AFFINE AND PROJECTIVE PLANES

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Introduction

The aim of this work is to suggest a setting for the discussion and classification of finite projective planes. In the past, two classification schemes have been put forward: the Lenz–Barlotti Classification (see [8, §3.1]) and a more restricted classification of translation planes proposed by Ostrom (see [19]).

Our approach is rather different from either of these two and rests heavily on the results and techniques of algebraic coding theory and, in particular, on the work of Philippe Delsarte.

The approach is to study a finite projective plane \( \Pi \) via its various affine parts and, to this end, we introduce the notion of the hull of an affine plane \( \pi \): the hull turns out to be the code generated, over an appropriate finite field \( \mathbb{F}_p \), by all differences of those pairs of rows of an incidence matrix that represent parallel lines of \( \pi \). In all known cases \( p \) will be the prime involved in the order, \( p' \), of the affine plane \( \pi \); in general it is any prime dividing the order \( n \) of the plane.

Letting \( H \) be such a hull, we have \( H \subset \mathbb{F}_p^n \), and the orthogonal \( H^\perp \) (in the usual inner product in \( \mathbb{F}_p^n \)) is a linear \((n', k)\) code over \( \mathbb{F}_p \) with minimum weight \( n \) and amongst its minimal-weight vectors one finds the affine plane \( \pi \) one began with. But, there can be (and frequently are) other affine planes to be found amongst the minimal-weight vectors of \( H^\perp = B \).

For example: if \( \pi = AG_2(4) \), the desarguesian affine plane of order 4, \( B \) is the \((16,11)\) extended binary Hamming code; this latter code contains 112 copies of \( AG_2(4) \) amongst its 140 weight-4 vectors.

We investigate two central questions:

(1) To what extent does the hull of an affine plane determine the plane?

(2) How can the various \((n', k)\) codes over \( \mathbb{F}_p \) of minimum weight \( n \) help to classify affine (and hence projective) planes?

Toward this end we introduce a notion of “linear equivalence”: two affine

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planes are linearly equivalent if their hulls are isomorphic. We show that for well-behaved planes (which we will call “tame”, all desarguesian planes being tame) linear equivalence implies isomorphism—at least when the order is odd.

In principle the theory proposed here is capable of producing planes, but one must find appropriate \((n^*, k)\) codes over \(F_p\) with minimum weight \(n\) in order to do so. A given plane automatically produces such a code and hence the theory is better suited to uncovering new planes using known planes than to building them out of whole cloth. But when the order in question is a power of a prime, algebraic coding theory produces an intensively-studied collection of standard codes all of which contain planes amongst their minimal-weight vectors.

One such standard choice is the code generated by the characteristic functions of all \(s\)-dimensional flats (i.e. subspaces and their translates) of a \(2s\)-dimensional vector space over \(F_p\). Let \(B\) be this \((q^s, k)\) code over \(F_p\), where \(q = p^r\). The minimum weight of \(B\) is \(q\), the minimal-weight vectors are known, and the dimension of \(B\) is computable. The design given by the minimal-weight vectors of \(B\) contains every translation plane of order \(q\) as a subdesign. The rank over \(F_p\) of the incidence matrix of any translation plane of order \(q\) is bounded above by \(q^2 + q - k\) and any two such planes meeting this bound are linearly equivalent. These matters are explored in some detail in Sections 4 and 5.

We also bring the theory to bear on the Hamada–Sachar Conjecture which asserts that every projective plane of order \(q = p^r\) has rank at least \((p^r + 1)^2 + 1\), with equality if and only if the plane is desarguesian. We show that this Conjecture is true for a class of odd-order projective planes.

By way of illustrating the classification of planes we have in mind we discuss briefly the four projective planes of order 9. They are, besides \(\text{PG}_2(9)\), a translation plane \(\Omega\), its dual, \(\Omega_{\text{dual}}\), and \(\Psi\), a self-dual plane first discovered by Veblen and Wedderburn. (In fact, Veblen and Wedderburn essentially knew all three of these non-desarguesian planes; \(\Psi\) is the first of an infinite class of non-desarguesian planes called Hughes planes; the only translation planes of order 9 are \(\text{PG}_2(9)\) and \(\Omega\).)

The \(B\) described above is an \((81, 50)\) ternary code with \(2 \times 1170\) weight-9 vectors. Amongst its minimal-weight vectors both \(\text{AG}_2(9)\) and \(\Omega^T\) can be found. (Here \(T\) is the translation line of \(\Omega\) and \(\Omega^T\) is the affine plane formed by taking \(T\) to be the line at infinity.)

Besides this standard \(B\) there is an \((81, 48)\) ternary code \(B'\) with minimum weight 9 and \(2 \times 306\) minimal-weight vectors; amongst its minimal-weight vectors both \(\Psi^R\), \(R\) a “real” line of \(\Psi\), and \(\Omega_{\text{dual}}^L\), where \(L\) is a line through the translation point of \(\Omega_{\text{dual}}\), can be found. These two ternary codes classify, if you will, the four projective planes, \(B\) classifying the translation planes and \(B'\) the non-translation planes: any two affine planes linear over (this notion is defined in the Glossary at the end of the paper) \(B\) or over \(B'\) can be obtained from one another by a derivation. The other three affine planes of order 9 each produce a ternary code, but these are less interesting. A fuller account can be found in Appendix II.
One finds in the existing literature (see, especially, [19]) vague feelings about certain planes being “very different” but no means to quantify these feelings and, indeed, there was not even a language in which to express them properly. The authors hope that the language and tools of algebraic coding theory will provide the necessary means.

This paper might have been entitled “On the theory of designs” and, indeed, the genesis of the ideas was the particular designs on 28 points given by the hermitian and Ree unitals. We have couched the definitions in this more general setting not only because of the possible usefulness in the theory of designs but also because if a projective plane of composite order arrives, the theory will be ready to greet it.

The paper is organized into seven sections. In Section 1 we define the hull of an arbitrary design at an arbitrary prime $p$ and present several examples. Section 2 explores the tight connection between the hull of a projective plane and the hull of an affine part. Section 3 is a detailed discussion of the affine case, Section 4 introduces linear equivalence and tameness, Section 5 is largely concerned with translation planes, and Section 6 discusses the Hamada–Sachar Conjecture and presents our partial solution. The final Section is a concluding discussion and considers possible avenues for further exploration. The appendices discuss the work of Delsarte and some results concerning the affine planes of order 9 that we obtained by computations using the Cayley package on the Birmingham University computer (VAX). Finally, we have included a Glossary of terms frequently used in the paper.

1. The hull of a design

Let $D$ be a design. $D$ consists of a set $P$ of points and a collection $B$ of subsets of $P$ (called blocks), all of the same cardinality, satisfying the single condition: any two distinct points of $P$ are contained in precisely $\lambda$ members of $B$, where $\lambda$ is some fixed, positive integer. Normally one denotes the cardinality of $P$ by $v$ (because of the origin of the notion in the design of agricultural experiments, the “$v$” standing for varieties) and the cardinality of the blocks in $B$ by $k$. In the modern literature such an object would be called a $2-(v, k, \lambda)$ design or a design of type $S_2(2, k, v)$. We will not usually employ either notation; moreover, $|P|$ will be denoted by $N$ and will usually be $n^2 + n + 1$ or $n^2$ and $k$ will usually be $n + 1$ or $n$ since our main objects of study here are finite projective and affine planes.

An incidence matrix of $D$ is a $|B|$ by $|P|$ matrix, $A$, of 0’s and 1’s where $a_{b,P} = 1$ if and only if the block $b$ of $B$ contains the point $P$ of $P$. Clearly $A$ determines $D$ uniquely and any incidence matrix of $D$ can be obtained from any other by row and column permutations.

Now, let $p$ be any prime, and set $C_p(D) = \text{row space } (A)$ over $F_p$. Then $C_p(D) \subseteq F_p^N \cong F_p^{|P|}$; it is an $(N, \text{dim } C_p(D))$ code over $F_p$, i.e. a subspace of the
$N$-dimensional space $F_p^N$ of row vectors with entries in $F_p$ of dimension the rank of $A$ over $F_p$. For any subset $X$ of $P$ we denote by $v^X$ the vector with 1's at the points in $X$ and 0's elsewhere; thus the rows of an incidence matrix of $D$ are simply the vectors $v^b$, $b \in B$. Viewing $F_p^N$ as $F_p^p$, $v^X$ is the characteristic function of $X$.

Using the standard inner product in $F_p^N$, $(v, w) = \sum_{Q \subseteq P} v_Q w_Q$ where $v = (\ldots, v_Q, \ldots)$ and $w = (\ldots, w_Q, \ldots)$, any code $C \subseteq F_p$ has a dual, namely $C^\perp = \{v \in F_p^N \mid (v, c) = 0 \text{ for all } c \in C\}$.

**Definition 1.** The hull at $p$ of a design $D$, denoted by $\text{Hull}_p(D)$, is $C_p(D) \cap C_p(D)^\perp$.

**Examples.** (1) If $\Pi$ is a projective plane of order $n$ and $p$ is a prime dividing $n$, then $\text{Hull}_p(\Pi) = \{c \in C_p(\Pi) \mid \sum_Q c_Q = 0\}$ (see [22]).

(2) If $D$ is either the Ree or hermitian unital on 28 points then, for $p = 2$, the hull is a $(28, 7)$ binary code invariant under the symplectic group $Sp_4(2)$, with weight distribution $1 + 63(X^{12} + X^{16}) + X^{28}$ (see [1], §6).

(3) If $D$ is any biplane of even order (i.e. a $2 - (N, n + 2, 2)$ design with $N = 1 + (\binom{n}{2}^2)$ where $n$ is an even integer), then $\text{Hull}_2(D) = C_2(D)$. For example the hull of the biplane of order 2 is the $(7, 3)$ even-weight subcode of the $(7, 4)$ Hamming code.

(4) There are exactly three biplanes of order 4 viz. $D_6$, $D_7$, and $D_8$. One can, by properly choosing the incidence matrices (see [3]), arrange matters so that

$$\text{Hull}_2(D_6) \subset \text{Hull}_2(D_7) \subset \text{Hull}_2(D_8).$$

These three codes are binary $(16, 6)$, $(16, 7)$, and $(16, 8)$ codes having minimum weights 6, 4, and 4 respectively. For $\text{Hull}_2(D_6) = C_2(D_6)$ the minimal-weight vectors are precisely the rows of the incidence matrix.

The central questions we want to address are: (1) When can we recover $D$ from $\text{Hull}_p(D)$? (2) How can we use coding theory to classify designs?

For a projective plane $\Pi$ of order $n$, if $p$ divides $n$, then the plane can be recovered from its hull; as we shall see in the next section $\text{Hull}_p(\Pi)^\perp$ has minimum weight $n + 1$ and its minimal-weight vectors are essentially the lines of the plane. (Here, and in the remainder of the paper, we will not make a verbal distinction between $X$ and $v^X$; thus “line” will refer to both the set of points of the plane and to the vector that is its characteristic function, depending on the context.)

Clearly, Example 2 above shows that one cannot, in general, recover the design from its hull. In that case $\text{Hull}_2(D)^\perp$ is a $(28, 21)$ binary code with minimum weight 4 and there are 315 minimal-weight vectors. The supports of these vectors form a design with $\lambda = 5$ and both the hermitian and Ree unitals are subdesigns with $\lambda = 1$. These two designs are not isomorphic and one must use group-
theoretic methods to recover them from the hull. The hulls of this example are \(\{0\}\) for any prime other than 2 and hence of no use in recovering the design.

Example 4 also shows that one cannot always recover the design from its hull. \(\text{Hull}_2(D_7) = C_2(D_7)\) contains, amongst its weight-6 vectors, a collection making up \(D_6\) as well as a collection making up \(D_7\) and in \(\text{Hull}_2(D_8)\) one sees all three biplanes. Here \(\text{Hull}_2(D_8)^{\perp}\) contains \(\text{Hull}_2(D_8)\) and one can obtain all three biplanes from \(\text{Hull}_2(D_8)\). One sees in this small example a microcosm of the idea behind the extraction of non-desarguesian projective planes from the classical desarguesian plane; we treat this aspect of the subject in detail in Section 5.

Although the prime \(p\) can be chosen arbitrarily, \(\text{Hull}_p(D)\) will be \(\{0\}\) except for finitely many primes, and the only primes of any interest are those dividing the order, \(r - \lambda\), of the design (see [13], or [21]). Here \(r\) is the number of blocks containing a fixed point. Thus, for \(AG_2(p^s)\), the desarguesian affine plane of order \(p^s\), only \(p\) will yield an interesting hull. Put another way, if \(\text{Hull}_p(D) = \{0\}\) or \(F_{p^s}J\), where \(J\) is the vector all of whose coordinates are 1 (a situation which obtains whenever \(p\) does not divide the order of \(D\)), then \((\text{Hull}_p(D))^{\perp}\) will capture all possible designs for the block size in question and hence these hulls will be useless as classifying tools.

It should be clear that the hull of a design is independent of the choice of an incidence matrix. All are obtained from one by the natural action of \(\text{Sym}(N)\) on \(F_p^N\). In the language of algebraic coding theory, the hull is obtained uniquely up to strict code isomorphism: here we use "strict" to indicate that one need not employ monomial matrices but merely permutation matrices to achieve the isomorphism.

Although one should, perhaps, work universally over \(Z\), reducing mod \(p\) when required, we do not do so, partly because previous such attempts have not led to startling results, but mostly because the tools of algebraic coding theory were fashioned for use over finite fields and it is precisely these tools that give the theory its power.

2. The passage from the projective to the affine hull

We begin with a proposition summarizing some well-known and easily proven results, but we cast them in our terms and, for the convenience of the reader, sketch their proofs; for further details see [22].

**Proposition 1.** Let \(\Pi\) be a projective plane of order \(n\) and let \(p\) be a prime dividing \(n\). Set \(H = \text{Hull}_p(\Pi)\) and \(B = H^\perp\). If \(k\) is the dimension of \(C_p(\Pi)\), then \(B\) is an \((n^2 + n + 1, n^2 + n + 2 - k)\) code with minimum weight \(n + 1\). The minimal-weight vectors of \(B\) are precisely the vectors of the form \(\alpha u^L\), where \(\alpha\) is a scalar and \(L\) is a line of \(\Pi\). Moreover, \(H = \{c \in C_p(\Pi) \mid \sum_{Q} c_Q = 0\}\) and is generated by vectors of the form \(v^L - v^M\) where \(L\) and \(M\) are lines of \(\Pi\).
Proof. We first prove the last assertion. Since, for any three lines, \( L, M, \) and \( N \), of \( \Pi \), \( (v^N, v^L - v^M) = 0 \) we have that \( v^L - v^M \) is in \( C_\rho(\Pi)^\perp \) for any two lines, \( L \) and \( M \); these vectors are clearly in \( C_\rho(\Pi) \) and hence in \( H \). Since \( n + 1 = 1 \mod p \), \( \sum_L v^L = J \), the all-one vector. Thus \( J \in C_\rho(\Pi) \). Further, \( (v^L - J, v^M) = 0 \) for all \( L \) and \( M \), so \( v^L - J \in H \) for all \( L \), and \( C_\rho(\Pi) = H \oplus F\rho J \), since \( J \) is not in \( H \). Thus the space spanned by the vectors \( v^L - J \) is a subspace of \( H \) which is a subspace of \( \{ c \in C_\rho(\Pi) \mid \sum_Q c_Q = 0 \} \), and as all these have codimension 1 in \( C_\rho(\Pi) \), they must be equal. Further, since for a line \( L \) and a fixed point \( Q \) not on \( L \), \( v^L - J = cM_{\rho Q} (v^L - v^M) \), the set \( \{ v^L - v^M \mid L \) and \( M \) lines of \( \Pi \} \) also spans \( H \).

It follows that \( b \in B \) if and only if \( (b, v^L - v^M) = 0 \) for any two lines, \( L \) and \( M \), of \( \Pi \) and hence that \( \alpha = (b, v^L) \) is independent of \( L \). If \( \alpha = 0 \) (i.e. if \( b \in C_\rho(\Pi)^\perp \)), it follows easily that \( \text{wt} \, b \geq n + 2 \), where \( \text{wt} \, v \) is the number of non-zero coordinates of the vector \( v \). Since \( C_\rho(\Pi) \subset R \), the minimum weight of \( B \) is at most \( n + 1 \). Thus for a minimal-weight vector \( b \) and all \( L \), \( \alpha = (b, v^L) \neq 0 \). For such a vector \( b \), let \( L_0 \) be a line of \( \Pi \) meeting \( \text{Supp} \, b \) at least twice, where \( \text{Supp} \, b \) is the subset \( \{ Q \mid b_Q \neq 0 \} \). If \( L_0 \subset \text{Supp} \, b \), then \( L_0 = \text{Supp} \, b \) and, choosing a point \( Q \) off \( L_0 \), we have for all lines \( L \) through \( Q \), \( (b, v^L) = \alpha = b_{L_0 \cap L} \); i.e., \( b = \alpha v^{L_0} \). If \( L_0 \notin \text{Supp} \, b \), let \( Q \) be a point of \( L_0 \) not in \( \text{Supp} \, b \). Every line through \( Q \) meets \( \text{Supp} \, b \) since \( (b, v^L) \neq 0 \) and thus the \( n + 1 \) lines through \( Q \) force \( b \) to have weight at least \( n + 2 \). To get the dimension of \( B \), we use that fact that \( \dim H = k - 1 \) since \( C_\rho(\Pi) = H \oplus F\rho J \). Thus, \( B \) has dimension \( n^2 + n + 1 - (k - 1) \), as required. □

Remark. For a projective plane \( \Pi \) of order \( n \) and a prime \( p \) dividing \( n \) we have \( \text{Hull}_p(\Pi) \subset C_\rho(\Pi) \subset \text{Hull}_p(\Pi)^\perp \). So for any code \( B \) with \( C_\rho(\Pi) \subset B \subset \text{Hull}_p(\Pi)^\perp \) the minimum weight of \( B \) is \( n + 1 \) and the minimal-weight vectors are precisely the scalar multiples of the lines of \( \Pi \). Hence any such \( B \) uniquely determines \( \Pi \); as we shall see, the situation is entirely different in the affine case, to which we now turn.

Let \( L \) be any line of the projective plane \( \Pi \) and set \( \pi = \Pi^L \), the affine plane determined by taking \( L \) to be the line at infinity. If \( p \) divides the order \( n \) of \( \Pi \) and \( \pi \), then there is a natural projection of \( C_\rho(\Pi) \) onto \( C_\rho(\pi) \) obtained by ignoring the coordinate places corresponding to the points of \( L \). The restriction to \( B \) of the linear transformation from \( F_p^{n+1} \) to \( F_p^n \) that effects this natural projection we denote by \( T \). (Recall that \( B = \text{Hull}_p(\Pi)^\perp \).) Since \( B \) has minimum weight \( n + 1 \) the kernel of \( T \) is one-dimensional and consists precisely of the scalar multiples of \( v^L \). Thus,

\[
\dim C_\rho(\pi) = \dim C_\rho(\Pi) - 1;
\]

also

\[
\dim C_\rho(\pi)^\perp = n^2 - \dim C_\rho(\pi).
\]
Clearly, if \( v \in C_p(\Pi) \) and \( v_Q = 0 \) for \( Q \) on \( L \), \( T(v) \in C_p(\pi) \). Setting
\[
D = \{ v \in C_p(\Pi) \mid v_Q = 0 \text{ for } Q \text{ on } L \}
\]
we have that \( T(D) \subseteq C_p(\pi) \).

The dimension of \( D \) is easy to compute; if \( v \in C_p(\Pi) \) and \( v_Q = 0 \) for all but one point \( Q_0 \) on \( L \), then \( v_{Q_0} = 0 \) since \( (v', v) = \sum_{Q \in L} v_{Q} \), and requiring that \( v_Q = 0 \) for one point \( Q \) on \( L \) lowers the dimension by one the first \( n \) times. Hence,
\[
\dim D = n^2 + n + 1 - \dim C_p(\Pi) - n = n^2 - \dim C_p(\pi)
\]
and therefore \( T(D) = C_p(\pi) \) since \( D \cap \ker T = \{0\} \) and \( \dim T(D) = \dim(D) \).

Observe next that
\[
B = \text{Hull}_p(\Pi) = C_p(\Pi) + C_p(\Pi) = C_p(\Pi) + D
\]
since any \( v \in C_p(\Pi) \) can be written as the sum of an element of \( D \) and an appropriate linear combination of the vectors \( v^\mathcal{M} \in C_p(\Pi) \), where \( M \) runs through the lines of \( \Pi \) except \( L \). (More precisely, one takes for each \( P \) on \( L \) a line \( L_P \) through \( P \) with \( L_P \neq L \) and, if \( v \in C_p(\Pi) \), forms \( v - \sum_{P \in L} v_P v^L = w \); clearly \( w_P = 0 \) for \( P \in L \) and hence \( w \in D \) since \( \sum_{P \in L} v_P v^L \in C(\Pi) \) in view of the fact that \( \sum_{P \in L} v_P = (v, v^L) = 0 \).) Hence
\[
T(C_p(\Pi) + C_p(\Pi) = C_p(\pi) + C_p(\pi)
\]
or, in other words,
\[
T(\text{Hull}_p(\Pi)) = \text{Hull}_p(\pi)\]
Thus, \( \dim \text{Hull}_p(\pi) = \dim \text{Hull}_p(\Pi) + 1 = n^2 + 1 - (\dim C_p(\Pi) - 1) = n^2 + n - \dim C_p(\pi) \) and \( \dim \text{Hull}_p(\pi) = \dim C_p(\pi) - n \).

We are now in a position to state and finish the proof of the following result.

**Proposition 2.** Let \( \pi \) be an affine plane of order \( n \) and \( \Pi \) its projective completion with \( L_0 \) the line at infinity. Then, for a prime \( p \) dividing \( n \),
\[
\dim \text{Hull}_p(\pi) = \dim C_p(\pi) - n,
\]
both \( C_p(\pi) \) and \( \text{Hull}_p(\pi) \) are the images of the natural projection of \( C_p(\Pi) \) and \( \text{Hull}_p(\Pi) \) respectively into \( F_p^n \), \( \text{Hull}_p(\pi) \) is the projection of the subcode \( \{ c \in \text{Hull}_p(\Pi) \mid c_Q = 0 \text{ for } Q \text{ on } L_0 \} \), and \( \text{Hull}_p(\pi) \) is generated by \( v^l - v^m \) where \( l \) and \( m \) are any two parallel lines of \( \pi \).

**Proof.** Only the last two assertions need to be verified. If \( l \) and \( m \) are two parallel lines of \( \pi \), their extensions, \( L \) and \( M \), to \( \Pi \) meet at a point of \( L_0 \). Thus, \( v^L - v^M \) is not only in \( C_p(\Pi) \) but also in \( D \) (defined above), i.e. \( T(v^L - v^M) = v^l - v^m \in C_p(\pi) \). Since \( v^l - v^m \) is clearly in \( C_p(\pi) \) we need only show that these vectors form a generating set of \( \text{Hull}_p(\pi) \). Let \( E \) be the subspace of \( F_p^n \) generated by the set \( \{ v^l - v^m \mid l \) and \( m \) parallel lines of \( \pi \} \). Then \( J = -\sum_{i=1}^{n} (v^l - v^m) \) for any fixed \( l \) since there are \( n \) lines in a parallel class and \( p \) divides \( n \). Thus \( J \in E \subseteq \text{Hull}_p(\pi) \). Fix a point \( Q \) of \( \pi \) and choose \( n \) lines through \( Q, l_1, \ldots, l_n \), say. The subspace generated by \( E \) together with \( v^{l_1}, \ldots, v^{l_n} \) clearly contains the \((n+1)\)th line through \( Q \), namely \( J - \sum_{i=1}^{n} v^{l_i} \). Since every \( v^l - v^m, l \parallel m, \) is in \( E \)
this subspace must be $C_p(\pi)$. Thus $\dim E \geq \dim C_p(\pi) - n$; i.e., $E = \text{Hull}_p(\pi)$. Clearly $\text{Hull}_p(\pi)$ is the projection of $\{ c \in \text{Hull}_p(\Pi) \mid c_Q = 0 \text{ for } Q \text{ on } L_0 \}$. □

We turn next to the minimum weight of $B$, where, as in the projective case, we have $B = \text{Hull}_p(\pi)^\perp \supset C_p(\pi) \supset \text{Hull}_p(\pi)$. For the sake of economy of language we call a vector in $F^N$ a constant vector if it is a scalar multiple of a vector all of whose entries are 0 or 1. We can now state the result.

**Proposition 3.** If $\pi$ is an affine plane of order $n$ and $p$ is a prime dividing $n$, then the minimum weight of $B = \text{Hull}_p(\pi)^\perp$ is $n$ and all minimal-weight vectors are constant.

**Proof.** Clearly the minimum weight of $B$ is at most $n$ since $C_p(\pi)$ is contained in $R$. Let $b$ be a minimal-weight vector in $R$. Then $(b, v' - v'') = 0$ for every pair of parallel lines, $l$, and $m$, of $\pi$. Thus $\alpha = (b, v')$ depends only on the parallel class in which $l$ lies. Since $\text{wt} b \leq n$, if $Q \in \text{Supp } b$, not all the $n + 1$ lines through $Q$ can meet $\text{Supp } b$ again. Let $l$ be a line with $|l \cap \text{Supp } b| = 1$. Then $(b, v') = \alpha \neq 0$ where $\alpha = b_Q$. Since $(b, v'') = \alpha$ for $m$ parallel to $l$, $\text{wt } b \geq n$ and hence $\text{wt } b = n$ and $b$ is a constant vector, $\text{Supp } b$ being a transversal for that parallel class. □

We return to the ideas in this proof in the next section, but let us now remark that we also have the conclusion of Proposition 3 for $C_p(\pi)$ since $C_p(\pi) \subseteq B$. An important point to note here is that it can very well happen that $B$ will contain minimal-weight vectors that are not scalar multiples of lines. If we take $\text{PG}_2(q^2)$ where $q$ is an arbitrary prime power and let $S$ be the point set of a Baer subplane of $\Pi$, then $|S| = q^2 + q + 1$ and for every line $l$ of $\Pi$, $|l \cap S|$ is 1 or $q + 1$. Hence, taking $p = \text{char } F_q$ we have that $|S \cap L| = 1(\text{mod } p)$ for all $L$. It follows that $J - v^S$ is in $C_p(\Pi)^\perp$ and hence

$$v^S \in \text{Hull}_p(\Pi)^\perp = C_p(\Pi)^\perp \oplus F_p J.$$

If $L$ is a line of $\Pi$ meeting $S$ in $q + 1$ points and $\pi = \Pi^L$, then the projection of $\text{Hull}_p(\Pi)^\perp$ onto $\text{Hull}_p(\pi)^\perp$ takes $v^S$ to a constant vector of weight $q^2 = n$ of $\text{Hull}_p(\pi)^\perp$.

For $q = 2$, for example, $\text{Hull}_p(\text{AG}_2(4))$ is the $(16, 5)$ Reed–Muller code with weight distribution

$$1 + 30x^8 + x^{16}$$

and $\text{Hull}_p(\text{AG}_2(4))^\perp$ is the $(16, 11)$ binary extended Hamming code with 140 weight-4 vectors. These vectors form a Steiner quadruple system that can be partitioned into seven affine planes of order 4, a fact irrelevant to our present purposes, but, as a design, they contain 112 subdesigns which are affine planes of order 4, all of which have the same hull. The 20 vectors forming an affine plane of
order four having been chosen, the remaining 120 are Baer subplanes of that plane. (The reader should not make too much of this special case; although 2 is the prime of choice in algebraic coding theory, it frequently plays an irksome role here.)

3. The hull of an affine plane

Throughout this section we shall have at hand an affine plane $\pi$ of order $n$; $\Pi$ will be its projective completion with $L_0$ the line at infinity. We fix a prime $p$ dividing $n$ and set $H = \text{Hull}_p(\pi)$ and $B = H^\perp$. We know from the previous section that $B$ has a minimum weight $n$ and that all minimal-weight vectors are constant; further, amongst the minimal-weight vectors are the scalar multiples of the $n^2 + n$ lines of $\pi$. Our purpose here is to investigate the nature of those minimal-weight vectors of $B$ which are not scalar multiples of lines of $\pi$. In fact we want to relate these vectors to their preimages, under the natural projection, in $\text{Hull}_p(\Pi)^\perp$. Set $\tilde{B} = \text{Hull}_p(\Pi)^\perp$ and, as before, let $T : \tilde{B} \to B$ be the natural projection.

We begin with a technical result.

**Lemma 1.** Let $b$ be a minimal-weight vector in $B$. Then, for $p$ odd, there is a unique constant vector $\bar{b}$ in $B$ with $T(\bar{b}) = b$. Moreover $\text{wgt} \bar{b} = n + r$ with $r = 1 (\text{mod } p)$, $r < n$; $r$ is the number of parallel classes in $\pi$ for which $\text{Supp} b$ is not a transversal. When $p = 2$ there are two such vectors and uniqueness is achieved by requiring that $\bar{b}$ be not in $C_2(\Pi)^\perp$.

**Proof.** Since $\tilde{B} = C_p(\Pi)^\perp \oplus F_p J$ and $v^{L_0}$ is not in $C_p(\Pi)^\perp$, $\tilde{B} = C_p(\Pi)^\perp \oplus F_p v^{L_0}$. Thus if $c$ is any vector in $\tilde{B}$ with $T(c) = b$, all such are of the form $c + \alpha v^{L_0}$, $\alpha \in F_p$. Without loss of generality we assume that the non-zero coordinates of $b$ are 1's. By a proper choice of $\alpha$ we can find a $c' \in C_p(\Pi)^\perp$ with $T(c') = b$.

If $\vartheta$ is a parallel class of $\pi$ for which $\text{Supp} b$ is a transversal, then each extension of an $l$ in $\vartheta$ to $\Pi$ goes through a fixed point $Q$ on $L_0$ with, necessarily, $c'_Q = -1$. On the other hand, if $\vartheta$ is a parallel class of $\pi$ for which $l \cap \text{Supp} b = \phi$ for some $l$ in $\vartheta$, then for the corresponding $Q$ on $L_0$, $c'_Q = 0$. Clearly, for $p$ odd, the only constant vector amongst $c' + \alpha v^{L_0}$, $\alpha \in F_p$ is $\bar{b} = c' + v^{L_0}$, with $r$ as described. Here $r = 1 (\text{mod } p)$ since now $(\bar{b}, v^{L_0}) = 1$ for all $l \neq L_0$ of $\Pi$, which forces $(\bar{b}, J) = 1 = r (\text{mod } p)$ (and $r = n + 1$ is impossible since then $\text{wgt}(\bar{b} - v^{L_0}) = n$). The amendment for $p = 2$ should be obvious. \(\square\)

**Remarks.** (1) The constant vector $\bar{b}$ of the lemma has a support that is a "blocking set" of the projective plane. Thus, if $b$ is not a line of the affine plane (i.e. if $r \neq 1$), we have the inequalities

$$1 + \sqrt{n} \leq r < n$$
with \( r \equiv 1(\text{mod } p) \): see [5]. From the lemma and Proposition 4 (Section 5) one can deduce easily that a projective plane that is a translation plane of order \( q^m \) with kernel \( F_p \) has a blocking set of cardinality \( q^m + q^{m-1} + 1 \).

(2) It has been suggested that a projective plane of order 15 containing an extended Kirkman system might exist [10]. If so, the lemma shows that \( \text{Hull}_n(\pi)^\perp \), where \( \pi \) a particular affine part of the proposed plane, will not exhibit this system. But \( \text{Hull}_n(\pi)^\perp \) will and the 15 points of the Kirkman system will be a weight-15 vector that is not a line of the affine plane. In this case, \( \text{Hull}_n(\pi) \) would be a ternary \((225, 105)\) code and its orthogonal a \((225, 120)\) code with more than 240 weight-15 vectors. As in the case of prime order, this latter code is \( C_3(\pi) \) and thus the Kirkman system would have to be a linear combination of the rows of \( \pi \)’s incidence matrix (with, of course, coefficients in \( F_p \)).

The next lemma will, among other things, completely characterize the minimal-weight vectors in \( B \) for \( n = p \) and for \( n = p^2 \).

**Lemma 2.** Suppose \( n = sp \) where \( p \) is a prime and \( 1 \leq s \leq p \). Then for an affine plane of order \( n \), with \( B \) as above, the minimal-weight vectors of \( B \) are scalar multiples of lines unless \( s = p \) where they are scalar multiples of either lines or Baer subplanes of \( \pi \).

**Proof.** Let \( \vec{b} \) be the constant vector of \( B \) described in Lemma 1. If \( r = 1 \), \( \vec{b} \) is a scalar multiple of a line of \( \Pi \) by Proposition 1 (Section 2), and hence \( b \) is a scalar multiple of a line of \( \pi \). Suppose \( b \) is a weight-\( n \) vector of \( B \), normalized so that its non-zero coordinates are 1, and suppose it is not a line of \( \pi \). Then, for some parallel class \( \theta \) of \( \pi \), we have \( l \in \theta \) with \( 1 < |l \cap \text{Supp } b| < sp \) and, in fact, for each \( l \in \theta \), \( l \cap \text{Supp } b \) is empty or \( |l \cap \text{Supp } b| \) is a multiple of \( p \). There must be at least two lines \( l_1 \) and \( l_2 \) in \( \theta \) with \( l_1 \cap \text{Supp } b \neq \phi \). Now if \( l \) is a line through a point of \( l_1 \cap \text{Supp } b \) and a point of \( l_2 \cap \text{Supp } b \), then \( |l \cap \text{Supp } b| \geq 2 \) and is hence a multiple of \( p \) and thus \( |l \cap \text{Supp } b| \geq p \). Clearly, then there are at least \( p \) lines in \( \theta \) meeting \( \text{Supp } b \) in a subset of cardinality at least \( p \). This is impossible unless \( s = p \) and, in this case, every line of \( \pi \) meets \( \text{Supp } b \) in 0, 1, or \( p \) points. Thus \( \text{Supp } b \) is a Baer subplane of \( \pi \). \( \square \)

**Corollary 1.** If \( \pi \) is an affine plane of prime order \( p \), the minimal-weight vectors of \( \text{Hull}_n(\pi)^\perp \) are precisely the scalar multiples of its lines and \( \text{Hull}_n(\pi)^\perp \) uniquely determines \( \pi \).

**Corollary 2.** If \( \pi \) is an affine plane of order \( p^2 \) where \( p \) is a prime, the minimal-weight vectors of \( \text{Hull}_n(\pi)^\perp \) are scalar multiples of either lines or Baer subplanes of \( \pi \).
Remark. For a projective plane $\Pi$ of prime order $p$, we have $\dim C_p(\Pi) = \frac{1}{2}(p^2 + p + 2)$ and $C_p(\Pi)^\perp \subseteq C_p(\Pi)$. In this case, therefore, $\text{Hull}_p(\Pi) = C_p(\Pi)^\perp$ and $\text{Hull}_p(\Pi)^\perp = C_p(\Pi)$. Going to the affine plane we have $\dim C_p(\pi) = \frac{1}{2}(p^2 + p)$, $\dim C_p(\pi)^\perp = \frac{1}{2}(p^2 - p)$ with, again, $\text{Hull}_p(\pi) = C_p(\pi)^\perp$ and $\text{Hull}_p(\pi)^\perp = C_p(\pi)$. The classification of projective or affine planes of prime order $p$ is tantamount to a classification of certain codes over $\mathbb{F}_p$ of block length $p^2 + p + 1$ and minimum weight $p + 1$ (the projective case) or block length $p^2$ and minimum weight $p$ (the affine case).

Examples. (1) For $p = 2$ the affine hull is a binary $(4,1)$ code; it is, of course, simply $F_2(1,1,1,1)$.

(2) For $p = 3$ the affine hull is a $(9,3)$ ternary code; it is, of course, unique and is generated by $(1,1,1,-1,-1,0,0,0)$, $(1,1,1,0,0,0,-1,-1,-1)$ and $(1,-1,0,1,-1,0,1,-1,0)$.

Corollary 3. If $\Pi$ is a projective plane of order $n$ and $p$ is a prime dividing $n$ with the minimum weight of $\text{Hull}_p(\Pi) = 2n$, then every affine part $\pi$ of $\Pi$ has the property that $C_p(\pi)$ has, as minimal-weight vectors, only the scalar multiples of the lines of $\pi$.

Proof. Let $b \in C_p(\pi)$ have weight $n$. Then the $\overline{b}$ of Lemma 1 can be chosen in $C_p(\Pi)$. If $r = 1$ then $\overline{b}$, and hence $b$, is a line. Assume $1 < r$; again by Lemma 1, $r < n$ and, normalizing so that $\overline{b}$'s non-zero coordinates are 1, we have $\sum_{Q \in \Pi} \overline{b}_Q = 1$ since $r = 1 \mod p$. Thus, $\sum_{Q} (\overline{b} - v^{t_0})_Q = 0$ and $\overline{b} - v^{t_0} \in C_p(\Pi)^\perp$. But $\text{wgt}(\overline{b} - v^{t_0}) = n + (n + 1 - r) = 2n + 1 - r < 2n$, a contradiction, since $\overline{b} - v^{t_0}$ is clearly in $C_p(\Pi)$, and hence in $\text{Hull}_p(\Pi)$, which has minimum weight $2n$. □

Corollary 4. The minimal-weight vectors of $C_p(AG_2(q))$ are scalar multiples of the lines of $AG_2(q)$, where $p = \text{Char} F_q$.

Proof. It follows from the work of Delsarte (see Appendix I) that $\text{Hull}_p(AG_2(q))$ has minimum weight $2q$. But then the same is true for $\text{Hull}_p(PG_2(q))$ since, if not, and $c$ is a vector in $\text{Hull}_p(PG_2(q))$ with $0 < \text{wgt } c < 2n$, one finds a line $L$ of $PG_2(q)$ with $L \cap \text{Supp } c = \phi$ (such a line exists for otherwise $|L \cap \text{Supp } c| \geq 2$ for all lines $l$ and $\text{wgt } c \geq 2(n + 1)$) and the affine plane determined by taking $L$ to be the line at infinity has a vector, $T(c)$, (where $T$ is the natural projection) in $C_p(AG_2(q) \cap C_p(AG_2(q))^\perp = \text{Hull}_p(AG_2(q))$ of weight less than $2q$. □

Remarks. (1) This corollary has been proved in a different setting by Delsarte, Goethals, and MacWilliams: see Appendix I.

(2) We do not have an example of an affine plane whose code has weight-$n$ vectors whose supports are not lines.
4. Linear equivalence and tameness

Suppose $D_1$ and $D_2$ are two designs with the same parameters. Then they have the same order, $n$ say, and for a prime $p$ dividing $n$, both $\text{Hull}_p(D_1)$ and $\text{Hull}_p(D_2)$ are codes in $F_p^n$.

**Definition 2.** $D_1$ and $D_2$ are said to be linearly equivalent (or, more properly, "linearly equivalent at $p"\) if $\text{Hull}_p(D_1)$ is code-isomorphic to $\text{Hull}_p(D_2)$. That is, if there is an element $S$ of $\text{Sym}(N)$ acting naturally on $F_p^n$ with $S(\text{Hull}_p(D_1)) = \text{Hull}_p(D_2)$.

As Example 2 of Section 1 shows, $D_1$ and $D_2$ may be linearly equivalent even when $C_p(D_1)$ and $C_p(D_2)$ have different dimensions. Thus linear equivalence is a coarser equivalence relation than design isomorphism. Clearly, by Proposition 1 (Section 2) projective planes are linearly equivalent if and only if they are isomorphic. For affine planes the situation is different. Since our interest is in projective planes studied via their affine parts, we will want to know when two projective planes are isomorphic under the assumption that one has an affine part linearly equivalent to an affine part of the other. We therefore introduce the notion of "tameness".

**Definition 3.** A projective plane $\Pi$ of order $n$ is said to be tame (or, more properly, "tame at $p"\) if $\text{Hull}_p(\Pi)$ has minimum weight $2n$ and the minimal-weight vectors are precisely the scalar multiples of the vectors of the form $v^L - v^M$, where $L$ and $M$ are lines of the plane.

**Theorem 1.** Suppose $\Pi$ and $\Sigma$ are projective planes of order $n$ and that $\Pi$ is tame at $p$, where $p$ is odd. If for some line $L$ of $\Pi$ and some line $M$ of $\Sigma$, $\Pi^L$ is linearly equivalent at $p$ to $\Sigma^M$, then $\Pi$ and $\Sigma$ are isomorphic.

**Proof.** Set $\pi = \Pi^L$ and $\sigma = \Sigma^M$. We may as well assume that $\text{Hull}_p(\pi) = \text{Hull}_p(\sigma)$ in $F_p^n$. Since $\Pi$ is tame at $p$ and the natural projection, $T: \text{Hull}_p(\Pi)^{\perp} \rightarrow \text{Hull}_p(\pi)^{\perp}$, maps $\{c \in \text{Hull}_p(\Pi) \mid c_Q = 0 \text{ for all } Q \text{ on } L\}$ isomorphically onto $\text{Hull}_p(\pi)$, the minimal-weight vectors of $\text{Hull}_p(\pi)$ are precisely the scalar multiples of vectors of the form $v^l - v^m$ where $l$ and $m$ are parallel lines of $\pi$. Since any affine plane $\rho$, $\text{Hull}_p(\rho)$ contains (indeed is generated by) all vectors of this form, and since clearly, when $p$ is odd, $l$ and $m$ can be recovered from $v^l - v^m$, the lines of $\pi$ and $\sigma$ are precisely the same. But $\Pi$ and $\Sigma$ are uniquely determined by $\pi$ and $\sigma$, so we are through. 

**Remarks.** (1) A projective plane need not be tame: as we shall see the translation plane of order 9 that is non-desarguesian is not tame at 3.
(2) It is conceivable that a projective plane could be tame at one prime but not at another. For instance, should there be a projective plane of order $2p$ where $p$ is an odd prime (necessarily congruent to 5 modulo 8) and should that plane have an oval, then it would not be tame at 2, but probably would be tame at $p$. One therefore notes with interest that a (almost surely non-existent) projective plane of order 10 cannot have an oval [16].

(3) Both the projective planes of order 4 and 8 are tame, but, as we mentioned, there are 112 affine planes of order 4 contained in the $(16,11)$ extended binary Hamming code. They are all, of course, isomorphic but the hull does not locate a unique plane and it is conceivable that for order 16, say, there are two non-isomorphic tame planes that are linearly equivalent. It is also possible that the Hamada–Sachar Conjecture (see Section 6) is false for $q$ a power of 2, even for translation planes.

5. Planes of prime power order

It should be clear from Section 2 and 3 that in order to discover and classify planes of order $n$ one should have at one's disposal linear codes of block length $n^2$ over $F_p$ (where $p$ divides $n$) with minimum weight $n$, and a sufficiently rich structure of minimal-weight vectors to accommodate $C_p(\pi)$'s for various affine planes $\pi$. Of course, given an affine plane $\pi$, $B = \text{Hull}_p(\pi)^\perp$ is such a code. Fortunately, for $n = p^e$ there are very standard choices for such codes $B$ and, more importantly, these codes have been intensively studied by algebraic coding theorists, in particular, by Delsarte [6] and Delsarte et al. [7].

Before discussing these linear codes and the affine planes connected with them we make three more general definitions.

Let $B$ be an arbitrary code of block length $n^2$ over $F_p$ (where $p$ divides $n$) with minimum weight $n$.

**Definition 4.** An affine plane $\pi$ of order $n$ is said to be contained in $B$ if $C_p(\pi)$ is code isomorphic to a subcode of $B$.

**Definition 5.** An affine plane $\pi$ of order $n$ is said to be linear over $B$ if $C_p(\pi)$ is code isomorphic to a subcode $C$ of $B$ where $B \subseteq C + C^\perp$.

**Definition 6.** $B$ is said to be linearly closed if, for every affine plane $\pi$ linear over $B$ we have $B = C + C^\perp$, $C$ being the subcode of $B$ isomorphic to $C_p(\pi)$.

*Example.* If one takes the Veblen–Wedderburn plane $\Psi$ of order 9 and a “real” line $R$ and sets $\psi = \Psi^R$, then the vectors in $\text{Hull}(\psi)^\perp$ of the form $v^S$, where $S$ is a line or Baer subplane of $\psi$, generate an $(81, 48)$ ternary code $B$ over which $\psi$ is linear. Moreover, $\omega = \Omega^L_{\text{dual}}$ is also linear over $B$ where $\Omega$ is the non-desarguesian
translational plane of order 9, \( \Omega_{\text{dual}} \) its dual, and \( L \) a line through the translation point. Both \( \text{Hull}(\psi)^{\perp} \) and \( \text{Hull}(\omega)^{\perp} \) are \((81, 50)\) ternary codes containing \( B \); both are linearly closed. They are not code-isomorphic since \( \text{Hull}(\psi)^{\perp} \) contains \( 2 \times 306 \) minimal-weight vectors while \( \text{Hull}(\omega)^{\perp} \) contains \( 2 \times 522 \). See Appendix II for further details.

Remark. \( B = \text{Hull}(\text{AG}_2(q))^{\perp} \) is linearly closed since \( B \subseteq \text{Hull}(\pi)^{\perp} \) implies \( \text{Hull}(\pi) \subseteq \text{Hull}(\text{AG}_2(q)) \) and the minimal-weight vectors of \( \text{Hull}(\text{AG}_2(q)) \) are precisely those vectors of the form \( u^l - v^m \) where \( l \) and \( m \) are parallel lines of \( \text{AG}_2(q) \); this forces the generators of \( \text{Hull}(\pi) \) to be these vectors and hence \( \text{Hull}(\pi) = \text{Hull}(\text{AG}_2(q)) \) or \( B = \text{Hull}(\pi)^{\perp} \). In fact, as in the proof of Theorem 1, we have, when \( q \) is odd, that \( \pi \) must be \( \text{AG}_3(q) \).

The code \( B \) of the above example is not a standard choice and we turn now to such codes. Set \( q = p^s \) and let \( F \) be an arbitrary subfield of \( F_q \). Let \( V \) be a 2-dimensional vector space over \( F_q \); consider \( V \) as a vector space over \( F \). It will, of course, have dimension \( 2[F_q : F] \), where \( [K : F] \) denotes the degree of \( K \) over \( F \). Set \( m = [F_q : F] \) and consider the collection, \( L_F \), of all \( m \)-dimensional subspaces over \( F \) and their translates under the addition in \( V \), i.e. all the \( m \)-dimensional cosets in the affine space of \( V \), \( \text{AG}_n(F) \).

Now \( F_p^V \) is a \( q^2 \)-dimensional vector space over \( F_p \) with a standard basis \( \{ \delta^u \mid u \in V \} \) where \( \delta^u \) is that function on \( V \) which is 1 at \( u \) and 0 otherwise. We use this basis to identify \( F_p^V \) with \( F_p^q \). Each \( X \in L_F \) is a subset of \( V \) of cardinality \( q \) and defines a vector \( u^X \) of \( F_p^q \) given by \( \Sigma_{u \in X} \delta^u \); i.e. \( v^X \) is the characteristic function of \( X \) in \( F_p^q \). Let \( B(F_q \mid F) \) be the subspace of \( F_p^q \) spanned by \( \{ u^X \mid X \in L_F \} \). \( B(F_q \mid F) \) is a code of length \( q^2 \) over \( F_p \) with minimum weight \( q \), its minimal-weight vectors are scalar multiples of the vectors \( v^X \), and its dimension is computable (see Appendix I).

If \( F \) and \( K \) are two subfields of \( F_q \), \( q = p^s \), with \( F \subseteq K \), then clearly, \( B(F_q \mid K) \subseteq B(F_q \mid F) \). At one extreme, \( F = F_q \), \( L_F \) consists of the lines of the affine plane \( \text{AG}_2(q) \) and \( B(F_q \mid F_q) = C_p(\text{AG}_2(q)) \). At the other extreme, \( F = F_p \), \( L_F \) consists of all \( s \)-dimensional subspace and their translates, where \( V \) is viewed as a 2-dimensional vector space over \( F_p \).

Examples. (1) For \( q = 9 \) the only interesting standard choice is \( B(F_3 \mid F_3) \); it is an \((81, 50)\) ternary code with \( 2 \times 1170 \) weight-9 vectors. \( B(F_3 \mid F_3) \) is code-isomorphic to \( \text{Hull}(\Omega^T)^{\perp} \) where \( \Omega \) is the non-desarguesian translation plane of order 9 and \( T \) its translation line. \( \text{Hull}(\Omega^T) \) has many vectors of weight 18 that are not of the form \( u^l - v^m \) where \( l \) and \( m \) are parallel lines. Thus \( \Omega \) is not tame.

(2) For \( q = 16 \) we have \( F_2 \subseteq F_4 \subseteq F_{16} \) with \( C(\text{AG}_2(16)) = B(F_{16} \mid F_{16}) \subseteq B(F_{16} \mid F_4) \), \( B(F_{16} \mid F_4) \) is a code of length \( 163 \) with \( 2 \times 163 \) weight-9 vectors, and the dimensions are 81, 129 and 163 respectively. There are precisely two translation planes linear over \( B(F_{16} \mid F_4) \): see [15]. Their codes each have dimension 97. We do not know whether or not they are linearly equivalent.
Proposition 4. Let q be a power of the prime p. Then an affine plane of order q is linear over \( B(F_q | F_p) \) if and only if it is a translation plane.

Proof. A translation plane of order \( q = p^s \) is given by a collection of \( q + 1 \) \( s \)-dimensional subspaces of \( V \) (the so-called spread) with the property that the intersection of any two of the chosen subspaces is \( \{0\} \). The plane is given by taking \( V \) to be the point set and the lines to be the subsets corresponding to the spread, and all their translates. We first show that \( B(F_q | F_p) \) contains every vector of the form \( v^X - v^Y \) where \( X \) and \( Y \) are both translates of the same \( s \)-dimensional subspace over \( F_p \). Since \( B(F_q | F_p) \) is generated by vectors of the form \( v^Z \), we must only show that \( (v^X - v^Y, v^Z) = 0 \) for all \( Z \), where, again, \( Z \) is a translate of an \( s \)-dimensional subspace. Because of the affine invariance we can assume \( Z \) is an \( s \)-dimensional subspace; set \( X = a + S \) and \( Y = b + S \) where \( S \) is an \( s \)-dimensional subspace. If \( S \cap Z = \{0\} \), then \( |X \cap Z| = |Y \cap Z| = 1 \) and the result is clear. Otherwise, \( S \cap Z \) being a subspace of positive dimension, \( X \cap Z \) and \( Y \cap Z \) are either empty or cosets of \( S \cap Z \) and hence of cardinality a positive power of \( p \), whence \( (v^X, v^Z) = 0 = (v^Y, v^Z) \); a fortiori \( (v^X - v^Y, v^Z) = 0 \).

Now suppose \( \pi \) is a translation plane of order \( q \). Then \( C(\pi) \subseteq B(F_q | F_p) \) and, by Proposition 2, \( \text{Hull}(\pi) \) is generated by vectors of the form \( v^X - v^Y \), since the weight-\( q \) vectors of \( B(F_q | F_p) \) are all of the form \( v^Z \) (see Appendix I). Hence \( \text{Hull}(\pi) \subseteq B(F_q | F_p) \) or \( B(F_q | F_p) \subseteq \text{Hull}(\pi) \) and \( \pi \) is linear over \( B(F_q | F_p) \).

Finally, suppose \( \pi \) is linear over \( B(F_q | F_p) \) but not a translation plane. Then \( B(F_q | F_p) \) is generated by vectors of the form \( v^l - v^m \) where \( l \) and \( m \) are parallel lines of \( \pi \), therec must be an \( s \)-dimensional subspace \( S \) with \( v^S = v^l \) where \( l \) is a line of \( \pi \) and an \( m \) in \( \pi \) parallel to \( l \) but not a translate of \( S \); i.e. \( v^m = v^Y \) with \( S \cap Y = \varnothing \) but \( Y \) not a translate of \( S \). If \( Y \) is a translate of the subspace \( T \) then clearly \( S \cap T = \{0\} \) and \( S \cap T \neq S \). Let \( U \) and \( U' \) be subspaces with \( S = (S \cap T) \oplus U \) and \( V = (S + T) \oplus U' \). Then \( U + U' = Z \), an \( s \)-dimensional subspace of \( V \), with \( Z \cap S \neq \{0\} \) and \( Z \cap T = \{0\} \). Hence \( (v^l - v^m, v^Z) = 1 \), a contradiction. \( \square \)

Remark. Since it follows from the work of Delsarte et al (see Appendix I) that the minimum weight of the code generated by vectors of the form \( v^X - v^Y \) is \( 2q \), the above proof shows that the hull of any translation plane of order \( q \) has minimum weight \( 2q \). Thus, when a translation plane fails to be tame (as, for example, the non-desarguesian translation plane of order 9 does) it is for the subtle reason that its hull has more vectors of weight \( 2q \) than it ought.

Although this result is very simple it yields, immediately, an upper bound on \( \text{dim}(C_p(\pi)) \) when \( \pi \) is an affine translation plane of order \( q \).
Theorem 2. If $\pi$ is an affine translation plane of order $q = p^s$ then 
$$\dim C_\pi(\pi) \leq q^2 + q - \dim(B(F_q \mid F_p)).$$
Moreover, any two affine translation planes meeting this upper bound are linearly equivalent.

Proof. If $\pi$ is a translation plane of order $q$ then, by Proposition 4, $\text{Hull}_p(\pi) \subseteq B(F_q \mid F_p) \subseteq \text{Hull}_p(\pi)^\perp$. Each inequality gives an upper bound. We use the second: $B(F_q \mid F_p) \subseteq \text{Hull}_p(\pi)^\perp$ implies $\text{Hull}_p(\pi) \subseteq B(F_q \mid F_p)$ and hence $\dim C_\pi(\pi) - q \leq \dim B(F_q \mid F_p)^\perp = q^2 - \dim B(F_q \mid F_p)$ yielding the inequality. If $\dim C_\pi(\pi) = q^2 + q - \dim B(F_q \mid F_p)$ then $\text{Hull}_p(\pi) = B(F_q \mid F_p)^\perp$; hence any two translation planes with $\dim C_\pi(\pi) = q^2 + q - \dim B(F_q \mid F_p)$ have isomorphic hulls or, in other words, they are linearly equivalent. □

Examples. (1) In general, the formula for $\dim B(F_q \mid F_p)$ is complicated (see [13] or [6] and Appendix I) but for $q = p^2$ we can deduce that 
$$p^4 + p^2 - \dim B(F_q \mid F_p) = \frac{1}{2}p^4 + p^2 - \frac{1}{3}p^3 - \frac{1}{3}p.$$ 
The only previously known bound (an easy one coming from orthogonality conditions) was 
$$\dim C_\pi(\pi) \leq \frac{1}{2}(p^4 + p^2)$$ 
for a plane of order $q = p^2$; this bound is, however, for any affine plane of order $p^2$. Observe that for $p = 3$ the bound is 40. All six known non-desarguesian affine planes of order 9 have dimension 40, while $\dim C_3(\text{AG}_2(9)) = 36$. For $p = 2$ the bound is 9 and $C_2(\text{AG}_2(4))$ has dimension 9. For $p = 5$ the bound is 295. Oakden and Czerwinski [18] have found that there are 22 isomorphism classes of translation planes of order 25. Our computations of their dimensions show that the highest value is 264, and that only 14 distinct values occur.

(2) There is another case where one can write down a rather pretty formula for the dimension: $\dim B(F_q \mid F_p) = 2^{2^s - 1} + \frac{1}{2}(\frac{2^s}{s})$. In this case we have that an affine translation plane $\Pi$ of order $2^s$ satisfies 
$$\dim C_\Pi(\pi) = 2^s(2^{s-1} + 1) - \frac{1}{2}\left(\frac{2^s}{s}\right).$$

(3) Just as the non-desarguesian translation plane of order 9 is not tame, any non-desarguesian translation plane meeting the upper bound will not be tame because of the existence of too many vectors of weight $2q$ in the hull.

We turn next to the so-called “2-dimensional case”, i.e. to translation planes of order $q^2$ contained in $B(F_q \mid F_p)$. For $q$ odd, those with a flag-transitive automorphism group seem to have been classified [4]. We simply show here that any two such planes can be obtained from one another by a “derivation”. (The reader need not know what a derivation is.)

Observe that we have changed notation: $F_q$ is our big field and $F = F_q; V$ is a
2-dimensional vector space over $F_{q^2}$.

$$|L_F| = \frac{q^2(q^4-1)(q^4-q)}{(q^2-1)(q^2-q)} = q^2(q^2 + 1)(q^2 + q + 1).$$

Thus $B(F_{q^2} | F_q)$ has precisely $(q-1)q^2(q^2 + 1)(q^2 + q + 1)$ weight-$q^2$ vectors. They yield a design with point set $V$ of cardinality $q^4$, with block size $q^2$ and $\lambda = q^2 + q + 1$. It follows, just as in the proof of Lemma 2, Corollary 2, that any affine plane of order $q^2$ linear over $B(F_{q^2} | F_q)$ has $q^4 + q^2$ of these vectors as lines and $q^4(q^2 + 1)(q + 1)$ of them as Baer subplanes. Thus each can be obtained from the other by a "derivation".

**Proposition 5.** Suppose $\pi$ and $\sigma$ are two affine planes of order $q^2$ that are linear over $B(F_{q^2} | F_q)$. Then $\sigma$ can be obtained from $\pi$ by using certain of $\pi$'s lines and certain of its Baer subplanes as the lines of $\sigma$.

**Remark.** We restricted ourselves to affines planes of order $q^2$ simply to make contact with the existing literature on translation planes. Any two planes of order $q$ that are contained in $B(F_q | F_q)$ are part of the design given by the minimal-weight vectors of this code and one can be obtained from the other by a "generalized derivation". There is nothing very deep or mysterious going on here and much of the existing literature on derivations, nets, etc. can be viewed, from the present perspective, as ingenious methods for choosing which lines to ignore and which Baer subplanes, or whatever, to include when going from one affine plane to another.

**Proposition 6.** Every affine plane of order $q^2$ that is linear over $B(F_{q^2} | F_q)$ has at least $q^4(q^2 + 1)(q + 1)$ desarguesian Baer subplanes.

**Proof.** Only the fact that the planes are desarguesian remains to be proved. If $v$ is a weight-$q^2$ vector of $B(F_{q^2} | F_q)$ and $H$ is the hull of the given affine plane of order $q^2$, then $v \in H^1$. Since all such $v$'s have supports that are $F_q$-subspaces or their translates, $|\text{Supp } v \cap l| = 0, 1, q$ or $q^2$ for every line $l$ of the plane. If this intersection is of cardinality $q^2$, $v = av^l$; otherwise it is a Baer subplane and moreover its structure is of a 2-dimensional vector space over $F_q$ with the 1-dimensional $F_q$-subspaces or their translates as lines; it is, therefore, isomorphic to $PG_2(q)$. □

**Caution.** The projective completion may very well have more Baer subplanes, for example, $PG_2(q^2)$ will. The point is that some Baer subplanes of a given projective plane $\Pi$ may meet a given line $L$ in only one point and thus, in $\text{Hull}_p(\Pi^1)$, they will appear as constant vectors of weight $q^2 + q$, rather than as minimal-weight vectors that are Baer subplanes.
Corollary 5. Any translation plane of order $p^2$ has at least $p^3(p^2 + 1)(p + 1)$ Baer subplanes.

Remark. For $p = 3$ this is the exact number for the non-desarguesian translation plane of order 9. In fact, all three known non-desarguesian projective planes of order 9 have 1080 Baer subplanes; $PG_2(9)$ has 7560 Baer subplanes. All these Baer subplanes are, of course, isomorphic to $PG_2(3)$.

6. The Hamada–Sachar conjecture

N. Hamada [14] made a sweeping conjecture that proved to be false [23] and, independently, Sachar [22] made a narrower conjecture that is still undecided. The present method throw a great deal of light on this narrower conjecture which we call the "Hamada–Sachar Conjecture". Here it is.

Conjecture. Every projective plane of order $p^r$, $p$ a prime, has $p$-rank $\geq (p^{r+1})^s + 1$ with equality if and only if it is desarguesian.

Remarks. (1) The $p$-rank of $II$ is the dimension of $C_p(II)$.
(2) It has been known for many years that $\dim C(PG_2(p^r)) = (p^{r+1})^s + 1$.
(3) It is clear from the dimensional relationships we have obtained that our methods should have something to say about the Hamada–Sachar conjecture.

We begin with a technical result.

Lemma 3. Suppose $B$ is a code of block length $n^2$ over $F_p$, where $p$ divides $n$, and suppose $\pi$ is an affine plane of order $n$ that is linear over $B$. Then, for any affine plane $\sigma$ contained in $B$, if a subcode $C$ of $B$ can be chosen with $C$ code-isomorphic to $C_p(\sigma)$ and $Hull_p(\pi) \subseteq C$, we have that

$$\dim C_p(\sigma) \geq \dim C_p(\pi)$$

with equality if and only if $\sigma$ is linearly equivalent to $\pi$.

Proof. Without loss of generality we may assume that

$$H = Hull_p(\pi) \subseteq C_p(\sigma) \subseteq B \subseteq H^\perp.$$ 

Since the first containment implies $C_p(\sigma) \subseteq H^\perp$ we have that

$$Hull_p(\sigma) \subseteq H^\perp$$

or, in other words, $H \subseteq Hull_p(\sigma)$. Since $\dim H = \dim C_p(\pi) - n$ and $\dim Hull_p(\sigma) = \dim C_p(\sigma) - n$ we have the required inequality with equality if and only if $H = Hull_p(\sigma)$; i.e. if and only if $\pi$ and $\sigma$ are linearly equivalent. □
There are, in view of the above lemma, two obstacles in the way to a proof of the Conjecture.

(1) If we take \( \pi = \text{AG}_2(q) \), how restrictive is the assumption that \( C(\sigma) \) contain a code isomorphic to \( \text{Hull}(\text{AG}_2(q)) \)?

(2) When does linear equivalence imply isomorphism?

The second question has already been addressed: tameness insures that linear equivalence implies isomorphism. We now invoke the work of Delsarte et al. (see Appendix I) which shows that we have the following result. (This work shows that \( \text{AG}_2(q) \) has the required property, but, as in the proof of Theorem 1 (Section 4), so does \( \text{PG}_2(q) \).

**Theorem 3.** Desarguesian projective planes are tame.

**Remarks.** (1) We do not have an example of a tame plane that is not desarguesian.

(2) The assumption that the hull of \( \text{AG}_2(q) \) be contained in the code of an affine plane does not seem to be a great restriction; it may be that every translation plane satisfies this condition. In any case here is what we can presently prove.

**Theorem 4.** Let \( \sigma \) be an affine plane of odd order \( q = p^s \) contained in \( B(F_q | F_p) \). Then, if \( C(\sigma) \) contains a subcode isomorphic to \( \text{Hull}(\text{AG}_2(q)) \), we have that

\[
\dim C(\sigma) \geq \binom{p+1}{2}^s
\]

with equality if and only if \( \sigma \) is isomorphic to \( \text{AG}_2(q) \).

**Proof.** The theorem follows from Lemma 3 and Theorems 1 (Section 4) and 3. □

**Remarks.** (1) We have, of course, worked with the affine version of the Conjecture but since an affine plane uniquely determines its projective completion we have proven the Conjecture for a class of projective planes.

(2) It might be possible to get the inequality for any plane via the methods of algebraic coding theory, but it is difficult to see how one could get the isomorphism without working inside the tower

\( H = \text{Hull}(\text{AG}_2(q)) \subset B(F_q | F_p) \subset H^+ \).

(3) The automorphism group of \( B(F_{p^s} | F_p) \) contains \( \text{AGL}_{2s}(p) \) and hence there are an enormous, easily computable, number of copies of \( \text{Hull}(\text{AG}_2(p^s)) \) in \( B(F_{p^s} | F_p) \), so it seems reasonable to suppose that any \( C(\sigma) \) contained in \( B(F_{p^s} | F_p) \) will contain one of them.
7. Concluding remarks

As remarked in Section 3, the theory presented here will not identify, under linear equivalence, any two distinct affine planes of order \( p \), when \( p \) is a prime; instead, it simply changes the problem of classifying planes of prime order into a coding theory problem. It has long been conjectured that there is a unique projective plane of each prime order (namely \( \text{PG}_2(p) \)) but there does not seem to be any good reason for believing that conjecture in this coding-theoretic setting.

But, already for order \( n = p^2 \), it is known that the number of distinct affine planes of that order goes to infinity with \( p \) [11]. One often hears despair from finite geometers concerning the possibility of classifying even the translation planes because they are so ubiquitous.

We presume that linear equivalence will identify non-isomorphic affine planes and hence be an aid in classification. It may be that the proper place for the configurational and group-theoretic methods of classification is within an equivalence class. We have not explored that avenue here.

More generally one can, from the present perspective, begin with a pair of codes \( B \) and \( \bar{B} \), each of block length \( n^2 \) and minimum weight \( n \), with

\[
B \subseteq \bar{B} \subseteq F_{p^2}^{n^2},
\]

where, of course, \( p \) is a prime dividing \( n \). Then one can study the class of affine planes of order \( n \) that are contained in \( B \) and are linear over \( \bar{B} \). By varying \( B \) and \( \bar{B} \) one can capture fewer planes or more planes. For example, with \( B = B(F_q | F_p) \) and any \( \bar{B} \), the only affine plane contained in \( B \) is \( \text{AG}_2(q) \). Taking \( B = \bar{B} = B(F_q | F_p) \), captures all translation planes and is, hence, a poor choice. But one could choose a subfield \( F \) of \( F_q \) and \( \bar{B} \supseteq B(F_q | F) \) to capture fewer. Even \( B = B(F_q | F_3) \), with any strictly larger \( \bar{B} \), would cut out the non-desarguesian affine translation plane of order 9.

Clearly there is here great scope for computer studies. Order 9 is surely easy; we have done a great deal with that already using Cayley and clearly orders 16 and 25 are within range [4]. We have computed the dimensions of the 22 codes generated by the affine translation planes of order 25. The dimensions and their frequencies are: 225 (the desarguesian plane), 238 (1), 250 (1), 252 (1), 254 (1), 255 (1), 256 (1), 257 (3), 258 (4), 259 (2), 260 (2), 261 (2), 263 and 264. The flag-transitive planes (see [12]) have dimensions 257 and 261. The Lorimer plane of order 16 has dimension 105; it was incorrectly reported to have dimension 121 in [21]. The three non-desarguesian translation planes of order 16 given in [15] each have dimension 97.

On the theoretical side, one ought to have for translation planes \( \pi \) of order \( q = p^r \) that

\[
\binom{p+1}{2}^r \leq \dim C(\pi) \leq q^2 + q - \dim B(F_q | F_p),
\]
with the lower bound implying \( \pi = \AG_2(q) \). As remarked in Section 6, what remains to be shown is that incidence matrices for \( \AG_2(q) \) and \( \pi \) can be chosen so that \( \Hull(\AG_2(q)) \subset C(\pi) \). One will also need to investigate the structure of the set of linearly equivalent planes meeting the upper bound. To do this one has a very large affine group as an aid, since \( B(F_q \mid F_p) \) is invariant under \( \AGL_p(p) \). For those translation planes whose dimensions lie strictly between the bounds, one could choose \( B = \Hull(\pi_0) \) for a fixed affine plane \( \pi_0 \) and \( B = B(F_q \mid F) \) for a fixed intermediate field and then investigate the structure of those affine planes contained in \( B \) and linear over \( B \); this would give a relative theory.

Another theoretical line of investigation is to determine the structure of a given projective plane \( \Pi \) when, for two distinct lines, \( L \) and \( M \), both \( \Pi^L \) and \( \Pi^M \) are linear over a given \( B \); for example it is true that \( \Pi \) is desarguesian if both \( \Pi^L \) and \( \Pi^M \) are linear over \( B(F_q \mid F_p) \).

References

Appendix 1. The standard geometric codes

We shall show in this appendix how we have used the results of Delsarte [6] and Delsarte, Goethals and MacWilliams [7] in our work. Essentially this requires the identification of their notation with the codes we consider.

For the purposes of this appendix we will introduce some additional notation of our own. We will have $AG_2(q)$, where $q = p^r$, inside $V_2(p)$, and if $t$ divides $s$, we will set $s = th$ and $m = 2s = 2ht$. For any $r$ with $1 \leq r \leq 2h$, we set

$L(r, t) = \{ X \mid X \text{ is an } r \text{- flat over } F_{p^t} \text{ inside } V_2(p) \}$;

$D(r, t) = \text{the design of point set the vectors of } V_2(p), \text{ block set } L(r, t)$;

$C(D(r, t)) = \text{the code of } D(r, t) \text{ over } F_p$;

$\delta(C) = \text{the minimum weight of the code } C$.

Thus, if $F = F_{p^r}$, $L(h, t) = L_F$ of Section 5, and $C(D(h, t)) = B(F_q \mid F_p)$. We look first at [6]. For the variables $m_i$ in [6; (1), (2)] we will take throughout

$m_i = t$ for all $i$ where $1 \leq i \leq 2h$.

Then $F_i - F_{p^t}$ for $1 \leq i \leq 2h$, $F - F_{p^t}$. The codes $C_u(m_1, \ldots, m_r)$ over $F_p$ are defined in [6; 2.1] and we will write

$C_u(m_1, \ldots, m_r) = C_u([t]^r)$ for $1 \leq r \leq 2h$.

For the variable $u$, we will take cases:

(1) $u = 2h - 1$. Here

$C_{p^t-1}([t]^h) = C(D(r, t))$, and, by [6; (16)],

$\delta(C_{p^t-1}([t]^h)) = p^u$.

(2) $u = 1$. Here $C_{h-1}([t]^h)$ is the subcode of $C(D(r-1, t))$ spanned by vectors $vX - vY$ where $X$ and $Y$ are parallel $(r - 1)$-flats in $L(r-1, t)$. By [6; (16)]

$\delta(C_{h-1}([t]^h)) = 2p^{u-1}h$

We take $r = h + 1$, then $C_{t}([t]^{h+1}) \subseteq C_{p^t-1}([t]^h)$, and

$\delta(C_{t}([t]^{h+1})) = 2p^h = 2q$. 

In particular, if $t = s$, $h = 1$, then $C_t([s]^2) = \text{Hull}(\text{AG}_1(q))$, by Proposition 2, Section 2. By [6; 2.5], for $1 \leq r \leq 2h$,

$$C_{p^r-1}([t]^{2h-r+1}) \subseteq C_t([t]^{r})^\perp,$$

and these codes have the same minimum weight, by [6; (20)], i.e.

$$\delta(C_t([t]^{r})^\perp) = p^{2h-(r-1)}.$$

If we take $r = h + 1$, $2h - r + 1 = h$, then

$$C_{p^r-1}([t]^{h}) \subseteq C_t([t]^{h+1})^\perp,$$

i.e. $C_{p^r-1}([t]^{h})^\perp \supseteq C_t([t]^{h+1})$.

Thus $C_t([t]^{h+1}) \subseteq C_{p^r-1}([t]^{h}) \cap C_{p^r-1}([t]^{h})^\perp = \text{Hull}(D(h, t))$, with equality certainly when $h = 1$, $t = s$.

A lower bound is obtained in [6] for $\delta(C_{p^r-1}([t]^{h+1}))$ [6; (18)], which is improved in [7] when $r = h$. Here

$$\delta(C_{p^r-1}([t]^{h+1})) \geq \frac{p^{2h} - (r^2 - 1)}{p^{r-1}} + 1$$

and combining this with (6) above gives, for $1 \leq r \leq 2h$,

$$2q^2p^{-r} \geq \delta(C_{p^r-1}([t]^{r})^\perp) \geq \frac{p^{2h} - (r-1)}{p^{r-1}} + 1. \tag{7}$$

When $r = h$, this becomes

$$2q \geq \delta(B(F_q | F_{p^h})) \geq q + p^{(h-1)} + \cdots + p + 2. \tag{8}$$

In particular, for $t = s$, $h = 1$:

$$2q \geq \delta(B(F_q | F_{p}^s)) \geq q + 2,$$

which is best possible for $p = 2$; for $t = 1$, $s = h$,

$$2q \geq \delta(B(F_q | F_{p}^s)) \geq q + (p - 1) + \cdots + p + 2$$

which is improved to equality with $2q$ in [7]; see below.

All the codes $C_t([t]^{r})$ are invariant under the small affine group $\text{AGL}_1(q^2)$ in $\text{AGL}_{2s}(p)$ acting on $V_{2s}(p)$, by [6; 2.2.2]. Copies of $\text{AGL}_{2s}(p^t)$ will preserve the codes defined over $F_{p^t}$, for each divisor $t$ of $s$, where $s = th$, as before.

Finally, a formula for the calculation of the dimension of $C_t([t]^{r})$ is given in [6; (15)] in terms of the radix-$p$ representation of integers $z$ satisfying $1 \leq z \leq q^2 - 1$. This formula for calculating the dimension is not easy to use, but for some cases it is simple: thus for $p = 2$ we have

$$\dim(C_t([t]^{r})) = \sum_{i=0}^{m-1} \binom{m}{i}$$

for $t = 1$, $1 \leq r \leq 2h = 2s = m$, which for $r = s$, becomes

$$\dim(C_t([t]^{r})) = \dim(B(F_{p^s} | F_{p})) = 2^{2s-1} + \left(\frac{2s}{2}\right). \tag{9}$$

The notation in the second paper [7] is different. We will continue to write $q = p^s$, $s = th$, $m = 2s$, and interpret the notation of [7] in terms of our notation, and that of [6]. The codes in question are denoted by $B_{s}(m', q')$, and are defined in [7; 4.2.2].

For $v = h(p^t - 1)$, $m' = 2h$, $q' = p^t$, we have

$$B_{s}(2h, p^t) = C_{p^r-1}([t]^{h}) = B(F_q | F_{p}^t). \tag{10}$$

and for $v = (h - 1)(p^t - 1) + (p^t - 2)$, $m = 2h$, $q' = p^t$, we have

$$B_{s}(2h, p^t) = C_t([t]^{h+1}). \tag{11}$$
The minimum weight of these codes is as determined in [6], but here we have the additional Theorem 2.6.3 that identifies the minimal weight vectors, i.e.

(i) the vectors of weight $q$ of $C_{p^e-1}([t]^h)$ are multiples of the vectors $u^L$ for $L \in L(h, t)$;
(ii) the vectors of weight $2q$ of $C_{p^e}([t]^h)$ are multiples of the vectors $u^L - v^M$ where $L, M \in L(h, t)$, and $L$ and $M$ are parallel.

Further, in [7; 4.3] an improvement of the lower bound (7) is deduced to give the following:

$$2q \geq \delta(C_{p^e-1}([t]^h)) \geq q(1 + p^{1-\epsilon}).$$

In particular, with $t = 1$, $h = s$

$$2q \geq \delta(B(F_q | F_p)^+) \geq 2q, \quad \text{i.e.}$$
$$\delta(B(F_q | F_p)^+) = 2q,$$

and for $t = s$, $h = 1$

$$2q \geq \delta(C(AG_2(q))^+) \geq q + p.$$

Appendix II. The planes of order 9

There are four projective planes of order 9: the desarguesian plane $PG_2(q)$; a translation plane $\Omega$; its dual, $\Omega^T$; and a self-dual plane $\Psi$, discovered by Veblen and Wedderburn [24] (the first of an infinite class of odd-order planes known as the Hughes planes). These projective planes produce seven non-isomorphic affine planes: $AG_2(9)$; $\Omega^T$ and $\Omega^T$ where $T$ is the translation line of $\Omega$ and $L$ any other line; $\Omega^T_{\text{dual}}$ and $\Omega^T_{\text{dual}}$ where $L$ is any one of the ten lines through the translation point.
of $\Omega_{\text{dual}}$ and $M$ is any one of the remaining 81 lines of $\Omega_{\text{dual}}$: $\Psi^R$ and $\Psi^C$ where $R$ is any real line of $\Psi$ and $C$ any complex line. We are using here the notation of [20] but the reader may wish to consult [9] for details concerning $\Psi$ and $\Omega_{\text{dual}}$. The facts that follow, except for the containment assertions, can be deduced from [9] and the theory we have presented; we have independently verified the computer results interactively using CAYLEY implemented on the University of Birmingham’s VAX. The answers came virtually instantaneously. All codes in this Appendix are ternary.

Sachar had computed $\dim C(\Omega)$ and $\dim C(\Psi)$ and found them to be 41, facts we also verified. Thus, $\dim \text{Hull}(\Omega^F) = \dim \text{Hull}(\Omega^F_\text{dual}) = \dim \text{Hull}(\Omega^M_\text{dual}) = \dim \text{Hull}(\Psi^R) = \dim \text{Hull}(\Psi^C) = 31$ while $\dim \text{Hull}(\text{AG}_2(9)) = 27$. It follows from the results of Section 5 that

$$\text{Hull}(\phi^F)^\perp = B(F_3 | F_3)$$

and, in what follows, we denote this $(81, 50)$ ternary code by $B$. It has $2 \times 1170$ weight-9 vectors and both $C(\text{AG}_2(9))$ and $C(\Omega^F)$ are linear over $B$. Given the 90 lines of either of those planes, the 1080 remaining supports of weight-9 vectors become their Baer subplanes. Either plane can, of course, be “derived” from the other. Combinatorially speaking, each is a subdesign with $\lambda = 1$ of the design of supports of weight-9 vectors (whose $\lambda$ is 13). The Hasse diagram, if one chooses the subcodes of $B$ properly, is given on the previous page.

We have written the dimension on the right. $B$ is the classifying code for the two translation planes of order 9.

Consider next $\text{Hull}(\Psi^R)^\perp$. It has $2 \times 306$ weight-9 vectors; the supports are the 90 lines of $\Psi^R$ and its 216 Baer subplanes; that $\Psi^R$ has 216 Baer subplanes follows from an easy counting argument and the facts contained in [9]. The weight-9 vectors generate an $(81, 48)$ ternary code which we call $B'$. It is the classifying code for both $\Psi^R$ and $\Omega^C_{\text{dual}}$: by properly choosing the subcodes of $B'$ one gets the following Hasse diagram:

\[
\begin{array}{c}
F_3^{81} \downarrow & & \downarrow & & \downarrow \\
\text{Hull}(\Psi^R)^\perp & & \text{Hull}(\Omega^C_{\text{dual}})^\perp & & 50 \\
 & & \downarrow & & \\
 & & \text{B}' & & 48 \\
 & & \downarrow & & \\
C(\Psi^R) + C(\Omega^C_{\text{dual}})^\perp & & 45 \\
 & & \downarrow & & \\
C(\Psi^R) & & C(\Omega^C_{\text{dual}}) & & 40 \\
 & & \downarrow & & \\
C(\Psi^R) \cap C(\Omega^C_{\text{dual}}) & & 35 \\
 & & \downarrow & & \\
B'^\perp & & 33 \\
 & & \downarrow & & \\
\text{Hull}(\Psi^R) & & \text{Hull}(\Omega^C_{\text{dual}}) & & 31 \\
\end{array}
\]

Again, the dimensions are on the right. $\Psi^R$ and $\Omega^C_{\text{dual}}$ are not linearly equivalent since $\Omega^C_{\text{dual}}$ has 432 Baer subplanes and thus $\text{Hull}(\Omega^C_{\text{dual}})^\perp$ has $2 \times 522$ weight-9 vectors while $\text{Hull}(\Psi^R)^\perp$ has but $2 \times 306$. Once again $\Psi^R$ and $\Omega^C_{\text{dual}}$ can be “derived” from another. Combinatorially speaking, the 306 supports of weight 9-vectors of $B'$ (which do NOT form a design) contain designs isomorphic to $\Psi^R$ and $\Omega^C_{\text{dual}}$. 

\[\text{Affine and projective planes}\]
We have now accounted for four of the seven known affine planes of order 9. The remaining three have Hasse diagrams of the form given below.

For \( \Psi^C \), \( \text{Hull}(\Psi^C) \) has 2 \times 234 minimal-weight vectors that generate it (corresponding to 90 lines and 144 Baer subplanes); \( \text{Hull}(\Omega^2_{\text{ext}}) \) has 2 \times 2210 minimal-weight vectors (corresponding to 90 lines and 120 Baer subplanes); \( \text{Hull}(\Omega^2) \) has 2 \times 234 minimal-weight vectors corresponding to 90 lines and 144 Baer subplanes.

It is conceivable that \( \text{Hull}(\Psi^C) \) is isomorphic to \( \text{Hull}(\Omega^2) \) since each of their duals has 2 \times 234 minimal-weight vectors. We searched (but not exhaustively) for \( \Omega^2 \) amongst the minimal-weight vectors of \( \Psi^C \) but did not find this affine plane.

Probably an exhaustive search should be undertaken, for if these hulls are isomorphic, it would explain the "surprising" (see [20], p. 166 bottom) fact that each of the three non-desarguesian projective planes of order 9 has 1080 Baer subplanes.

\[ F^{81} \]
\[ \text{Hull}^\perp \]
\[ C \]
\[ \text{Hull} \]

Glossary

**Affine part**: An affine part of a projective plane \( \Pi \) is any affine plane \( \pi = \Pi^L \) obtained from \( \Pi \) by deleting \( L \) for the line at infinity.

**AG_2(q)**: The desarguesian affine plane over \( F_q \).

**AGL_n(q)**: The affine general linear group of dimension \( n \) over \( F_q \).

**Baer subplane**: A subplane of order \( n \) of a projective or affine plane of order \( n' \).

**B(F, q)**: The code over \( F_q \) of the design \( L^q \), where \( F \) is a subfield of \( F_q \) and \( q = p^s \). (Section 5).

**Blocking set**: A set \( X \) of points of a projective plane \( \Pi \) which is such that every line of \( \Pi \) meets \( X \), but no line is contained in \( X \).

**Char** \( F \): The characteristic of the field \( F \).

**Code**: An \( (N, k) \) code \( C \) over a field \( F \) is a subspace of dimension \( k \) of \( F^N \), where \( N \) is the code length.

**Code isomorphism**: Two codes, \( C_1 \) and \( C_2 \), of the same length \( N \) over \( F \), are code isomorphic if there is a permutation of the \( N \) coordinate places that maps \( C_1 \) to \( C_2 \).

**C_{\pi}(D)**: The code over \( F_\pi \) formed by the row space of a \( [B] \) by \( |P| \) incidence matrix of the design \( D \). (Section 1).

**Containment of affine planes**: An affine plane \( \pi \) of order \( n \) is contained in an \( (n^2, k) \) code \( B \) over \( F_\pi \) of minimum weight \( n \), if there exists \( C \subseteq B \) with \( C = C_\pi(\pi) \). (Section 5, Definition 4).

**Constant vector**: A vector that is a scalar multiple of a vector all of whose entries are 0 or 1.

**Design**: A 2-\((v, k, \lambda)\) design \( D \) with point set \( P \), block set \( B \), and \( |P| = v \). (Section 1).

**\( F^N \)**: The vector space of all \( N \)-tuples, where \( N = |P| \), over the field \( F \).

**\( F_q \)**: The Galois field of \( q \) elements.

\( \text{Hull}_\pi(D) \): The hull of the design \( D \) over the field \( F_\pi \), i.e. \( C_{\pi}(D) \cap C_{\pi}(D)^\perp \).

\( \text{Hull}(D) = \text{Hull}_\pi(D) \) when the order of \( D \) is a power of \( p \). (Section 1, Definition 1).
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J: That vector of a code all of whose entries are 1.

[K:F]: The degree of the extension field \( K \) over the field \( F \).

\( \text{Ker } T \): The kernel of the linear transformation \( T \).

\( L_F \): The set of all \( m \)-flats over the field \( F \) in the affine space \( AG_m(F) \). (Section 5).

Linear equivalence of designs: \( D_1 \) is linearly equivalent to \( D_2 \) if \( \text{Hull}_p(D_1) = \text{Hull}_p(D_2) \). (Section 4, Definition 2).

Linear over: An affine plane \( \pi \) of order \( n \) is linear over a code \( B \) if there exists \( C = C_p(\pi) \) such that \( C \subseteq B \subseteq C + C^\perp \). (Section 4, Definition 5).

Minimum weight of a code: The smallest weight a non-zero vector of the code can have.

Order of a design \( D \): If the number of blocks through a point of \( D \) is \( r \), then the order of \( D \) is \( r - \lambda \).

\( \text{PG}_d(q) \): The desarguesian projective plane over \( F_q \).

\( \text{PGL}_d(q) \): The projective general linear group of dimension \( n \) over the field \( F_q \).

\( p \)-Rank of a design: The dimension of the code \( C_p(D) \).

Projective completion: A projective completion of an affine plane \( \pi \) is any projective plane \( \Pi \) for which \( \pi \) is an affine part.

Subcode: A subcode of a code \( C \) is any subspace of \( C \).

Support, \( \text{Supp } v \): If \( v \) is a vector of \( F^N \) then \( \text{Supp } v \) is the set of coordinate planes where \( v \) has a non-zero coordinate.

Tame projective plane: A projective plane \( \Pi \) of order \( n \) is tame at \( p \) if \( \text{Hull}_p(\Pi) \) has minimum weight \( 2n \), and the minimal-weight vectors are the scalar multiples of the vectors \( v^L - v^M \), where \( L \) and \( M \) are lines of \( \Pi \). (Section 4, Definition 3).

Transversal: A transversal of a parallel class \( \vartheta \) of an affine plane \( \pi \), is a set of points of \( \pi \) that meets each member of \( \vartheta \) once.

\( V_d(F) \), \( V_d(q) \): The vector space of dimension \( n \) over the field \( F \), or \( F_q \).

\( v^X \): If \( X \) is a subset of the set of \( N \) coordinate places of \( F^N \), then \( v^X \) is the vector formed by setting the coordinate at \( Q \) to be 1 if \( Q \in X \), 0 if \( Q \notin X \). No verbal distinction is made between \( X \) and \( v^X \); thus, a “line” will be either a subset of points or the vector that is the characteristic function of that set, depending on the context.

Weight, \( \text{wgt } v \): If \( v \) is a vector then \( \text{wgt } v \) is the cardinality of \( \text{Supp } v \).

\( |X| \): the cardinality of the set \( X \).

\( (x, y) \): The standard inner product of the vectors \( x \) and \( y \) of \( F^N \).