Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Linear Algebra and its Applications 426 (2007) 706–715
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

The linear dependence problem for power linear maps ☆

Dayan Liu, Xiankun Du, Xiaosong Sun *

School of Mathematics, Jilin University, Changchun 130012, China

Received 16 February 2007; accepted 31 May 2007
Available online 16 June 2007
Submitted by R.A. Horn

Abstract

Let $B_l, l = 1, \ldots, k$, be $m \times n_l$ complex matrices and let $x^{[l]} \in \mathbb{C}^{n_l}, l = 1, \ldots, k$, be complex vector variables. We show that the components of the map $H = \left(B_1 x^{[1]}\right)^{(d_1)} \circ \cdots \circ \left(B_k x^{[k]}\right)^{(d_k)}$ are linearly dependent over $\mathbb{C}$ if and only if $\det\left(B_1 B_1^*\right)^{(d_1)} \circ \cdots \circ \left(B_k B_k^*\right)^{(d_k)} = 0$, where $\circ$ means the Hadamard product, X^* and $X^{(d)}$ denote the conjugate transpose and the dth Hadamard power of a matrix X, respectively. Connections are established between the Homogenous Dependence Problem (HDP(n, d)), which arises in the study of the Jacobian Conjecture, and the dependence problem for power linear maps (PLDP(n, d)). An algorithm is given to compute counterexamples to PLDP(n, d) from those to HDP(n, d), and counterexamples to PLDP(n, 3) are obtained for all $n \geqslant 67$.

AMS classification: 15A03; 14R15

Keywords: Jacobian conjecture; Power linear maps; Homogeneous dependence problem

1. Introduction

Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. F_i is called the ith component of F. The Jacobian Conjecture asserts that F is invertible if the Jacobian determinant of F is a

☆ This work is supported by the National Natural Science Foundation of China (No. 10471055) and "985 Project" of Jilin University.
* Corresponding author.
E-mail address: sunxs@jlu.edu.cn (X. Sun).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2007.05.047

nonzero constant. Bass et al. [1] proved that it suffices to investigate the Jacobian conjecture for polynomial maps of the form $F = x + H$ with JH nilpotent and H homogeneous of degree 3, i.e., each component H_i is either zero or homogeneous of degree 3. The studies of these special polynomial maps led to the following problem; see [18, Section 7.1], [3,4,11,16].

1.1. Homogeneous Dependence Problem (HDP(n, d))

Let $H = (H_1, \dots, H_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map of degree $d \geqslant 2$ such that JH is nilpotent. Does it follow that $H_1, \dots, H_n$ are linearly dependent over $\mathbb{C}$; equivalently, are the rows of JH linearly dependent over $\mathbb{C}$?

Affirmative answers are known when $n \leqslant 3$, d arbitrary [3,19]; $n = 4$, $d = 2$ [13] or $d = 3$ [11]. Counterexamples are given by de Bondt for all dimensions $n \geqslant 5$, including counterexamples for all dimensions $n \geqslant 10$ with $d = 3$ [4]. But the following problems are still open:

Problem 1. Does HDP(n, 2) have an affirmative answer for all $n > 4$? See [16, Conjecture 11.3], [18, Question 7.4.16] or [4].

Problem 2. Does HDP(n, d) have an affirmative answer for all power linear maps: $H = \left(\left(\sum_{j=1}^n a_{1j}x_j\right)^d, \dots, \left(\sum_{j=1}^n a_{nj}x_j\right)^d\right)$?

In the paper, we solve Problem 2 and reduce Problem 1 to the case in which H is a power linear map of degree 2, which is also called a *quadratic linear map*. The motivation to consider the linear dependence problem for power linear maps is that Drużkowski [5] showed that it suffices to investigate the Jacobian Conjecture for all the maps $F = \left(x_1 + \left(\sum_{j=1}^n a_{1j}x_j\right)^3, \dots, x_n + \left(\sum_{j=1}^n a_{nj}x_j\right)^3\right)$. To describe the structure of these maps, it is necessary to consider the linear dependence problem for power linear maps. For studies of power linear maps we refer the reader to [2,6,12,14].

For $m \times n$ complex matrices $A = (a_{ij})$ and $B = (b_{ij})$ and a positive integer d, the matrix $(a_{ij}b_{ij})$ is called the *Hadamard product* of A and B, denoted by $A \circ B$, and the matrix (a_{ij}^d) is called the dth *Hadamard power* of A, denoted by $A^{(d)}$. We view the vectors in $\mathbb{C}^n$ as column vectors, and let $x = (x_1, \dots, x_n)^{\mathrm{T}}$, $F = (F_1, \dots, F_n)^{\mathrm{T}}$ and $A = (a_{ij})_{n\times n}$. Then $F = \left(x_1 + \left(\sum_{j=1}^n a_{1j}x_j\right)^d, \dots, x_n + \left(\sum_{j=1}^n a_{nj}x_j\right)^d\right)^{\mathrm{T}}$ can be written as $F_A(x) = x + (Ax)^{(d)}$, and $H = \left(\left(\sum_{j=1}^n a_{1j}x_j\right)^d, \dots, \left(\sum_{j=1}^n a_{nj}x_j\right)^d\right)^{\mathrm{T}}$ can be written as $H = (Ax)^{(d)}$.

Definition 1. The map $F_A(x) = x + (Ax)^{(d)}$, $d \geqslant 2$, is called a Drużkowski map of degree d, or a special power linear map of degree d, and $H = (Ax)^{(d)}$ is called a power linear map of degree d.

Problem 2 can be restated as follows,

1.2. Dependence Problem for Power Linear Maps (PLDP(n, d))

Let $H = (Ax)^{(d)} : \mathbb{C}^n \to \mathbb{C}^n$ be a power linear map such that JH is nilpotent. Does it follow that $H_1 = \left(\sum_{j=1}^n a_{1j}x_j\right)^d, \dots, H_n = \left(\sum_{j=1}^n a_{nj}x_j\right)^d$ are linearly dependent over $\mathbb{C}$; equivalently, are the rows of JH linearly dependent over $\mathbb{C}$?

In general, we describe the linear dependency of the following maps:

$$H = \left(B_1 x^{[1]}\right)^{(d_1)} \circ \cdots \circ \left(B_k x^{[k]}\right)^{(d_k)},$$

where $B_l = \left(b_{ij}^{[l]}\right)_{m\times n_l}$, $l = 1, \ldots, k$, are complex matrices, $x^{[l]} = \left(x_1^{[l]}, \ldots, x_{n_l}^{[l]}\right)^{\mathrm{T}}$, $l = 1, \ldots, k$, are complex vector variables, and $d_1, \ldots, d_k$ are positive integers. We show that the components of H are linearly dependent over $\mathbb{C}$ if and only if $\det\left(B_1 B_1^*\right)^{(d_1)} \circ \cdots \circ \left(B_k B_k^*\right)^{(d_k)} = 0$, where X^* denotes the conjugate transpose of a matrix X. In particular, the components of $H = (Ax)^{(d)}$ are linearly dependent if and only if $\det(AA^*)^{(d)} = 0$.

Using this result, we establish connections between HDP(n, d) and PLDP(n, d) though Gorni–Zampieri pairing. In fact, we prove that, for a fixed d, HDP(n, d) has an affirmative answer for all $n \geqslant 1$ if and only if PLDP(n, d) has an affirmative answer for all $n \geqslant 1$. As a consequence, counterexamples to PLDP(n, d) exist, too. An algorithm is given to compute counterexamples to PLDP(n_1, d) from counterexamples to HDP(n_0, d), and when $d = 3$, a concrete program is given and a counterexample to PLDP(67, 3) is computed explicitly, from which counterexamples to PLDP(n, 3) are derived for all $n \geqslant 67$.

2. Linear dependency of $H = \left(B_1 x^{[1]}\right)^{(d_1)} \circ \cdots \circ \left(B_k x^{[k]}\right)^{(d_k)}$

We start with some basic facts about Hadamard products of matrices; for details we refer the reader to [10, Chapter 5].

A Hermitian matrix B is said to be *positive semidefinite* if $x^* B x \geqslant 0$ for all $x \in \mathbb{C}^n$. If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ are positive semidefinite then so is $A \circ B$. If, in addition, B is positive definite and A has no diagonal entry equal to 0, then $A \circ B$ is positive definite [10, Theorem 5.2.1].

Now we introduce the power similarity of power linear maps; for details see [12,14].

Let A and B be $n \times n$ complex matrices; let $F_A = x + (Ax)^{(d)}$ and $F_B = x + (Bx)^{(d)}$ be special power linear maps of degree $d \geqslant 2$. F_A is said to be *power similar* to F_B, denoted by $F_A \overset{d}{\sim} F_B$ (or $A \overset{d}{\sim} B$), if there exists some $T \in Gl_n(\mathbb{C})$ such that $F_B = T^{-1} F_A T$.

We can verify that $F_A \overset{d}{\sim} F_B$ (or $A \overset{d}{\sim} B$) if and only if there exists some $T \in Gl_n(\mathbb{C})$ such that $T^{-1}(ATx)^{(d)} = (Bx)^{(d)}$. The invariants of power similarity can be used to classify power linear maps. The dimension of $\operatorname{span}(x \mapsto (Ax)^{(d)})$ is a power similarity invariant, where by the span of a map we mean the subspace spanned by the image of the map. Gorni and Tutaj-Gasińska [9] studied this invariant and conjectured that $\operatorname{span}(x \mapsto (Ax)^{(d)}) = \operatorname{range}(AA^*)^{(d)}$. The authors proved the conjecture in [17] and recently Qiu and Zhan [15] generalized it to the following form.

Theorem 2 [15, Theorem 6]. *Let B_l, $l = 1, \ldots, k$, be $n \times n$ complex matrices. Then*

$$\operatorname{span}\left\{\left(B_1 x^{[1]}\right) \circ \left(B_2 x^{[2]}\right) \circ \cdots \circ \left(B_k x^{[k]}\right) \middle| x^{[l]} \in \mathbb{C}^n\right\}$$
$$= \operatorname{range}\left(\left(B_1 B_1^*\right) \circ \left(B_2 B_2^*\right) \circ \cdots \circ \left(B_k B_k^*\right)\right).$$

Theorem 2 can be generalized to the following form.

Theorem 3. *Let B_l, $l = 1, \ldots, k$, be $n \times n$ complex matrices. Then*

$$\operatorname{span}\left\{\left(B_1 x^{[1]}\right)^{(d_1)} \circ \left(B_2 x^{[2]}\right)^{(d_2)} \circ \cdots \circ \left(B_k x^{[k]}\right)^{(d_k)} \middle| x^{[l]} \in \mathbb{C}^n\right\}$$
$$= \operatorname{range}\left(\left(B_1 B_1^*\right)^{(d_1)} \circ \left(B_2 B_2^*\right)^{(d_2)} \circ \cdots \circ \left(B_k B_k^*\right)^{(d_k)}\right).$$

Proof

$$\begin{aligned}&\operatorname{span}\{(B_1x^{[1]})^{(d_1)}\circ(B_2x^{[2]})^{(d_2)}\circ\cdots\circ(B_kx^{[k]})^{(d_k)}\mid x^{[l]}\in\mathbb{C}^n\}\\&\quad=\operatorname{span}\{(B_1B_1^*y^{[1]})^{(d_1)}\circ(B_2B_2^*y^{[2]})^{(d_2)}\circ\cdots\circ(B_kB_k^*y^{[k]})^{(d_k)}\mid y^{[l]}\in\mathbb{C}^n\}\\&\quad=\operatorname{range}((B_1B_1^*)^{(d_1)}\circ(B_2B_2^*)^{(d_2)}\circ\cdots\circ(B_kB_k^*)^{(d_k)}),\end{aligned}$$

where the final step is by [15, Theorem 4]. □

Two sets of vectors $\alpha_1,\dots,\alpha_s$ and $\beta_1,\dots,\beta_s$ are said to have *the same linear dependency over* $\mathbb{C}$, if for any $c_1,\dots,c_s\in\mathbb{C}$, $c_1\alpha_1+\cdots+c_s\alpha_s=0$ if and only if $c_1\beta_1+\cdots+c_s\beta_s=0$.

Theorem 4. *Let B_l, $l=1,\dots,k$, be $m\times n_l$ complex matrices. Let $x^{[l]}\in\mathbb{C}^{n_l}$, $l=1,\dots,k$, be complex vector variables and let $H=(B_1x^{[1]})^{(d_1)}\circ\cdots\circ(B_kx^{[k]})^{(d_k)}$. Then the components of H have the same linear dependency as the rows of matrix $(B_1B_1^*)^{(d_1)}\circ\cdots\circ(B_kB_k^*)^{(d_k)}$. In particular, the components of H are linearly dependent over $\mathbb{C}$ if and only if* $\det(B_1B_1^*)^{(d_1)}\circ\cdots\circ(B_kB_k^*)^{(d_k)}=0$.

Proof. We can assume that $n_1=\max\{n_1,\dots,n_k\}$. Let $H=(H_1,\dots,H_m)^{\mathrm{T}}$. For any $c_1,\dots,c_m\in\mathbb{C}$,

$$\begin{aligned}&c_1H_1+\cdots+c_mH_m=0\iff(c_1,\dots,c_m)H=0\\&\quad\iff(c_1,\dots,c_m)((B_1x^{[1]})^{(d_1)}\circ\cdots\circ(B_kx^{[k]})^{(d_k)})=0\quad\forall x^{[l]}\in\mathbb{C}^{n_l}.\end{aligned}$$

First we suppose that $m=n_1=\cdots=n_k$. By Theorem 3,

$$\begin{aligned}&(c_1,\dots,c_m)((B_1x^{[1]})^{(d_1)}\circ\cdots\circ(B_kx^{[k]})^{(d_k)})=0\quad\forall x^{[l]}\in\mathbb{C}^{n_l}\\&\quad\iff(c_1,\dots,c_m)((B_1B_1^*)^{(d_1)}\circ\cdots\circ(B_kB_k^*)^{(d_k)})=0,\end{aligned}$$

and so,

$$c_1H_1+\cdots+c_mH_m=0\iff(c_1,\dots,c_m)((B_1B_1^*)^{(d_1)}\circ\cdots\circ(B_kB_k^*)^{(d_k)})=0.$$

If $m\geqslant n_1$, let $B_l'=(B_l,0_{m\times(m-n_l)})_{m\times m}$, $l=1,\dots,k$, and add new variables such that each $x^{[l]}\in\mathbb{C}^m$. Let $H'=(B_1'x^{[1]})^{(d_1)}\circ\cdots\circ(B_k'x^{[k]})^{(d_k)}$. Then the components of H' have the same linear dependency as the rows of the matrix

$$(B_1'B_1'^*)^{(d_1)}\circ\cdots\circ(B_k'B_k'^*)^{(d_k)}=(B_1B_1^*)^{(d_1)}\circ\cdots\circ(B_kB_k^*)^{(d_k)}.$$

Since $H=H'$, the components of H have the same linear dependency as the rows of the matrix $(B_1B_1^*)^{(d_1)}\circ\cdots\circ(B_kB_k^*)^{(d_k)}$.

If $m<n_1$, let $B_l''=\begin{pmatrix}B_l\\0\end{pmatrix}_{n_1\times n_l}$, $l=1,\dots,k$, and let $H''=(B_1''x^{[1]})^{(d_1)}\circ\cdots\circ(B_k''x^{[k]})^{(d_k)}$. By the proof of the case $m\geqslant n_1$, we know that the components of H'' have the same linear dependency as the rows of the matrix

$$(B_1''(B_1'')^*)^{(d_1)}\circ\cdots\circ(B_k''(B_k'')^*)^{(d_k)}=\begin{pmatrix}(B_1B_1^*)^{(d_1)}\circ\cdots\circ(B_kB_k^*)^{(d_k)}&0\\0&0\end{pmatrix}.$$

Since the components of H are just the first m components of H'', the components of H have the same dependency as the rows of the matrix $\left(B_1B_1^*\right)^{(d_1)} \circ \cdots \circ \left(B_kB_k^*\right)^{(d_k)}$. □

Corollary 5. *Let $H = (Ax)^{(d)} : \mathbb{C}^n \to \mathbb{C}^n$ be a power linear map. Then the components of H have the same linear dependency as the rows of the matrix $(AA^*)^{(d)}$. In particular, the components of H are linearly dependent if and only if* $\det(AA^*)^{(d)} = 0$.

Proposition 6. *Let $H = \left(B_1x^{[1]}\right)^{(d_1)} \circ \cdots \circ \left(B_kx^{[k]}\right)^{(d_k)}$ and let $H' = \left(B_1x^{[1]}\right)^{(d_1')} \circ \cdots \circ \left(B_kx^{[k]}\right)^{(d_k')}$, where $1 \leqslant d_l' \leqslant d_l,\ l = 1, \ldots, k$. If the components of H are linearly dependent, then the components of H' are linearly dependent.*

Proof. If B_l has a zero row for some l, the proposition is true. Now assume that each B_l has no zero row. Since the components of H are linearly dependent, by Theorem 4, $\det\left(B_1B_1^*\right)^{(d_1)} \circ \cdots \circ \left(B_kB_k^*\right)^{(d_k)} = 0$. Suppose that the components of H' are linearly independent. Then $\det\left(B_1B_1^*\right)^{(d_1')} \circ \cdots \circ \left(B_kB_k^*\right)^{(d_k')} \neq 0$. Since each $B_lB_l^*$ is positive semidefinite so is $P = \left(B_1B_1^*\right)^{(d_1')} \circ \cdots \circ \left(B_kB_k^*\right)^{(d_k')}$, and so P is positive definite. Since each B_l has no zero row, each $B_lB_l^*$ has no diagonal entry equal to 0, and so $Q = \left(B_1B_1^*\right)^{(d_1-d_1')} \circ \cdots \circ \left(B_kB_k^*\right)^{(d_k-d_k')}$ has no zero row. By [10, Theorem 5.2.1], $P \circ Q = \left(B_1B_1^*\right)^{(d_1)} \circ \cdots \circ \left(B_kB_k^*\right)^{(d_k)}$ is also positive definite, which contradicts $\det\left(B_1B_1^*\right)^{(d_1)} \circ \cdots \circ \left(B_kB_k^*\right)^{(d_k)} = 0$. Thus, the components of H' are linearly dependent. □

Corollary 7. *If the components of $(Ax)^{(d)}$ are linearly independent, then the components of $(Ax)^{(k)}$ are linearly independent for every integer $k \geqslant d$. In particular, if the rows of A are linearly independent, then so are the components of $(Ax)^{(k)}$ for every integer $k \geqslant 1$.*

3. Connections between HDP(n, d) and PLDP(n, d)

To describe the connections between HDP(n, d) and PLDP(n, d), we need the following definition; see [18, Section 6.4] or [8].

Definition 8. Let $f = x + H : \mathbb{C}^n \to \mathbb{C}^n$, where H is homogeneous of degree $d \geqslant 2$ and $F_A = y + (Ay)^{(d)} : \mathbb{C}^N \to \mathbb{C}^N$ with $N > n$. We say that f and F_A are a Gorni–Zampieri pair through the matrices $B \in M_{n,N}(\mathbb{C})$ and $C \in M_{N,n}(\mathbb{C})$ if

1. $f(x) = BF_A(Cx),\ \forall x \in \mathbb{C}^n$;
2. $BC = I_n$;
3. $\ker B = \ker A$.

Remark 9. Gorni and Zampieri [8] introduced the Gorni–Zampieri pair for $d = 3$, but it can be easily generalized to every integer $d \geqslant 2$. The crucial point is that any monomial of degree $d (\geqslant 2)$ can be written as a finite sum of polynomials $(q_1x_1 + \cdots + q_nx_n)^d$, where $q_i \in \mathbb{Q},\ i = 1, \ldots, n$; see [18, Exercise 5.2.7].

We can verify that $f(x) = BF_A(Cx),\ \forall x \in \mathbb{C}^n$, if and only if $H = B(ACx)^{(d)}$. Observe that $BC = I_n$ and $\ker B = \ker A$ implies that $\mathrm{rk}\, B = \mathrm{rk}\, C = \mathrm{rk}\, A = n$, where $\mathrm{rk}\, X$ denotes the rank of a matrix X.

For any $f = x + H : \mathbb{C}^n \to \mathbb{C}^n$, where H is homogeneous of degree $d \geqslant 2$, there exists some $N > n$ and a special power linear map $F_A = x + (Ay)^{(d)} : \mathbb{C}^N \to \mathbb{C}^N$, such that f and F_A are a Gorni–Zampieri pair. Conversely, for any $F_A = y + (Ay)^{(d)} : \mathbb{C}^N \to \mathbb{C}^N$ with $n := \text{rk}\, A < N$, there exists $f = x + H : \mathbb{C}^n \to \mathbb{C}^n$, where H is homogeneous of degree $d \geqslant 2$, such that f and F_A are a Gorni–Zampieri pair.

If f and F_A are a Gorni–Zampieri pair, then $\det Jf$ is a nonzero constant if and only if $\det JF_A$ is a nonzero constant; equivalently, JH is nilpotent if and only if $J(Ay)^{(d)}$ is nilpotent, and f is a polynomial automorphism if and only if F_A is a polynomial automorphism. For the details, see [18, Section 6.4].

Lemma 10 [18, Lemma 6.4.4]. *Let $f = x + H : \mathbb{C}^n \to \mathbb{C}^n$ and $F_A = y + (Ay)^{(d)} : \mathbb{C}^N \to \mathbb{C}^N$ with $N > n$ be a Gorni–Zampieri pair through the matrices $B \in M_{n,N}(\mathbb{C})$ and $C \in M_{N,n}(\mathbb{C})$. Then $ACB = A$.*

Lemma 11. *Let A, B be $n \times n$ complex matrices such that $A \overset{d}{\sim} B$ and let $T \in Gl_n(\mathbb{C})$ be such that $F_B = T^{-1} F_A T$. Then there exists some $P \in Gl_n(\mathbb{C})$ such that $PAT = B$.*

Proof. Note that $F_B = T^{-1} F_A T$ implies that $T^{-1}(ATx)^{(d)} = (Bx)^{(d)}$. The system of linear equations $ATx = 0$ and $Bx = 0$ have the same set of solutions. Thus AT and B have the same row-reduced echelon form, whence there exists $P \in Gl_n(\mathbb{C})$ such that $PAT = B$. $\square$

Lemma 12. *Let A, B be $n \times n$ complex matrices such that $A \overset{d}{\sim} B$. Then $\text{rk}\, A = \text{rk}\, B$, $\text{rk}(BB^*)^{(d)} = \text{rk}\,(AA^*)^{(d)}$ and $\text{rk}\, B(BB^*)^{(d)} = \text{rk} A(AA^*)^{(d)}$.*

Proof. By Lemma 11, there exists some $P \in Gl_n(\mathbb{C})$ such that $PAT = B$. Thus $\text{rk}\, A = \text{rk}\, B$. By Theorem 3, for any $n \times n$ complex matrix C, $\text{range}(CC^*)^{(d)} = \text{span}(x \mapsto (Cx)^{(d)})$, and so $\text{range}\, C(CC^*)^{(d)} = \text{span}(x \mapsto C(Cx)^{(d)})$. Then

$$\begin{aligned}
\text{rk}(BB^*)^{(d)} &= \dim \text{span}(x \longmapsto (Bx)^{(d)}) \\
&= \dim \text{span}(x \longmapsto T^{-1}(ATx)^{(d)}) \\
&= \dim \text{span}(x \longmapsto (ATx)^{(d)}) \\
&= \dim \text{span}(x \longmapsto (Ax)^{(d)}) \\
&= \text{rk}(AA^*)^{(d)}
\end{aligned}$$

and

$$\begin{aligned}
\text{rk}\, B(BB^*)^{(d)} &= \dim \text{span}(x \longmapsto B(Bx)^{(d)}) \\
&= \dim \text{span}(x \longmapsto PATT^{-1}(ATx)^{(d)}) \\
&= \dim \text{span}(x \longmapsto PA(ATx)^{(d)}) \\
&= \dim \text{span}(x \longmapsto A(Ax)^{(d)}) \\
&= \text{rk}\, A(AA^*)^{(d)}. \qquad \square
\end{aligned}$$

Proposition 13. *Let $f = x + H : \mathbb{C}^n \to \mathbb{C}^n$ and $F_A = y + (Ay)^{(d)} : \mathbb{C}^N \to \mathbb{C}^N$ with $N > n$ be a Gorni–Zampieri pair. Then the components of H are linearly dependent if and only if $\text{rk}\, A(AA^*)^{(d)} < \text{rk}\, A$.*

Proof. There exist matrices $B \in M_{n,N}(\mathbb{C})$ and $C \in M_{N,n}(\mathbb{C})$ such that $f(x) = BF_A(Cx)$, $\forall x \in \mathbb{C}^n$, $BC = I_n$, and $\ker B = \ker A$. Then $H = B(ACx)^{(d)}$. By Lemma 10, $ACB = A$, so that $A^{\mathrm{T}} = B^{\mathrm{T}}C^{\mathrm{T}}A^{\mathrm{T}}$, which implies that range $A^{\mathrm{T}} \subseteq$ range B^{T}. Since $\operatorname{rk} A = \operatorname{rk} B = n$, range $A^{\mathrm{T}} =$ range B^{T}. Suppose that the components of H are linearly dependent. Then there exists a nonzero vector $\alpha = (a_1, \ldots, a_n)^{\mathrm{T}} \in \mathbb{C}^n$ such that $\alpha^{\mathrm{T}} H = 0$, i.e., $\alpha^{\mathrm{T}} B(ACx)^{(d)} = 0$. We have $\alpha^{\mathrm{T}} B \neq 0$, since $\operatorname{rk} B = n$. Because range $A^{\mathrm{T}} =$ range B^{T}, there exists a nonzero vector $\beta = (b_1, \ldots, b_N)^{\mathrm{T}} \in \mathbb{C}^N$ such that $\beta^{\mathrm{T}} A = \alpha^{\mathrm{T}} B$. Consequently $\beta^{\mathrm{T}} A(ACx)^{(d)} = 0$, whence $\beta^{\mathrm{T}} A(ACBy)^{(d)} = 0$, $\forall y \in \mathbb{C}^N$, and so $\beta^{\mathrm{T}} A(Ay)^{(d)} = 0$. By Corollary 5, $\beta^{\mathrm{T}} A(AA^*)^{(d)} = 0$. Thus $\operatorname{rk} A(AA^*)^{(d)} < \operatorname{rk} A$. Conversely, suppose that $\operatorname{rk} A(AA^*)^{(d)} < \operatorname{rk} A$. Then there exists a nonzero vector $\beta = (b_1, \ldots, b_N) \in \mathbb{C}^N$ such that $\beta^{\mathrm{T}} A(AA^*)^{(d)} = 0$ and $\beta^{\mathrm{T}} A \neq 0$. Since range $A^{\mathrm{T}} =$ range B^{T}, there exists a nonzero vector $\alpha = (a_1, \ldots, a_n)^{\mathrm{T}} \in \mathbb{C}^n$ such that $\beta^{\mathrm{T}} A = \alpha^{\mathrm{T}} B$. Thus $\alpha^{\mathrm{T}} B(AA^*)^{(d)} = 0$. By Corollary 5, $\alpha^{\mathrm{T}} B(Ay)^{(d)} = 0$, and so $\alpha^{\mathrm{T}} B(ACx)^{(d)} = 0$, i.e., $\alpha^{\mathrm{T}} H = 0$. □

Theorem 14. *Let* $f = x + H : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}$ *and* $F_A = y + (Ay)^{(d)} : \mathbb{C}^N \to \mathbb{C}^N$ *be a Gorni–Zampieri pair. If* H *is a counterexample to* HDP(n_0, d) *with* $\operatorname{rk}(AA^*)^{(d)} = n_1$, *then there exists a counterexample to* PLDP(n_1, d). *Furthermore, if* $2n_0 \leqslant n_1$, *then there exist counterexamples to* PLDP(k, d) *for all* $k \geqslant n_1$.

Proof. Suppose $f = x + H : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}$, where H is homogeneous of degree d, such that JH is nilpotent and the components of H are linearly independent. By Proposition 13, $\operatorname{rk} A(AA^*)^{(d)} = \operatorname{rk} A = n_0$. Let $\operatorname{rk}(AA^*)^{(d)} = n_1$. Then $n_0 \leqslant n_1 \leqslant N$. Let A_i be the ith row of A. Then $(Ay)^{(d)} = ((A_1y)^d, \ldots, (A_Ny)^d)^{\mathrm{T}}$.

If $n_1 = N$, then by Corollary 5 the components of $(Ay)^{(d)}$ are linearly independent, which implies that PLDP(n_1, d) has a negative answer.

If $n_1 < N$, after a suitable permutation we may assume that $(A_1y)^d, \ldots, (A_{n_1}y)^d$ are linearly independent, and there exists some $P \in Gl_N(\mathbb{C})$ such that

$$P^{-1}(Ay)^{(d)} = ((A_1y)^d, \ldots, (A_{n_1}y)^d, 0, \ldots, 0)^{\mathrm{T}} = ((A_1^{\mathrm{T}}, \ldots, A_{n_1}^{\mathrm{T}}, 0)^{\mathrm{T}} y)^{(d)}.$$

Let $U = (A_1^{\mathrm{T}}, \ldots, A_{n_1}^{\mathrm{T}}, 0)^{\mathrm{T}} P$. It can be written as $U = \begin{pmatrix} W & V \\ 0 & 0 \end{pmatrix}$, where $W \in M_{n_1}(\mathbb{C})$. Since $P^{-1}(APy)^{(d)} = (Uy)^{(d)}$, $A \overset{d}{\sim} U$. Thus

$$\operatorname{rk} U = \operatorname{rk} A = \operatorname{rk} A(AA^*)^{(d)} = \operatorname{rk} U(UU^*)^{(d)} = \operatorname{rk} W(WW^* + VV^*)^{(d)} \leqslant \operatorname{rk} W,$$

whence $\operatorname{rk} U = \operatorname{rk} W$. It follows that each column of V is a linear combination of the columns of W. Therefore, there exists a matrix R such that $V = WR$. Let $S = \begin{pmatrix} I_{n_1} & -R \\ 0 & I_{N-n_1} \end{pmatrix}$. Then

$$S^{-1}(USy)^{(d)} = S^{-1}\left(\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} y\right)^{(d)} = \left(\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} y\right)^{(d)},$$

which implies that $U \overset{d}{\sim} \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$. Consequently, $A \overset{d}{\sim} \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$. It follows from Lemma 12 that

$$\operatorname{rk}\left(\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}^*\right)^{(d)} = \operatorname{rk}(AA^*)^{(d)} = n_1.$$

Thus $\operatorname{rk}(WW^*)^{(d)} = n_1$. By Corollary 5, the components of $(Wx)^{(d)} : \mathbb{C}^{n_1} \to \mathbb{C}^{n_1}$ are linearly independent. Since $J(Ay)^{(d)}$ is nilpotent and $A \overset{d}{\sim} \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$, we have $J(Wx)^{(d)}$ is nilpotent. Consequently, there exists a counterexample to PLDP(n_1, d).

Now suppose $2n_0 \leqslant n_1$. Since $\mathrm{rk}\, W = \mathrm{rk}\, A = n_0$, there exists some $T \in Gl(n_1)$ such that $W = (W', 0)T$, where W' is an $n_1 \times n_0$ matrix. For any integer $n_1 < m \leqslant 2n_1$, let W'' be the matrix consisting of the first $m - n_1$ rows of W'. Let $V' = \begin{pmatrix} W'_{n_1\times n_0} & 0_{n_1\times(n_1-n_0)} & 0 \\ 0_{(m-n_1)\times(n_1-n_0)} & W''_{(m-n_1)\times n_0} & 0 \end{pmatrix}_{m\times m}$, and let $V = V'\begin{pmatrix} T & 0 \\ 0 & I_{m-n_1} \end{pmatrix}$. Then $V = \begin{pmatrix} W & 0 \\ (0, W'')T & 0 \end{pmatrix}$. Since $J(Wx)^{(d)}$ is nilpotent, $J(Vx)^{(d)}$ is nilpotent. Because the components of $(Wx)^{(d)}$ are linearly independent, so are the components of $(W'x)^{(d)}$ and $(W''x)^{(d)}$. Since $2n_0 \leqslant n_1$, $n_1 - n_0 \geqslant n_0$, whence the components of $\left(\begin{pmatrix} W' & 0 \\ 0 & W'' \end{pmatrix}x\right)^{(d)}$ are linearly independent, and so are the components of $(V'x)^{(d)}$. Since $\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} \in Gl(m)$, we see that the components of $(Vx)^{(d)}$ and $(V'x)^{(d)}$ have the same linearly dependency. Hence the components of $(Vx)^{(d)}$ are linearly independent. Thus there exist counterexamples to PLDP(m, d) for all $n_1 < m \leqslant 2n_1$. Let $V_i = \begin{pmatrix} \left.\begin{matrix} W & & \\ & \ddots & \\ & & W \end{matrix}\right\}i & \\ & V \end{pmatrix}$, $i \geqslant 1$. $J(V_i x)^{(d)}$ is nilpotent, but the components of $(V_i x)^{(d)}$ are linearly independent. Thus there exist counterexamples to PLDP(m, d) for all $(i + 1)n_1 < m \leqslant (i + 2)n_1$, and so there exist counterexamples to PLDP(k, d) for all $k \geqslant n_1$. $\square$

Corollary 15. *For a fixed integer $d \geqslant 2$,* PLDP*(n, d) has an affirmative answer for all integers $n \geqslant 1$ if and only if* HDP*(n, d) has an affirmative answer for all integers $n \geqslant 1$.*

Remark 16. For Problem 1, it is sufficient to investigate all the quadratic linear maps, i.e., power linear maps of degree 2.

4. Counterexamples to PLDP(n, d)

By Theorem 14, counterexamples to PLDP(n_1, d) can be obtained using counterexamples to HDP(n_0, d). Here is an algorithm to do so.

Step 1: Let $f = x + H$ be such that H is a counterexample to HDP(n_0, d). Calculate F_A from f.

Step 2: $X :=$ the transpose of $(AA^*)^{(d)}$, $n_1 := \mathrm{rk}X$. Compute the row-reduced echelon form of X and denote it by Q_0. Let $(i_1, \ldots, i_{n_1})$ be the indices of the columns of Q_0 in which the leading 1's lie.

$Q_1 :=$ the transpose of Q_0. Then the rows of Q_1 have the same linear dependency as the rows of $X := (AA^*)^{(d)}$, and so they have the same linear dependency as the components of H.

$Q_2 :=$ the matrix obtained by substituting zero rows for the $i_1, \ldots, i_{n_1}$ rows of Q_1. $P := Q_2 + I$. Then

$$P^{-1}(Ay)^{(d)} = (0, \ldots, (A_{i_1}y)^{(d)}, \ldots, (A_{i_{n_1}}y)^{(d)}), \ldots, 0).$$

$A_0 :=$ the $i_1, \ldots, i_{n_1}$ rows of A.

$W :=$ the $i_1, \ldots, i_{n_1}$ columns of A_0P.

Then $J(Wx)^{(d)}$ is nilpotent but the components of the power linear map $(Wx)^{(d)} : \mathbb{C}^{n_1} \to \mathbb{C}^{n_1}$ are linearly independent.

When $d = 3$, we give a Mathematica routine (version 5.1) to compute counterexamples to PLDP(n_1, d) using counterexamples to HDP(n_0, d). It uses the function **makePairing** (supplied by Wolfram Research Inc.; see [7]), which takes $f = x + H$ and returns matrices A, B, and C such that f and F_A are a Gorni–Zampieri pair though B and C.

```
<< LinearAlgebra ‘MatrixManipulation’;
Remove["‘*"];
a = makePairing[f][[1]];
(* makePairing[f] returns a list, the first
element of which is A *)
m = Length[a];
Print["N=", m];
Print["A=",a//MatrixForm];
x0 = a.Transpose[a];
x = Transpose[x0*x0*x0];
n1 = MatrixRank[x];
Print["n1=", n1];
q1 = Transpose[RowReduce[x]];
r = Table[0, {n1}];
For[i = 1; j = 1, i <= m, i++,
    If[TakeRows[q1, {i, i}] == TakeRows[IdentityMatrix[m], {j, j}],
    r = ReplacePart[r, i, j]; q1 = ReplacePart[q1, 0, {i, j}]; j++;]];
p = q1 + IdentityMatrix[m];
w1 = a[[r, All]].p;
w = w1[[All, r]];
Print["W=",w // MatrixForm];
```

In the following example, a counterexample to PLDP(67, 3) is computed explicitly from a counterexample to HDP(10, 3) given by M. de Bondt in [4], and counterexamples to PLDP(n, 3) are obtained for all $n \geqslant 67$.

Example 17. Let

$$H^{[n]}_{A,B} = \begin{pmatrix} B(Ax_1 - Bx_2) \\ A(Ax_1 - Bx_2) \\ B(Ax_3 - Bx_4) \\ A(Ax_3 - Bx_4) \\ \vdots \\ B(Ax_{n-1} - Bx_n) \\ A(Ax_{n-1} - Bx_n) \end{pmatrix} \; (n \text{ even}),$$

where A and B are symbolic constants.

$$a = x_1x_4 - x_2x_3, \quad b = x_3x_6 - x_4x_5.$$

Let $H = \left(H^{(6)}_{x_9,x_{10}}, x_9a, x_9b, x_8a - x_7b, x_9^3\right) : \mathbb{C}^{10} \to \mathbb{C}^{10}$. Then H is a counterexample to HDP(10, 3); see [4, Corollary 3.4].

Now, let $f = x + H$.

```
f = Function[{x1, x2, x3, x4, x5, x6, x7, x8, x9, x10},
    {x1 + x1 * x9 * x10 − x2 * x10^2, x2 + x1 * x9^2 − x2 * x9 * x10,
     x3 + x3 * x9 * x10 − x4 * x10^2, x4 + x3 * x9^2 − x4 * x9 * x10,
     x5 + x5 * x9 * x10 − x6 * x10^2, x6 + x5 * x9^2 − x6 * x9 * x10,
     x7 + x1 * x4 * x9 − x2 * x3 * x9, x8 + x3 * x6 * x9 − x4 * x5 * x9,
     x9 + x1 * x4 * x8 − x2 * x3 * x8 − x3 * x6 * x7 + x4 * x5 * x7, x10 + x9^3}].
```

Applying the previous program, we get $N = 75$, $n1 = 67$, and a 67×67 matrix W such that $(Wx)^{(3)}$ is a counterexample to PLDP(67, 3). Since $n_0 = 10$, $2n_0 \leqslant n_1$, by Theorem 14, there exist counterexamples to PLDP(n, 3) for all $n \geqslant 67$.

References

[1] H. Bass, E. Connel, D. Wright, The Jacobian Conjecture: reduction of degree and formal expansion of the inverse, Bull. Amer. Math. Soc. 7 (1982) 287–330.

[2] C.C. Cheng, Quadratic linear Keller maps, Linear Algebra Appl. 348 (2002) 203–207.

[3] M. de Bondt, A. van den Essen, The Jacobian Conjecture: linear triangularization for homogeneous polynomial maps in dimension three, J. Algebra 294 (2005) 294–306.

[4] M. de Bondt, Quasi-translations and counterexamples to the Homogeneous Dependence Problem, Proc. Amer. Math. Soc. 134 (2006) 2849–2856.

[5] L.M. Drużkowski, An effective approach to the Jacobian conjecture, Math. Ann. 264 (1983) 303–313.

[6] L.M. Drużkowski, The Jacobian Conjecture in case of rank or corank less than three, J. Pure Appl. Algebra 85 (1993) 233–244.

[7] G. Gorni, A nonlinearizable cubic-linear mapping, in: E. Hubbers (Ed.), Conference on Polynomial Maps and the Jacobian Conjecture, University of Nebraska-Lincoln, May 9–10, 1997.

[8] G. Gorni, G. Zampieri, On cubic-linear polynomial mappings, Indag. Math. 8 (4) (1997) 471–492.

[9] G. Gorni, H. Tutaj-Gasińska, On the entrywise powers of matrices, Comm. Algebra 32 (2) (2004) 495–520.

[10] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.

[11] E. Hubbers, Cubic similarity in dimension four, Master's thesis, University of Nijmegen, The Netherlands, 1994.

[12] E. Hubbers, Nilpotent Jacobians, Ph.D. thesis, University of Nijmegen, The Netherlands, 1998.

[13] G. Meisters, C. Olech, Strong nilpotence holds in dimension up to five only, Linear and Multilinear Algebra 30 (1991) 231–255.

[14] G. Meisters, Power-similarity: summary of first results, in: Conference on Polynomial Automorphisms, the C.I.R.M. Luminy, France, October 12–16, 1992.

[15] L. Qiu, X. Zhan, On the span of Hadamard products of vectors, Linear Algebra Appl. 422 (2007) 304–307.

[16] K. Rusek, Polynomial automorphisms, preprint 456, Institute of Mathematics, Polish Academy of Sciences, IMPAN, sniadeckich 8, P.O. Box 137, 00-950 Warsaw, Poland, May 1989.

[17] X. Sun, X. Du, D. Liu, On the range of a Hadamard power of a positive semidefinite matrix, Linear Algebra Appl. 416 (2006) 868–871.

[18] A. van den Essen, Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics, vol. 190, Birkhäuser, Basel, Boston, Berlin, 2000.

[19] D. Wright, The Jacobian Conjecture: linear triangularization for cubics in dimension three, Linear and Multilinear Algebra 34 (1993) 85–97.