring R, note that $k \leq \mathrm{Tdim}(R)$. Therefore we have that gl.dim $R \leq \mathrm{Tdim}(R) - 1$.

LEMMA 4.10. *Let $\{e_1, \dots, e_n\}$ be a set of orthogonal idempotents such that $e_1 + \cdots + e_n = 1$. Then the following conditions are equivalent*:

(i) *if $e_i x e_j R e_j y e_k = 0$ for some $x, y \in R$ and some $1 \leq i, j, k \leq n$, then either $e_i x e_j = 0$ or $e_j y e_k = 0$;*

(ii) *if $x e_j R e_j y = 0$ for some $x, y \in R$ and some $1 \leq j \leq n$, then either $x e_j = 0$ or $e_j y = 0$;*

(iii) *if $K e_j L = 0$ for some K and L ideals of R and some $1 \leq j \leq n$, then either $K e_j = 0$ or $e_j L = 0$.*

Proof. (i) $\Rightarrow$ (ii). Assume $x e_j R e_j y = 0$. If $e_i x e_j = 0$, for all $i = 1, \dots, n$ then $1 x e_j = \sum e_i x e_j = 0$. Hence $x e_j = 0$, and we are done. So suppose $e_m x e_j \neq 0$ for some $m \in \{1, \dots, n\}$. Then $e_j y e_k = 0$ for all $k = 1, \dots, n$. Hence $0 = (e_j y)(\sum e_i) = e_j y 1 = e_j y$.

(ii) $\Rightarrow$ (iii). Assume $K e_j L = 0$ and $e_j L \neq 0$. Let $y \in L$ such that $e_j y \neq 0$. Then $x e_j R e_j y = 0$, for all $x \in K$. So $x e_j = 0$, for all $x \in K$. Hence $K e_j = 0$.

(iii) $\Rightarrow$ (i). Assume $e_i x e_j R e_j y e_k = 0$. Since $e_j R = e_j R e_j R$, it follows that $(R e_i x e_j R) e_j (R e_j y e_k R) = 0$. Then either $R e_i x e_j R = 0$ or $R e_j y e_k R = 0$. Since R has a unity, either $e_i x e_j = 0$ or $e_j y e_k = 0$.

THEOREM 4.11. *Assume that R has a complete set of left triangulating idempotents with* $\mathrm{Tdim}(R) = n$. *Then the following are equivalent*:

(i) *R is quasi-Baer*;

(ii) *for any given complete set of left triangulating idempotents $\{b_1, \dots, b_n\}$ of R, if $b_i x b_j R b_j y b_k = 0$ for some $x, y \in R$ and some $1 \le i, j, k \le n$, then either $b_i x b_j = 0$ or $b_j y b_k = 0$*;

(iii) *there is a complete set of left triangulating idempotents $\{c_1, \dots, c_n\}$ of R such that if $c_i x c_j R c_j y c_k = 0$ for some $x, y \in R$ and some $1 \le i, j, k \le n$, then $c_i x c_j = 0$ or $c_j y c_k = 0$*;

(iv) *for any given complete set of left triangulating idempotents $\{b_1, \dots, b_n\}$, if $x b_j R b_j y = 0$ for some $x, y \in R$ and some $1 \le j \le n$, then either $x b_j = 0$ or $b_j y = 0$*;

(v) *for any given complete set of left triangulating idempotents $\{b_1, \dots, b_n\}$, if $K b_j L = 0$ for some K and L ideals of R and some $1 \le j \le n$, then either $K b_j = 0$ or $b_j L = 0$.*

Proof. (i) ⇒ (ii). Let $r(b_i x b_j R) = fR$, where $f \in \mathcal{S}_{\ell}(R)$. Then $b_j f b_j \in \mathcal{S}_{\ell}(b_j R b_j)$ by Lemma 1.4(ii). Since $\{b_1, \dots, b_n\}$ is a complete set of left triangulating idempotents, it follows that $\mathcal{S}_{\ell}(b_j R b_j) = \{0, b_j\}$. So either $b_j f b_j = 0$ or $b_j f b_j = b_j$. If $b_j f b_j = 0$, then since $b_j y b_k \in r(b_i x b_j R) = fR$, we have that $b_j y b_k = f b_j y b_k$. Thus $b_j y b_k = b_j b_j y b_k = b_j f b_j y b_k = 0$. On the other hand, if $b_j f b_j = b_j$, then since $b_i x b_j f = 0$, it follows that $0 = b_i x b_j f b_j = b_i x b_j$.

(ii) ⇒ (iii). This follows immediately because R has a complete set of triangulating idempotents.

(iii) ⇒ (i). This follows using the same method of proof as in (iv) ⇒ (i) in Theorem 1 of [Cl].

(ii) ⇔ (iv) ⇔ (v). These implications follow from Lemma 4.10.

Observe that Theorem 4.11 answers the question of Pollingher and Zaks which is implicit in their statement [PZ, p. 134], “Except for self-evident cases we don’t know of any intrinsic criterion for a triangular ring to be quasi-Baer.”

The following result extends Theorem 1 of [Cl] and Theorem 1 of [PZ].

THEOREM 4.12. *If R has a complete set of primitive idempotents, then the following are equivalent*:

(i) *R is quasi-Baer*;

(ii) *for any given complete set of primitive idempotents $\{e_1, \dots, e_n\}$, if $e_i x e_j R e_j y e_k = 0$ for some $x, y \in R$ and some $1 \le i, j, k \le n$, then either*

$e_i x e_j = 0$ or $e_j y e_k = 0$;

(iii) *there is a complete set of primitive idempotents $\{f_1, \ldots, f_m\}$ of R such that if $f_i x f_j R f_j y f_k = 0$ for some $x, y \in R$ and some $1 \le i, j, k \le m$, then $f_i x f_j = 0$ or $f_j y f_k = 0$;*

(iv) *for any given complete set of primitive idempotents $\{g_1, \ldots, g_p\}$, if $x g_j R g_j y = 0$ for some $x, y \in R$ and some $1 \le j \le p$, then either $x g_j = 0$ or $g_j y = 0$;*

(v) *for any given complete set of primitive idempotents $\{g_1, \ldots, g_p\}$, if $K g_j L = 0$ for some K and L ideals of R and some $1 \le j \le p$, then either $K g_j = 0$ or $g_j L = 0$.*

Proof. For $f \in \mathcal{S}_\ell(R)$ and a nonzero idempotent e, note that $efe \in \mathcal{S}_\ell(eRe)$, by Lemma 1.4(ii). In particular, if e is primitive, then $\mathcal{S}_\ell(eRe) = \{0, e\}$ and so it follows that either $efe = 0$ or $efe = e$. The proof can then be completed using an argument strictly analogous to that used in Theorem 4.11.

By Proposition 2.14(i), a quasi-Baer ring with a complete set of primitive idempotents has a complete set of left triangulating idempotents. But there is a quasi-Baer ring with a complete set of left triangulating idempotents which does not have a complete set of primitive idempotents. Let V be an infinite dimensional vector space over a field F and let $R = \mathrm{End}_F(V)$. Then as in Example 2.15, R is a prime ring; hence R is a quasi-Baer ring, and $\{1\}$ is a complete set of left triangulating idempotents. But R does not have a complete set of primitive idempotents.

As in [GS], a ring R is called a *piecewise domain* (or simply PWD) if there is a complete set of primitive idempotents $\{e_1, \ldots, e_n\}$ such that $xy = 0$ implies either $x = 0$ or $y = 0$ whenever $x \in e_i R e_j$ and $y \in e_j R e_k$, for $1 \le i, j, k \le n$.

COROLLARY 4.13. *Any PWD is a quasi-Baer ring.*

Proof. This is a direct consequence of Theorem 4.12.

This corollary may also be obtained from [PZ, Proposition 4; GS, p. 554, No. 3]. There are PWDs which are not Baer. For example, as mentioned before, let D be a domain which is not Prüfer; then $M_2(D)$ is a PWD, but not Baer. Also the ring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$, where $\mathbb{Z}$ is the ring of integers, is a PWD, which is not a Baer ring.

In light of Theorems 4.11 and 4.12, it is interesting to compare quasi-Baer rings having a complete set of left triangulating (or primitive) idempotents with piecewise domains. In fact, the equivalence of (i) and (iii) in Theorems 4.11 and 4.12 suggests calling a quasi-Baer ring with a complete set of left triangulating idempotents a *piecewise prime ring*, PWP ring (i.e., the piecewise prime generalization of a piecewise domain). Furthermore, in

[GS, p. 554] Gordon and Small posed the following question: "Can a PWD R possess a complete set $\{f_i\}_{i=1}^m$ of primitive orthogonal idempotents for which it is not true that $xy = 0$ implies $x = 0$ or $y = 0$ for some $x \in f_i R f_k$ and $y \in f_k R f_j$?" Moreover they stated, "To avoid ambiguity, we sometimes say that R is a PWD with respect to $\{e_i\}$." Theorems 4.11 and 4.12 show that if R is a PWP ring, then it is a PWP ring with respect to any complete set of left triangulating idempotents. Thereby it provides an answer to the "Question" of Gordon and Small for PWP rings.

Zalesskii and Neroslavskii [CH2, Example 14.17, p. 179] gave an example of a simple Noetherian ring R which is not a domain and in which 0 and 1 are the only idempotents. So this ring R is a PWP ring, but it is not a PWD. A right PP ring with a complete set of primitive idempotents and a right nonsingular ring which is a direct sum of uniform right ideals are PWDs [GS, p. 555]. So as a byproduct of Corollary 4.13, they are quasi-Baer.

LEMMA 4.14. *If R is a PWD and $0 \neq e \in \mathcal{S}_\ell(R) \cup \mathcal{S}_r(R)$, then the ring eRe is also a PWD.*

Proof. Assume R is a PWD with respect to a complete set of primitive idempotents $\{e_1, \ldots, e_n\}$. Then since $e \in \mathcal{S}_\ell(R)$, $e_i e = e e_i e$ is an idempotent for each i. Since e_i is primitive and $e_i e R \subseteq e_i R$, it follows that either $e_i e = 0$ or $e_i e R = e_i R$. If necessary reindex $\{e_1, \ldots, e_n\}$ so that $J = \{1, \ldots, r\}$ is the set of all indices such that $e_i e \neq 0$ for all $i \in J$. Then $e = (e_1 + \cdots + e_n)e = e_1 e + \cdots + e_r e$ and $eR = e_1 eR + \cdots + e_r eR = e_1 R + \cdots + e_r R$. Furthermore, by Lemma 2.13, $\{ee_1 e, \ldots, ee_r e\}$ is a complete set of primitive idempotents in the ring eRe. It can be easily checked that the mapping $\phi : eRe \to [e_i R e_j]$ defined by $\phi(exe) = [e_i x e_j]$, is a ring isomorphism. We will use E_{ij} for the r-by-r matrix units.

Now assume that $x \in (ee_i e)(eRe)(ee_j e)$ and $y \in (ee_j e)(eRe)(ee_k e)$ such that $xy = 0$ for $1 \leq i, j, k \leq r$. Put $x = (ee_i e)(eae)(ee_j e)$ and $y = (ee_j e)(ebe)(ee_k e)$. Then $0 = \phi(xy) = \phi(x)\phi(y) = e_i a e_j b e_k E_{ik}$, and so $(e_i a e_j)(e_j b e_k) = 0$. Since R is a PWD, either $e_i a e_j = 0$ or $e_j b e_k = 0$, and hence $\phi(x) = 0$ or $\phi(y) = 0$. So $x = 0$ or $y = 0$. Therefore eRe is a PWD with respect to the complete set of primitive idempotents $\{ee_1 e, \ldots, ee_r e\}$.

We next get as a corollary, the "Main Theorem" in [GS].

COROLLARY 4.15 [GS, Main Theorem]. *Assume that R is a PWD. Then*

$$R \cong \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix}, \tag{1}$$

where each R_i is a prime PWD and each R_{ij} is a left R_i-right R_j-bimodule. Furthermore

$$R_i \cong \begin{pmatrix} D_1 & \cdots & D_{1n_i} \\ \vdots & \ddots & \vdots \\ D_{n_i 1} & \cdots & D_{n_i} \end{pmatrix}, \qquad (2)$$

where each D_j is a domain and each D_{jk} is isomorphic as a right D_k-module to a nonzero right ideal in D_k and as a left D_j-module to a nonzero left ideal in D_j. The integer n is unique and the ring R_i is unique up to isomorphism.

Proof. Assume that R is a PWD with respect to a complete set of primitive idempotents $\{e_1, \ldots, e_m\}$. By Corollary 4.13, R is quasi-Baer. Thus the uniqueness of n and that of the ring R_i up to isomorphism follow from Theorem 4.4. From Proposition 2.14(i), there exists a complete set of left triangulating idempotents $\{b_1, \ldots, b_n\}$. We obtain (1) from Theorem 4.4, where $R_i = b_i R b_i$ is a prime ring. Lemma 4.14 yields $R_1 = b_1 R b_1$ and $(1 - b_1)R(1 - b_1)$ are PWDs. Since $0 \neq b_2 \in \mathcal{S}_\ell((1 - b_1)R(1 - b_1))$, Lemma 4.14 yields $R_2 = b_2 R b_2 = b_2(1 - b_1)R(1 - b_1)b_2$ is a PWD. By the same method, it can be shown that each $R_i = b_i R b_i$ is a PWD, for $i = 1, \ldots, n$. So there exists a complete set of primitive idempotents $\{c_1, \ldots, c_{n_i}\}$ for R_i such that $c_j x c_k y c_q = 0$ implies $c_j x c_k = 0$ or $c_k y c_q = 0$, for $x, y \in R_i$. Then (2) follows immediately, where D_{jk} is $c_j R_i c_k$. Hence each D_j is a domain. As in the proof of "Main Theorem" in [GS], let $0 \neq x \in c_j R_i c_k$; then $c_k R_i c_j$ is isomorphic as a right $c_j R_i c_j$-module to the nonzero right ideal $x c_k R_i c_j$ of $c_j R_i c_j$.

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