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Note

Hamiltonian paths containing a given arc, in almost regular bipartite tournaments

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Abstract

A tournament is an orientation of a complete graph, and in general a multipartite or c-partite tournament is an orientation of a complete c-partite graph. If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) \leqslant 1$, then D is called almost regular, and if $i_g(D) = 0$, then D is regular.

More than 10 years ago, Amar and Manoussakis and independently Wang proved that every arc of a regular bipartite tournament is contained in a directed Hamiltonian cycle. In this paper, we prove that every arc of an almost regular bipartite tournament T is contained in a directed Hamiltonian path if and only if the cardinalities of the partite sets differ by at most one and T is not isomorphic to $T_{3,3}$, where $T_{3,3}$ is an almost regular bipartite tournament with three vertices in each partite set.

As an application of this theorem and other results, we show that every arc of an almost regular c-partite tournament D with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \cdots = |V_c|$, is contained in a directed Hamiltonian path if and only if D is not isomorphic to $T_{3,3}$.

Keywords: Bipartite tournaments; Multipartite tournaments; Hamiltonian path; Almost regular bipartite tournaments

1. Terminology and introduction

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. Multipartite tournaments are well studied (see e.g., [2,4,8,11,15,17]). In particular, Gutin [7] gave a characterization of multipartite tournaments with a Hamiltonian path, Gutin [6] and Häggkvist and Manoussakis [9] presented a characterization of Hamiltonian bipartite tournaments, and Bang-Jensen, Gutin, and Yeo [3] showed that the Hamiltonian cycle problem is polynomial time solvable for multipartite tournaments.

We shall assume that the reader is familiar with standard terminology on directed graphs (see, e.g., [2]). In this paper, all digraphs are finite without loops or multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y. If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. By $d_D(X, Y) = d(X, Y)$ we denote the number of arcs from X to Y, i.e., $d(X, Y) = |\{xy \in E(D) : x \in X,\ y \in Y\}|$.

The *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the *outdegree* and the *indegree* of x, respectively.

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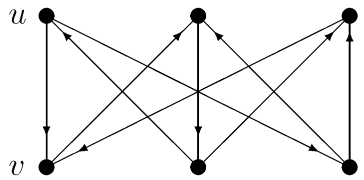

Fig. 1. The almost regular bipartite tournament $T_{3,3}$.

In [16], Yeo defines the *global irregularity* $i_g(D)$ of a digraph D by

$$i_g(D) = \max_{x\in V(D)} \{d_D^+(x), d_D^-(x)\} - \min_{y\in V(D)} \{d_D^+(y), d_D^-(y)\}.$$

If $i_g(D) \leqslant 1$ then D is called *almost regular*, and if $i_g(D) = 0$, then D is *regular*.

By a *cycle* or *path* we mean a directed cycle or directed path. A cycle of length k is also called a k-cycle. A cycle or path in a digraph D is *Hamiltonian* if it includes all the vertices of D. A set $X \subseteq V(D)$ of vertices is *independent* if no two vertices of X are adjacent. A set of $A \subseteq E(D)$ is *independent* if no two arcs of A are incident.

A bipartite tournament with the partite sets X and Y such that $|X| = |Y| = m$ is also called an $m \times m$ bipartite tournament. By $H(r_1, r_2, r_3, r_4)$ we define the bipartite tournament with the four pairwise disjoint independent sets of vertices B_1, B_2, B_3, B_4 with $|B_i| = r_i$ for $i = 1, 2, 3, 4$ such that $B_1 \to B_2 \to B_3 \to B_4 \to B_1$. The class $H^*(r_1, r_2, r_3, r_4)$ of bipartite tournaments originates from $H(r_1, r_2, r_3, r_4)$ by reversing some arcs between B_1 and B_4 or B_3 and B_4 such that $d(b_1, B_4) \leqslant 1$ and $d(B_4, b_3) \leqslant 1$ for every $b_1 \in B_1$ and $b_3 \in B_3$, or by reversing all the arcs.

In Section 2, we prove that every arc of an almost regular bipartite tournament T with the partite sets X, Y such that $|X| \leqslant |Y|$, is contained in a Hamiltonian path if and only if $|Y| \leqslant |X| + 1$ and T is not isomorphic to $T_{3,3}$ (cf. Fig. 1). As an application of this result, we show in Section 3 that every arc of an almost regular c-partite tournament D with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \cdots = |V_c|$, is contained in a Hamiltonian path if and only if D is not isomorphic to $T_{3,3}$.

2. Almost regular bipartite tournaments

The following results play an important role in the investigations of this section.

Theorem 2.1 (Amar and Manoussakis [1], Wang [13]). *Every arc of a regular bipartite tournament is contained in a Hamiltonian cycle.*

Theorem 2.2 (Wang [14]). *Let H be an $m \times m$ bipartite tournament. If $d_H^-(u) + d_H^+(v) \geqslant m - 1$ for every pair of vertices u and v satisfying $u \to v$, then H is Hamiltonian, unless H is isomorphic to $H((m+1)/2, (m+1)/2, (m-1)/2, (m-1)/2)$ when m is odd or $H(m/2, (m-2)/2, m/2, (m+2)/2)$ when m is even.*

Lemma 2.3 (Guo, Pinkernell, and Volkmann [5]). *Let C be an m-cycle of a strong c-partite tournament D. If there exists a vertex $y \in V(D) - V(C)$ with $N^+(y) \cap V(C) = \emptyset$ or $N^-(y) \cap V(C) = \emptyset$, then the vertex set $V(C)$ is contained in an $(m+1)$- or $(m+2)$-cycle.*

Theorem 2.4 (Zhang, Song, and Wang [18]). *Let H be an $(m+2) \times (m+2)$ bipartite tournament with $m \geqslant 3$. If $d_H^-(u) + d_H^+(v) \geqslant m$ for every pair of vertices u and v satisfying $uv \notin E(H)$, then H is Hamiltonian, or H is isomorphic to $H(k+2, k+2, k, k)$ when $m = 2k$, or $H \in H^*(m+2-t, t+1, t, m+1-t)$ with $(m-1)/2 \leqslant t \leqslant (m+2)/2$.*

Example 2.5. Let $T_{3,3}$ be the almost regular bipartite tournament presented in Fig. 1. Then it is straightforward to verify that the arc uv is not contained in a Hamiltonian path. (Note that $T_{3,3}$ has a Hamiltonian cycle.)

Theorem 2.6. *Let T be an almost regular bipartite tournament with the partite sets X, Y such that $1 \leqslant |X| \leqslant |Y|$. Every arc of T is contained in a Hamiltonian path if and only if $|Y| \leqslant |X| + 1$ and T is not isomorphic to $T_{3,3}$.*

Proof. Clearly, if $|Y| \geqslant |X| + 2$, then T does not have any Hamiltonian path. Thus, assume that $|Y| \leqslant |X| + 1$, and let uv be an arbitrary arc of T.

If T is regular, then the desired result follows immediately from Theorem 2.1. So, let now $i_g(T)=1$.

Firstly, assume that $|Y|=|X|+1$. If $|X|=1$, then the result is obvious. Let now $|Y|=|X|+1=m+1\geqslant 3$, $X=\{x_1,x_2,\ldots,x_m\}$, and $Y=\{y_1,y_2,\ldots,y_{m+1}\}$, and, without loss of generality, $u=y_{m+1}$ or $v=y_{m+1}$. The digraph $H=T-\{y_{m+1}\}$ is an $m\times m$ bipartite tournament, and since $i_g(T)=1$, it is easy to see that $d_H^-(w)+d_H^+(z)\geqslant m-1$ for every pair of vertices w and z satisfying $w\to z$. Hence, by Theorem 2.2, H is Hamiltonian, unless H is isomorphic to $H((m+1)/2,(m+1)/2,(m-1)/2,(m-1)/2)$ when m is odd or $H(m/2,(m-2)/2,m/2,(m+2)/2)$ when m is even. If H is Hamiltonian, then T contains a Hamiltonian path with the initial arc uv if $u=y_{m+1}$ and with the terminal arc uv if $v=y_{m+1}$.

If $m=2p+1$ is odd and $H=H(p+1,p+1,p,p)$ then let, without loss of generality, $B_1=\{y_1,y_2,\ldots,y_{p+1}\}$, $B_3=\{y_{p+2},y_{p+3},\ldots,y_{2p+1}\}$, $B_2=\{x_1,x_2,\ldots,x_{p+1}\}$, and $B_4=\{x_{p+2},x_{p+3},\ldots,x_{2p+1}\}$. Because of $d_T^+(x_i)=d_T^-(x_i)=p+1$ for $i=1,2,\ldots,2p+1=m$, we deduce that $B_2\to y_{m+1}\to B_4$. If, without loss of generality, $u=x_1$, if $v=y_{m+1}$ and $v=x_{p+2}$, if $u=y_{m+1}$, then

$$y_1x_1y_{m+1}x_{p+2}y_2x_2y_{p+2}x_{p+3}\ldots x_{2p}y_px_py_{2p}x_{2p+1}y_{p+1}x_{p+1}y_{2p+1}$$

is a Hamiltonian path, containing uv.

If $m=2p$ is even and $H=H(p,p-1,p,p+1)$, then because of $d_T^+(y_i)=d_T^-(y_i)=p$ for $i=1,2,\ldots,2p+1=m+1$ and $p\leqslant d_T^+(x_i),d_T^-(x_i)\leqslant p+1$ for $i=1,2,\ldots,2p=m$, it follows that $X=B_1\cup B_3$, $Y=B_2\cup B_4\cup\{y_{m+1}\}$, and $B_1\to y_{m+1}\to B_3$. Let $B_4=\{y_1,y_2,\ldots,y_{p+1}\}$, $B_2=\{y_{p+2},y_{p+3},\ldots,y_{2p}\}$, $B_1=\{x_1,x_2,\ldots,x_p\}$, and $B_3=\{x_{p+1},x_{p+2},\ldots,x_{2p}\}$. If, without loss of generality, $u=x_1$, if $v=y_{m+1}$ and $v=x_{p+1}$, if $u=y_{m+1}$, then

$$y_1x_1y_{m+1}x_{p+1}y_2x_2y_{p+2}x_{p+2}y_3\ldots y_px_py_{2p}x_{2p}y_{p+1}$$

is a Hamiltonian path, containing $uv=x_1y_{m+1}$.

Secondly, assume that $|X|=|Y|$. Since $i_g(T)=1$, we deduce that $|X|$ is odd, say $|X|=|Y|=2p+1$. If $|X|=|Y|=1$, then there is nothing to prove. If $|X|=|Y|=3$, then it is straightforward to verify that there exists only one exception, namely $T_{3,3}$.

Case 1: Let next $|X|=|Y|=5$, $X=\{u,x_1,x_2,x_3,x_4\}$, $Y=\{v,y_1,y_2,y_3,y_4\}$, and $H=T-\{u,v\}$. Since T is almost regular, we observe that $d_T^+(x),d_T^-(x)\geqslant 2$ for $x\in V(T)$.

Subcase 1.1: Assume that H is not strongly connected. We deduce that H has exactly two 4-cycles T_1 and T_2 as strong components such that, without loss of generality, there is no arc from T_2 to T_1. Since T is strongly connected, there exists an arc from v to T_1. Consequently, T contains a Hamiltonian path, starting with the arc uv.

Subcase 1.2: Assume that H is strongly connected. This implies that $d_H^+(x),d_H^-(x)\geqslant 1$ for $x\in V(H)$.

Subcase 1.2.1: Assume that H is Hamiltonian. It follows that T contains a Hamiltonian path, starting with the arc uv.

Subcase 1.2.2: Assume that the longest cycle in H has length 6. Let, without loss of generality, $x_1y_1x_2y_2x_3y_3x_1$ be such a longest cycle.

Subcase 1.2.2.1: Let $x_4\to y_4$. If $v\to x_4$ or $y_4\to u$, then it is no problem to find a Hamiltonian path through the arc uv in T. Therefore, assume now that $u\to y_4$ and $x_4\to v$. Since v has at least two positive neighbors, assume, without loss of generality, that $v\to\{x_1,x_2\}$. If $y_1\to x_4$, then $uvx_2y_2x_3y_3x_1y_1x_4y_4$ is a Hamiltonian path through uv. It remains to consider the case that $x_4\to y_1$. This yields $\{y_2,y_3\}\to x_4$, and then $uvx_1y_1x_2y_2x_3y_3x_4y_4$ is the desired Hamiltonian path.

Subcase 1.2.2.2: Let $y_4\to x_4$. Assume, without loss of generality, that $x_4\to y_1$. Since H has no Hamiltonian cycle, it follows that $y_4\to x_1$.

Subcase 1.2.2.2.1: Let $v\to x_4$. If $y_4\to u$, then $y_4uvx_4y_1x_2y_2x_3y_3x_1$ is the desired Hamiltonian path. So, let $u\to y_4$. If $u\to y_1$, then $y_3\to u$, and $y_4x_1y_1x_2y_2x_3y_3uvx_4$ is a Hamiltonian path. Thus, it remains $y_1\to u$. If $y_4\to x_2$, then $y_4x_2y_2x_3y_3x_1y_1uvx_4$ is a Hamiltonian path. Let now $x_2\to y_4$. If $v\to x_2$, then $y_4x_4y_1uvx_2y_2x_3y_3x_1$ is a Hamiltonian path. Thus, let $x_2\to v$. This leads to $y_3\to x_2$. If $v\to x_1$, then $y_2x_3y_3x_2y_4x_4y_1uvx_1$ is a Hamiltonian path. Otherwise, $x_1\to v$, and this implies $v\to x_3$. If $y_2\to u$, then $y_4x_4y_1x_2y_2uvx_3y_3x_1$ is a Hamiltonian path. Thus, let finally $u\to y_2$. This leads to $y_3\to u$, and we obtain the desired Hamiltonian path $y_4x_1y_1x_2y_2x_3y_3uvx_4$.

Subcase 1.2.2.2.2: Let $x_4\to v\to x_1$. We deduce that $x_1\to y_2$. If $y_3\to u$, then $y_4x_4y_1x_2y_2x_3y_3uvx_1$ is a Hamiltonian path. So, we assume $u\to y_3$. Now, $y_4\to u$ implies $x_2\to y_4$, and we have the Hamiltonian path $x_4y_1x_2y_4uvx_1y_2x_3y_3$. The case $u\to y_4$ yields $\{y_1,y_2\}\to u$. If $y_3\to x_2$, then $y_4x_4y_1uvx_1y_2x_3y_3x_2$ is a Hamiltonian path. If we assume that $x_2\to y_3$, then it follows $y_3\to x_4$. In the case $v\to x_3$, we find the Hamiltonian path $y_4x_4y_1x_2y_2uvx_3y_3x_1$. Thus, let finally $x_3\to v$. This leads to $v\to x_2$, and we obtain the desired Hamiltonian path $y_4x_4y_1uvx_2y_2x_3y_3x_1$.

Subcase 1.2.2.2.3: Let $x_4\to v$ and $x_1\to v$. It follows that $v\to\{x_2,x_3\}$. If $y_1\to u$, then $y_4x_4y_1uvx_2y_2x_3y_3x_1$ is a Hamiltonian path through uv. In the other case that $u\to y_1$, we deduce that $y_1\to x_3$. If $y_2\to u$, then $y_4x_4y_1x_2y_2uvx_3y_3x_1$ is a Hamiltonian path. Thus, we assume that $u\to y_2$. This implies $\{y_3,y_4\}\to u$ and this leads to $\{x_2,x_3\}\to y_4$. If $x_4\to y_3$, then $x_4y_3uvx_3y_4x_1y_1x_2y_2$ is a Hamiltonian path. In the last case that $y_3\to x_4$, we observe that $x_2\to y_3$. If $x_4\to y_2$, then $x_1y_1x_2y_4uvx_3y_3x_4y_2$ is a Hamiltonian path, and if $y_2\to x_4$, then $y_4uvx_2y_2x_4y_1x_3y_3x_1$ is a Hamiltonian path through uv.

Subcase 1.2.3: Assume that the longest cycle in H has length 4. Let, without loss of generality, $C = x_1y_1x_2y_2x_1$ be such a longest cycle. According to Lemma 2.3, each of the vertices x_3, y_3, x_4, y_4 has exactly one positive and one negative neighbor in C.

Subcase 1.2.3.1: Let $x_4y_1x_3y_2x_4$ and $x_1y_4x_2y_3x_1$ are both 4-cycles. This implies $x_4 \to y_4$, because otherwise there exists the 6-cycle $y_4x_4y_1x_2y_2x_1y_4$, $y_4 \to x_3$, because otherwise there exists the 6-cycle $x_3y_4x_2y_2x_1y_1x_3$. It follows that $x_4 \to y_3$, because otherwise there exists the 8-cycle $y_3x_4y_4x_3y_2x_1y_1x_2y_3$. This yields $v \to x_4$ and $uvx_4y_4x_3y_2x_1y_1x_2y_3$ is a Hamiltonian path, containing the arc uv.

Subcase 1.2.3.2: Let $x_4y_1x_3y_2x_4$ be a 4-cycle, and let $x_1y_4x_2y_3x_1$ be not a 4-cycle. We can assume, without loss of generality, that $x_1 \to y_4 \to x_2$ and $x_1 \to y_3 \to x_2$. This implies $x_4 \to y_4$, because otherwise there exists the 6-cycle $y_4x_4y_1x_2y_2x_1y_4$, $x_4 \to y_3$, because otherwise there exists the 6-cycle $y_3x_4y_1x_2y_2x_1y_3$, and $y_4 \to x_3$, because otherwise there exists the 6-cycle $x_3y_4x_2y_2x_1y_1x_3$. It follows that $y_3 \to x_3$, because otherwise there exists the 6-cycle $x_3y_3x_2y_2x_4y_4x_3$. This yields $v \to x_4$.

If $y_3 \to u$, then $y_3uvx_4y_4x_3y_2x_1y_1x_2$ is a Hamiltonian path, containing the arc uv. If $y_4 \to u$, then $y_4uvx_4y_3x_3y_2x_1y_1x_2$ is a Hamiltonian path, containing the arc uv. However, if $u \to \{y_3, y_4\}$, then we deduce that $y_1 \to u$, and we arrive at the Hamiltonian path $y_3x_2y_2x_1y_1uvx_4y_4x_3$ through uv.

Subcase 1.2.3.3: Let neither $x_4y_1x_3y_2x_4$ nor $x_1y_4x_2y_3x_1$ be a 4-cycle. We can assume, without loss of generality, that $y_2 \to x_4 \to y_1$, $y_2 \to x_3 \to y_1$, $x_1 \to y_4 \to x_2$, and $x_1 \to y_3 \to x_2$. This implies $x_4 \to y_4$, because otherwise there exists the 6-cycle $y_4x_4y_1x_2y_2x_1y_4$ and $x_3 \to y_4$, because otherwise there exists the 6-cycle $y_4x_3y_1x_2y_2x_1y_4$. It follows that $x_4 \to y_3$, because otherwise there exists the 6-cycle $y_3x_4y_4x_2y_2x_1y_3$. This yields $x_3 \to y_3$, because otherwise there exists the 6-cycle $x_4y_3x_3y_1x_2y_2x_4$. We deduce that $y_4 \to u$ and $v \to x_1$, and this leads to the Hamiltonian path $x_4y_4uvx_1y_1x_2y_2x_3y_3$, containing the arc uv.

Case 2: Let $|X| = |Y| = 2p + 1 = m + 3 \geqslant 7$, $H = T - \{u, v\}$, $X = \{u, x_1, x_2, \ldots, x_{m+2}\}$, and $Y = \{v, y_1, y_2, \ldots, y_{m+2}\}$. As T is almost regular, we observe that $p \leqslant d_T^+(x), d_T^-(x) \leqslant p + 1$ for all $x \in V(T)$. If we define $H = T - \{u, v\}$, then $p - 1 \leqslant d_H^+(x), d_H^-(x) \leqslant p + 1$ for all $x \in V(H)$, and therefore $d_H^-(w) + d_H^+(z) \geqslant 2p - 2 = m$ for every pair of vertices w and z satisfying $wz \notin E(H)$. It follows, according to Theorem 2.4, that H is Hamiltonian, or H is isomorphic to $H(p+1, p+1, p-1, p-1)$ or $H \in H^*(p+1, p, p-1, p)$. If H is Hamiltonian, then T contains a Hamiltonian path with the terminal arc uv.

If $H = H(p+1, p+1, p-1, p-1)$, then because of $p \leqslant d_T^+(x), d_T^-(x) \leqslant p + 1$ for all $x \in V(T)$, it follows that $X = B_1 \cup B_3 \cup \{u\}$, $Y = B_2 \cup B_4 \cup \{v\}$, $B_2 \to u \to B_4$, and $B_3 \to v \to B_1$. If $B_1 = \{x_1, x_2, \ldots, x_{p+1}\}$, $B_2 = \{y_1, y_2, \ldots, y_{p+1}\}$, $B_3 = \{x_{p+2}, x_{p+3}, \ldots, x_{2p}\}$, and $B_4 = \{y_{p+2}, y_{p+3}, \ldots, y_{2p}\}$, then

$$x_1y_1uvx_2y_2x_{p+2}y_{p+2}x_3 \ldots x_py_px_{2p}y_{2p}x_{p+1}y_{p+1}$$

is a Hamiltonian path containing the arc uv.

Subcase 2.1: Let $H \in H^*(p+1, p, p-1, p)$ such that $B_1 = \{y_1, y_2, \ldots, y_{p+1}\}$, $B_2 = \{x_1, x_2, \ldots, x_p\}$, $B_3 = \{y_{p+2}, y_{p+3}, \ldots, y_{2p}\}$, and $B_4 = \{x_{p+1}, x_{p+2}, \ldots, x_{2p}\}$. Because of $p \leqslant d_T^+(x), d_T^-(x) \leqslant p+1$ for all $x \in V(T)$, we conclude that $B_2 \to v \to B_4$, and there exists a vertex, say $y_{p+1} \in B_1$, with the property that $y_{p+1} \to u$.

Subcase 2.1.1: Assume that $d_H(x, B_1 - \{y_{p+1}\}) \geqslant 1$ for all $x \in B_4$. In this case, the hypothesis $d_H(b_1, B_4) \leqslant 1$ for every $b_1 \in B_1$ and the Marriage Theorem yield p independent arcs, say $x_{p+i}y_i$ for $i = 1, 2, \ldots, p$, from B_4 to B_1. Furthermore, the hypothesis $d_H(B_4, b_3) \leqslant 1$ for every $b_3 \in B_3$ leads easily to $p - 1$ independent arcs, say $y_{p+i}x_{p+i}$ for $i = 2, 3, \ldots, p$, from B_3 to B_4. Altogether, we see that

$$y_{p+1}uvx_{p+1}y_1x_1y_{p+2}x_{p+2}y_2x_2 \ldots y_{p-1}x_{p-1}y_{2p}x_{2p}y_px_p$$

is a Hamiltonian path through uv.

Subcase 2.1.2: Assume next that there exists a vertex, say $x_{p+1} \in B_4$, such that $d_H(x_{p+1}, B_1 - \{y_{p+1}\}) = 0$. This implies $((B_1 - \{y_{p+1}\}) \cup \{v\}) \to x_{p+1}$. Because of $p \leqslant d_T^+(x), d_T^-(x) \leqslant p+1$ for all $x \in V(T)$, it follows that $x_{p+1} \to (B_3 \cup \{y_{p+1}\})$, $u \to (B_1 - \{y_{p+1}\})$, and $B_3 \to u$. In addition, in view of the hypotheses $d_H(b_1, B_4) \leqslant 1$ for every $b_1 \in B_1$ and $d_H(B_4, b_3) \leqslant 1$ for every $b_3 \in B_3$, we deduce that $(B_4 - \{x_{p+1}\}) \to (B_1 - \{y_{p+1}\})$ and $B_3 \to (B_4 - \{x_{p+1}\})$. This shows that

$$y_1x_1y_{p+2}uvx_{p+2}y_2x_{p+1}y_{p+1}x_2y_{p+3}x_{p+3}y_3x_3y_{p+4} \ldots y_{p-1}x_{p-1}y_{2p}x_{2p}y_px_p$$

is a Hamiltonian path through uv.

Subcase 2.2: Let $H \in H^*(p+1, p, p-1, p)$ such that $B_1 = \{x_1, x_2, \ldots, x_{p+1}\}$, $B_2 = \{y_1, y_2, \ldots, y_p\}$, $B_3 = \{x_{p+2}, x_{p+3}, \ldots, x_{2p}\}$, and $B_4 = \{y_{p+1}, y_{p+2}, \ldots, y_{2p}\}$. Because of $p \leqslant d_T^+(x), d_T^-(x) \leqslant p + 1$ for all $x \in V(T)$, we have $B_2 \to u$, and there exists a vertex, say $x_{p+1} \in B_1$, with the property $v \to x_{p+1}$.

Subcase 2.2.1: Assume that $d_H(y, B_1 - \{x_{p+1}\}) \geqslant 1$ for all $y \in B_4$. In this case, the hypothesis $d_H(b_1, B_4) \leqslant 1$ for every $b_1 \in B_1$ and the Marriage Theorem yield p independent arcs, say $y_{p+i}x_i$ for $i = 1, 2, \ldots, p$, from B_4 to B_1. Furthermore, the hypothesis $d_H(B_4, b_3) \leqslant 1$ for every $b_3 \in B_3$ leads easily to $p-1$ independent arcs, say $x_{p+i}y_{p+i}$ for $i = 2, 3, \dots, p$, from B_3 to B_4. This implies that

$$y_{p+1}x_1y_1uvx_{p+1}y_2x_{p+2}y_{p+2}x_2y_3x_{p+3}y_{p+3}x_3y_4 \dots x_{p-1}y_px_{2p}y_{2p}x_p$$

is a Hamiltonian path through uv.

Subcase 2.2.2: Next, assume that there exists a vertex, say $y_{p+1} \in B_4$, such that $d_H(y_{p+1}, B_1 - \{x_{p+1}\}) = 0$. Because of $p \leqslant d_T^+(x), d_T^-(x) \leqslant p+1$ for all $x \in V(T)$, it follows that $v \to B_1$. In addition, the fact that $(B_1 - \{x_{p+1}\}) \to (B_2 - \{y_{p+1}\})$ leads to $(B_4 - \{y_{p+1}\}) \to (B_1 - \{x_{p+1}\})$.

Subcase 2.2.2.1: Let $y_{p+1} \to x_{p+1}$. Since y_{p+1} has at least $p-2$ positive neighbors in B_3, the condition $d_H(B_4, b_3) \leqslant 1$ for every $b_3 \in B_3$ leads easily to $p-1$ independent arcs, say $x_{p+i}y_{p+i}$ for $i = 2, 3, \dots, p$, from B_3 to $B_4 - \{y_{p+1}\}$. Thus,

$$x_1y_{p+1}x_{p+1}y_1uvx_2y_2x_{p+2}y_{p+2}x_3y_3x_{p+3}y_{p+3} \dots x_{p-1}y_{p-1}x_{2p-1}y_{2p-1}x_py_px_{2p}y_{2p}$$

is a Hamiltonian path through uv.

Subcase 2.2.2.2: Let $x_{p+1} \to y_{p+1}$. We deduce that $y_{p+1} \to (B_3 \cup \{u\})$, $v \to B_1$, $B_3 \to (B_4 - \{y_{p+1}\})$, and $(B_4 - \{y_{p+1}\}) \to B_1$. Consequently,

$$x_{p+1}y_{p+1}uvx_1y_1x_{p+2}y_{p+2}x_2y_2x_{p+3}y_{p+3} \dots x_{p-1}y_{p-1}x_{2p}y_{2p}x_py_p$$

is a Hamiltonian path through uv.

If we finally reserve all arcs of $H(p+1, p, p-1, p)$, then we arrive at a symmetric situation, and the proof of Theorem 2.6 is complete. □

3. Almost regular multipartite tournaments

Theorem 3.1 (Volkmann, Yeo [12]). *Every arc of a regular c-partite tournament D is contained in a Hamiltonian path of D.*

Theorem 3.2 (Volkmann, Yeo [12]). *Let D be a c-partite tournament with partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| \leqslant |V_2| \leqslant \dots \leqslant |V_c|$, and let P be a path of length q in D. If*

$$|V(D)| \geqslant 2i_g(D) + 3q + 2|V_c| + |V_{c-1}| - 2,$$

then there exists a Hamiltonian path in D, starting with the path P.

Theorem 3.3 (Jakobsen [10]). *If T is an almost regular tournament of order $n \geqslant 8$, then every arc of T is contained in an m-cycle for each $m \in \{4, 5, \dots, n\}$.*

Theorem 3.4. *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| = |V_2| = \dots = |V_c|$. Then each arc of D is contained in a Hamiltonian path if and only if D is not isomorphic to $T_{3,3}$.*

Proof. If D is regular, then we deduce from Theorem 3.1 that every arc is contained in a Hamiltonian path. In the remaining case that $i_g(D) = 1$, it follows, because of $|V_1| = |V_2| = \dots = |V_c| = r$, that c is even and r is odd. If $c = 2$, then Theorem 2.6 yields the desired result. In the case $c \geqslant 4$ and $r \geqslant 3$, we observe that

$$\begin{aligned} |V(D)| &= |V_1| + |V_2| + \dots + |V_c| \\ &\geqslant |V_{c-3}| + |V_{c-2}| + |V_{c-1}| + |V_c| \\ &\geqslant 2 - 2 + 3 + 2|V_c| + |V_{c-1}| \\ &= 2i_g(D) + 3 + 2|V_c| + |V_{c-1}| - 2. \end{aligned}$$

Applying Theorem 3.2 with $q = 1$, we see that there exists a Hamiltonian path in D, starting with an arbitrary arc. Finally, let D be a tournament. If $c \geqslant 8$, then, according to Theorem 3.3, every arc is even contained in a Hamiltonian cycle. If $c = 6$, then we can apply again Theorem 3.2, and if $c = 4$, then it is a simple matter to obtain the desired result. □

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