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Marcinkiewicz Integrals with Rough Homogeneous Kernels of Degree Zero in Product Domains[1]

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The L^2-boundedness of the Marcinkiewicz integrals in product domains with component-wise homogeneous kernels which belong to a certain Orlicz space and satisfy the cancellation property is studied. © 2001 Academic Press

Key Words: Marcinkiewicz integral; Orlicz space; product domain.

1. INTRODUCTION

Stein [15] defined a higher dimensional analogue of the Marcinkiewicz integral by

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_s f(x)|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}, \tag{1.1}$$

where

$$F_s f(x) = \int_{|x-y|<s} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy,$$

and Ω is a homogeneous function of degree zero whose restriction to S^{n-1} belongs to $L^1(S^{n-1})$ and satisfies the *cancellation* property,

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0.$$

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Here, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. The continuity of Marcinkiewicz integrals is very useful in harmonic analysis [4–6, 16, 17, 22]. Stein [15] proved that if $\Omega|_{S^{n-1}}$ belongs to the Lipschitz space $\Lambda^\alpha(S^{n-1})$ of order α $(0 < \alpha \le 1)$, then

$$\big|\{x \in \mathbb{R}^n : \mu_\Omega f(x) > \lambda\}\big| \le \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}, \tag{1.2}$$

and

$$\|\mu_\Omega f\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}, \tag{1.3}$$

where $1 < p \le 2$, and if Ω is an integrable odd function, then

$$\|\mu_\Omega f\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}, \tag{1.4}$$

for $2 < p < \infty$.

This result was further improved by Walsh [19]. Among other things, he proved:

THEOREM 1.1 (Walsh). *Let $n \ge 2$ and let $\Omega : \mathbb{R}^n \to \mathbb{C}$ be a homogeneous function of degree zero such that*

$$\int_{S^{n-1}} \Omega(y')\, d\sigma(y') = 0.$$

(1) *If $\Omega|_{S^{n-1}} \in L(\log^+ L)^{1/2}(S^{n-1})$, then μ_Ω is bounded in $L^2(\mathbb{R}^n)$.*

(2) *Let $1 < p < \infty$ and let p' be the conjugate of p. If $\Omega|_{S^{n-1}} \in L(\log^+ L)^{1/r}(\log^+\log^+ L)^{2-4/r'}(S^{n-1})$, with $r = \min\{p, p'\}$, then μ_Ω is bounded in $L^p(\mathbb{R}^n)$.*

Mapping properties of μ_Ω on other function spaces were also studied. Torchinsky and Wang [18] considered the weighted L^p-boundedness of μ_Ω and showed that if $\Omega|_{S^{n-1}} \in \Lambda^\alpha(S^{n-1})$, then for ω satisfying an A_p condition, μ_Ω is bounded on $L^p(\omega)$. Recently, this was extended to rougher kernels by Ding *et al.* [9]. Mapping properties of μ_Ω on *BMO* or Campanato spaces have been studied in [3, 7, 10, 14, 20].

On the other hand, the Marcinkiewicz integral defined on product domains has also been studied.

To be more specific, let $k \ge 1$ be an integer, $n_1, \dots, n_k \ge 2$ be integers, and $\Omega : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{C}$ be a *component-wise* homogeneous function of degree zero with the cancellation property:

$$\int_{S^{n_j-1}} \Omega(y_1', \dots, y_k')\, d\sigma(y_j') = 0 \qquad \text{for } j = 1, \dots, k. \tag{1.5}$$

The Marcinkiewicz integral $\tilde{\mu}_\Omega$ is defined by

$$\tilde{\mu}_\Omega f(x_1,\dots,x_k) = \left(\int_0^\infty \cdots \int_0^\infty |F_{s_1,\dots,s_k} f(x_1,\dots,x_k)|^2 \frac{ds_1 \cdots ds_k}{s_1^3 \cdots s_k^3}\right)^{\frac{1}{2}}, \tag{1.6}$$

where

$$F_{s_1,\dots,s_k} f(x_1,\dots,x_k) = \int_{|x_1-y_1|\le s_1} \cdots \int_{|x_k-y_k|\le s_k} \frac{\Omega(x_1-y_1,\dots,x_k-y_k)}{|x_1-y_1|^{n_1-1}\cdots|x_k-y_k|^{n_k-1}} \times f(y_1,\dots,y_k)\,dy_1\cdots dy_k.$$

Ding [8] showed the L^2-boundedness of $\tilde{\mu}_\Omega$ in case $k=2$ under the hypothesis of

$$\Omega|_{S^{n_1-1}\times S^{n_2-1}} \in L(\log^+ L)^2(S^{n_1-1}\times S^{n_2-1}).$$

In this paper, we improve the above result. Namely, we will prove:

THEOREM 1.2. *Let $k\ge 1$ be an integer, $n_1,\dots,n_k\ge 2$ be integers, and Ω: $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}\to\mathbb{C}$ be a component-wise homogeneous function of degree zero with the cancellation property* (1.5). *If $\Omega|_{S^{n_1-1}\times\cdots\times S^{n_k-1}}\in L(\log^+ L)^{k/2}(S^{n_1-1}\times\cdots\times S^{n_k-1})$, then the Marcinkiewicz integral operator $\tilde{\mu}_\Omega$ defined by* (1.6) *is bounded in $L^2(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k})$.*

This paper is organized as follows: in Section 2, elementary properties on Orlicz spaces are discussed and the proof of the main theorem appears in Section 3.

2. PRELIMINARIES ON ORLICZ SPACES

Let $\Phi:[0,\infty)\to[0,\infty)$ be a function with the following properties:

(1) Φ is convex;

(2) $\Phi(0)=0$; and that

(3) $\lim_{t\to\infty}\Phi(t)/t=\infty$.

DEFINITION 2.1. (1) For a measurable function $f:\mathbb{R}^n\to\mathbb{C}$, we define $\|f\|_{L^\Phi(\mathbb{R}^n)}$ (the Luxemberg norm) by

$$\|f\|_{L^\Phi(\mathbb{R}^n)} = \inf\left\{\lambda\in(0,\infty): \int_{\mathbb{R}^n}\Phi(\lambda^{-1}|f(x)|)\,dx\le 1\right\}.$$

(2) The Orlicz sapce $L^\Phi(\mathbb{R}^n)$ is defined by

$$\left\{f:\mathbb{R}^n\to\mathbb{C}: f \text{ is measurable and } \|f\|_{L^\Phi(\mathbb{R}^n)}<\infty\right\}.$$

Remark 2.2. (1) $L^\Phi(\mathbb{R}^n)$ is a Banach space.

(2) The following analogue of Hölder's inequality is available; see, for instance, [22].

LEMMA 2.3. *Let $\Phi:[0,\infty)\to[0,\infty)$ and $\Psi:[0,\infty)\to[0,\infty)$ be Young's pair in the sense that*

(1) *Φ and Ψ are convex*;

(2) *$\Phi':[0,\infty)\to[0,\infty)$ and $\Psi':[0,\infty)\to[0,\infty)$ are inverse to each other*;

(3) $\Phi(0)=\Psi(0)=0$; *and that*

(4) $\lim_{t\to\infty}\Phi(t)/t=\lim_{t\to\infty}\Psi(t)/t=\infty$.

If $f\in L^{\Phi}(\mathbb{R}^n)$ *and* $g\in L^{\Psi}(\mathbb{R}^n)$, *then* $fg\in L^1(\mathbb{R}^n)$ *and we have*

$$\|fg\|_{L^1(\mathbb{R}^n)}\le\|f\|_{L^{\Phi}(\mathbb{R}^n)}\|g\|_{L^{\Psi}(\mathbb{R}^n)}.$$

3. PROOF OF THE MAIN THEOREM

3.1. *Simplifying Notations.* For simplicity, we will use the following notations:

(1) $\mathbf{1}=(1,\dots,1)$, $\mathbf{n}=(n_1,\dots,n_k)\in\mathbb{N}^k$;

(2) For $j\in\{1,\dots,k\}$, $x\in\mathbb{R}^{n_j}$, and $s>0$, we write

$$B_j(x,s)=\{y\in\mathbb{R}^{n_j}:|y-x|<s\};$$

(3) For $j\in\{1,\dots,k\}$, $\xi'\in S^{n_j-1}$, and $s>0$, we write

$$C_s^{(j)}(\xi')=\left\{y'\in S^{n_j-1}:|y'\cdot\xi'|\le s^{-\frac{1}{2}}\right\};$$

(4) For $\mathbf{s}=(s_1,\dots,s_k)\in(\mathbb{R}_+)^k$ and $\mathbf{y}=(y_1,\dots,y_k)\in\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$, we write

$$\begin{aligned}\mathbf{sy}&=(s_1y_1,\dots,s_ky_k),\\ \mathbf{s}^{-\mathbf{1}}\mathbf{y}&=(s_1^{-1}y_1,\dots,s_k^{-1}y_k),\\ \mathbf{s}^{\mathbf{m}}&=\prod_{j=1}^k s_j^{m_j},\qquad \mathbf{m}=(m_1,\dots,m_k)\in\mathbb{N}^k,\\ \widetilde{\Omega}_{\mathbf{s}}(\mathbf{y})&=\frac{\Omega(\mathbf{y})}{|y_1|^{n_1-1}\cdots|y_k|^{n_k-1}}\cdot\chi_{B_1(0,s_1)\times\cdots\times B_k(0,s_k)}(\mathbf{y}).\end{aligned}$$

3.2. *A Reduction.* We can write

$$F_{\mathbf{s}}f(\mathbf{x})=\widetilde{\Omega}_{\mathbf{s}}*f(\mathbf{x}).$$

From the homogeneity of Ω, we see

$$\widetilde{\Omega}_{\mathbf{s}}(\mathbf{y})=\mathbf{s}^{\mathbf{1}}[\mathbf{s}^{\mathbf{n}}]^{-1}\widetilde{\Omega}_{\mathbf{1}}(\mathbf{s}^{-\mathbf{1}}\mathbf{y}).$$

Thus, we obtain the following formula for the Fourier multiplier of $F_{\mathbf{s}}$:

$$\widehat{\widetilde{\Omega}_{\mathbf{s}}}(\xi)=\mathbf{s}^{\mathbf{1}}\widehat{\widetilde{\Omega}_{\mathbf{1}}}(\mathbf{s}\xi).$$

Plancherel's theorem allows us to write

$$\begin{aligned}
\|\tilde{\mu}_\Omega f\|_{L^2}^2 &= \int_{\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}} \left(\int_{(\mathbb{R}_+)^k} |F_s f(\mathbf{x})|^2 \frac{d\mathbf{s}}{(\mathbf{s^1})^3} \right) d\mathbf{x} \\
&= \int_{(\mathbb{R}_+)^k} \left\| \widehat{F_s f} \right\|_{L^2}^2 \frac{d\mathbf{s}}{(\mathbf{s^1})^3} \\
&= \int_{(\mathbb{R}_+)^k} \left\| \widehat{\widetilde{\Omega}_s} \hat{f} \right\|_{L^2}^2 \frac{d\mathbf{s}}{(\mathbf{s^1})^3} \\
&= \int_{\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}} \left(\int_{(\mathbb{R}_+)^k} \left| \widehat{\widetilde{\Omega}_{\mathbf{s}}}(\xi) \right|^2 \frac{d\mathbf{s}}{(\mathbf{s^1})^3} \right) \left| \hat{f}(\xi) \right|^2 d\xi \\
&= \int_{\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}} \left(\int_{(\mathbb{R}_+)^k} \left| \widehat{\widetilde{\Omega}_{\mathbf{1}}}(\mathbf{s}\xi) \right|^2 \frac{d\mathbf{s}}{\mathbf{s^1}} \right) \left| \hat{f}(\xi) \right|^2 d\xi \\
&= \int_{\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}} \left(\int_{(\mathbb{R}_+)^k} \left| \widehat{\widetilde{\Omega}_{\mathbf{1}}}(\mathbf{s}\xi') \right|^2 \frac{d\mathbf{s}}{\mathbf{s^1}} \right) \left| \hat{f}(\xi) \right|^2 d\xi.
\end{aligned}$$

Therefore, it suffices to verify the uniform boundedness of

$$I(\xi') \equiv \int_{(\mathbb{R}_+)^k} \left| \widehat{\widetilde{\Omega}_{\mathbf{1}}}(\mathbf{s}\xi') \right|^2 \frac{d\mathbf{s}}{\mathbf{s^1}} = \int_{(\mathbb{R}_+)^k} \left| \widehat{\widetilde{\Omega}_{\mathbf{1}}}(s_1\xi_1', \dots, s_k\xi_k') \right|^2 \frac{ds_1 \cdots ds_k}{s_1 \cdots s_k}$$

in $\xi' = (\xi_1', \dots, \xi_k') \in S^{n_1-1} \times \cdots \times S^{n_k-1}$.

3.3. *Estimates on* $I(\xi')$. We let

$$R_0 = (0,1), \qquad R_1 = (1, \infty),$$

and write

$$I(\xi') \equiv \sum_{\alpha_1, \dots, \alpha_k = 0,1} I_{\alpha_1, \dots, \alpha_k}(\xi'),$$

with

$$I_{\alpha_1, \dots, \alpha_k}(\xi') = \int_{R\alpha_1 \times \cdots \times R\alpha_k} \left| \widehat{\widetilde{\Omega}_{\mathbf{1}}}(\mathbf{s}\xi') \right|^2 \frac{d\mathbf{s}}{\mathbf{s^1}}.$$

It suffices to verify the uniform boundedness of $I_{\alpha_1, \dots, \alpha_k}(\xi')$ in $\xi' \in S^{n_1-1} \times \cdots \times S^{n_k-1}$, for each choice of $\alpha_1, \dots, \alpha_k = 0, 1$. Write

$$J = \{1, \dots, k\},$$

and let

$$\begin{aligned}
J_0 &= \{ j \in \{1, \dots, k\} : \alpha_j = 0 \} \\
J_1 &= \{ j \in \{1, \dots, k\} : \alpha_j = 1 \}.
\end{aligned}$$

By the cancellation property and homogeneity condition, one can see

$$\begin{aligned}\left|\widehat{\widetilde{\Omega}_{\mathbf{1}}}(\mathbf{s}\xi')\right| &= \left|\int_{\prod_{j\in J}B_j(0,1)}\left[\prod_{\nu\in J_0}\left(e^{-2\pi i s_\nu \xi'_\nu\cdot y_\nu}-1\right)\right]\right.\\ &\qquad\left.\times\left[\prod_{l\in J_1}e^{-2\pi i s_l\xi'_l\cdot y_l}\right]\widetilde{\Omega}_{\mathbf{1}}(y_1,\dots,y_k)\prod_{j\in J}dy_j\right|\\ &\le \int_{\prod_{\nu\in J_0}B_\nu(0,1)}\left[\prod_{\nu\in J_0}\left|e^{-2\pi i s_\nu\xi'_\nu\cdot y_\nu}-1\right|\right]\\ &\qquad\times\left|\int_{\prod_{l\in J_1}B_l(0,1)}\left[\prod_{l\in J_1}e^{-2\pi i s_l\xi'_l\cdot y_l}\right]\widetilde{\Omega}_{\mathbf{1}}(y_1,\dots,y_k)\prod_{l\in J_1}dy_l\right|\prod_{\nu\in J_0}dy_\nu.\end{aligned}$$

Switching to polar coordinates, we obtain

$$\left|\widehat{\widetilde{\Omega}_{\mathbf{1}}}(\mathbf{s}\xi')\right| \lesssim \left[\prod_{\nu\in J_0}s_\nu\right]\int_{\prod_{j\in J}S^{n}j^{-1}}\left[\prod_{l\in J_1}\left|\frac{\sin\pi s_l y'_l\cdot\xi'_l}{s_l y'_l\cdot\xi'_l}\right|\right]|\Omega(y'_1,\dots,y'_k)|\prod_{j\in J}d\sigma(y'_j).$$

Observe that for each $l\in J_1$,

$$\left|\frac{\sin\pi s_l y'_l\cdot\xi'_l}{s_l y'_l\cdot\xi'_l}\right| \lesssim s_l^{-\frac{1}{2}}+\chi_{C^{(l)}_{s_l}(\xi'_l)}(y'_l)$$

holds.

Therefore, it suffices to estimate integrals of the form

$$\widetilde{I}_{J_0,J_1^{(1)},J_1^{(2)}}(\xi') = \int_{\prod_{\nu\in J_0}R_0}\int_{\prod_{l\in J_1^{(1)}}R_1}\mathscr{F}(\check{s})\left[\prod_{l\in J_1^{(1)}}\frac{ds_l}{s_l^2}\right]\prod_{\nu\in J_0}s_\nu ds_\nu$$

with

$$\mathscr{F}(\check{s}) = \int_{\prod_{l\in J_1^{(2)}}R_1}\left\|\left[\prod_{l\in J_1^{(2)}}\chi_{C^{(l)}_{s_l}(\xi'_l)}(y'_l)\right]\Omega(y'_1,\dots,y'_k)\right\|^2_{L^1(\prod_{j\in J}S^{n_j-1})}\prod_{l\in J_1^{(2)}}\frac{ds_l}{s_l},$$

where $\{J_0,J_1^{(1)},J_1^{(2)}\}$ is any partition of J. An application of Minkowski's inequality on integrals gives us

$$\mathscr{F}(\check{s})^{\frac{1}{2}} \lesssim \int_{\prod_{j\in J}S^{n_j-1}}\left[\prod_{l\in J_1^{(2)}}\left(\log\frac{1}{|y'_l\cdot\xi'_l|}\right)^{\frac{1}{2}}\right]|\Omega(y'_1,\dots,y'_k)|\prod_{j\in J}d\sigma(y'_j).$$

We have the following lemma:

LEMMA 3.1. *Let $K\ge 1$ be an integer, $n_1,\dots,n_k\ge 2$ be integers, $\xi'_1\in S^{n_1-1},\dots,\xi'_K\in S^{n_K-1}$, and $\Phi,\Psi:(0,\infty)\to(0,\infty)$ be a Young's pair with*

$$\Phi(u)=u(\log^+ u)^{\frac{K}{2}},\qquad \textit{for large } u>0.$$

Then

$$\prod_{j=1}^{K}\left(\log\frac{1}{|y_j'\cdot\xi_j'|}\right)^{\frac{1}{2}}\in L^{\Psi}(S^{n_1-1}\times\cdots\times S^{n_K-1}),$$

and moreover there exists a constant $C(K)$ *such that*

$$\left\|\prod_{j=1}^{K}\left(\log\frac{1}{|y_j'\cdot\xi_j'|}\right)^{\frac{1}{2}}\right\|_{L^{\Psi}(S^{n_1-1}\times\cdots\times S^{n_K-1})}\leq C$$

holds uniformly in $\xi_1',\ldots,\xi_K'\in S^{n_1-1}\times\cdots\times S^{n_K-1}$.

Proof of Lemma 3.1. Observe that there exists a constant C such that for large $u>0$

$$\Psi(u)\lesssim\exp\left(Cu^{\frac{2}{K}}\right) \tag{3.1}$$

holds. For $\lambda\gtrsim 1$, we have

$$\begin{aligned}
&\Psi\left(\lambda^{-1}\left(\prod_{j=1}^{K}\log\frac{1}{|y_j'\cdot\xi_j'|}\right)^{\frac{1}{2}}\right)\\
&\quad\lesssim\exp\left[C\left[\lambda^{-1}\left(\prod_{j=1}^{K}\log\frac{1}{|y_j'\cdot\xi_j'|}\right)^{\frac{1}{2}}\right]^{\frac{2}{K}}\right]\leq\exp\left[CK^{-1}\lambda^{-\frac{2}{K}}\sum_{j=1}^{K}\log\frac{1}{|y_j'\cdot\xi_j'|}\right]\\
&\quad\leq\prod_{j=1}^{K}\exp\left[CK^{-1}\lambda^{-\frac{2}{K}}\log\frac{1}{|y_j'\cdot\xi_j'|}\right]=\prod_{j=1}^{K}|y_j'\cdot\xi_j'|^{-CK^{-1}\lambda^{-\frac{2}{K}}}
\end{aligned}$$

from the geometric-arithmetic mean inequality and (3.1).

Uniform integrability of $\Psi(\lambda^{-1}\prod_{j=1}^{K}\log(1/|y_j'\cdot\xi_j'|))$ on $S^{n_1-1}\times\cdots\times S^{n_K-1}$ is clear and the lemma is proved. ∎

Applying Hölder's inequality (Lemma 2.3) followed by Lemma 3.1, we obtain a uniform estimate on $\mathcal{F}(\check{s})$.

Therefore, we obtain the uniform boundedness of $I(\xi')$ and the proof of the theorem is now complete.

Remark 3.2. In view of the sharpness of Walsh's result, one can see that the size condition on $\Omega|_{S^{n_1-1}\times\cdots\times S^{n_k-1}}$ is sharp.

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