Marcinkiewicz Integrals with Rough Homogeneous Kernels of Degree Zero in Product Domains

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The $L^2$-boundedness of the Marcinkiewicz integrals in product domains with component-wise homogeneous kernels which belong to a certain Orlicz space and satisfy the cancellation property is studied. © 2001 Academic Press

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1. INTRODUCTION

Stein [15] defined a higher dimensional analogue of the Marcinkiewicz integral by

$$
\mu_{\Omega f}(x) = \left( \int_0^\infty |F_s f(x)|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}},
$$

where

$$
F_s f(x) = \int_{|x-y|<s} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \, dy,
$$

and $\Omega$ is a homogeneous function of degree zero whose restriction to $S^{n-1}$ belongs to $L^1(S^{n-1})$ and satisfies the cancellation property,

$$
\int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0.
$$

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Here, $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. The continuity of Marcinkiewicz integrals is very useful in harmonic analysis [4–6, 16, 17, 22]. Stein [15] proved that if $\Omega|_{S^{n-1}}$ belongs to the Lipschitz space $\Lambda^\alpha(S^{n-1})$ of order $\alpha (0 < \alpha \leq 1)$, then

$$\int_{S^{n-1}} \Omega(y') \, d\sigma(y') = 0.$$

(1) If $\Omega|_{S^{n-1}} \in L(\log^+ L)^{1/2}(S^{n-1})$, then $\mu_\Omega$ is bounded in $L^2(\mathbb{R}^n)$.

(2) Let $1 < p < \infty$ and let $p'$ be the conjugate of $p$. If $\Omega|_{S^{n-1}} \in L(\log^+ L)^{1/2}(\log^+ \log^+ L)^{3/4}(S^{n-1})$, with $r = \min\{p, p'\}$, then $\mu_\Omega$ is bounded in $L^p(\mathbb{R}^n)$.

Mapping properties of $\mu_\Omega$ on other function spaces were also studied. Torchinsky and Wang [18] considered the weighted $L^p$-boundedness of $\mu_\Omega$ and showed that if $\Omega|_{S^{n-1}} \in \Lambda^\alpha(S^{n-1})$, then for $\omega$ satisfying an $A_p$ condition, $\mu_\Omega$ is bounded on $L^p(\omega)$. Recently, this was extended to rougher kernels by Ding et al. [9]. Mapping properties of $\mu_\Omega$ on BMO or Campanato spaces have been studied in [3, 7, 10, 14, 20].

On the other hand, the Marcinkiewicz integral defined on product domains has also been studied.

To be more specific, let $k \geq 1$ be an integer, $n_1, \ldots, n_k \geq 2$ be integers, and $\Omega : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \rightarrow \mathbb{C}$ be a component-wise homogeneous function of degree zero with the cancellation property:

$$\int_{S^{n_{j-1}}} \Omega(y'_1, \ldots, y'_{j-1}, y'_j, \ldots, y'_k) \, d\sigma(y'_j) = 0 \quad \text{for} \; j = 1, \ldots, k.$$

The Marcinkiewicz integral $\tilde{\mu}_\Omega$ is defined by

$$\tilde{\mu}_\Omega f(x_1, \ldots, x_k) = \left( \int_0^\infty \cdots \int_0^\infty |F_{s_1, \ldots, s_k} f(x_1, \ldots, x_k)|^2 \frac{ds_1 \cdots ds_k}{s_1^\frac{3}{2} \cdots s_k^\frac{3}{2}} \right)^{\frac{1}{2}},$$

where $F_{s_1, \ldots, s_k} f(x_1, \ldots, x_k) = \int_{S^{n_1-1}} \cdots \int_{S^{n_k-1}} f(x_1, \ldots, x_k) \chi_{B_{s_1} \times \cdots \times B_{s_k}}(y) \, d\sigma(y)$ for $s_1, \ldots, s_k > 0$. Recently, this was extended to rougher kernels by Ding et al. [9]. Mapping properties of $\tilde{\mu}_\Omega$ on BMO or Campanato spaces have been studied in [3, 7, 10, 14, 20].
where
\[
F_{s_1,\ldots,s_k} f(x_1,\ldots,x_k) = \int_{|x_1-y_1|\leq s_1} \cdots \int_{|x_k-y_k|\leq s_k} \frac{\Omega(x_1-y_1,\ldots,x_k-y_k)}{|x_1-y_1|^{s_1-1}\cdots|x_k-y_k|^{s_k-1}} x f(y_1,\ldots,y_k) dy_1 \cdots dy_k.
\]

Ding [8] showed the $L^2$-boundedness of $\tilde{\mu}_\Omega$ in case $k=2$ under the hypothesis of
\[
\Omega|_{S^{n-1} \times S^{n-1}} \in L(\log^+ L)^2(S^{n-1} \times S^{n-1}).
\]

In this paper, we improve the above result. Namely, we will prove:

**Theorem 1.2.** Let $k \geq 1$ be an integer, $n_1,\ldots,n_k \geq 2$ be integers, and $\Omega: \mathbb{R}^n_1 \times \cdots \times \mathbb{R}^n_k \to \mathbb{C}$ be a component-wise homogeneous function of degree zero with the cancellation property (1.5). If $\Omega|_{S^{n_1-1} \times \cdots \times S^{n_k-1}} \in L(\log^+ L)^{k/2}(S^{n_1-1} \times \cdots \times S^{n_k-1})$, then the Marcinkiewicz integral operator $\tilde{\mu}_\Omega$ defined by (1.6) is bounded in $L^2(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$.

This paper is organized as follows: in Section 2, elementary properties on Orlicz spaces are discussed and the proof of the main theorem appears in Section 3.

## 2. PRELIMINARIES ON ORLICZ SPACES

Let $\Phi: [0,\infty) \to [0,\infty)$ be a function with the following properties:

1. $\Phi$ is convex;
2. $\Phi(0)=0$; and that
3. $\lim_{t\to \infty} \Phi(t)/t = \infty$.

**Definition 2.1.** (1) For a measurable function $f: \mathbb{R}^n \to \mathbb{C}$, we define $\|f\|_{L^\Phi(\mathbb{R}^n)}$ (the Luxemburg norm) by
\[
\|f\|_{L^\Phi(\mathbb{R}^n)} = \inf \left\{ \lambda \in (0,\infty) : \int_{\mathbb{R}^n} \Phi(\lambda^{-1}|f(x)|) dx \leq 1 \right\}.
\]

(2) The Orlicz space $L^\Phi(\mathbb{R}^n)$ is defined by
\[
\{ f: \mathbb{R}^n \to \mathbb{C} : f \text{ is measurable and } \|f\|_{L^\Phi(\mathbb{R}^n)} < \infty \}.
\]

**Remark 2.2.** (1) $L^\Phi(\mathbb{R}^n)$ is a Banach space.

(2) The following analogue of Hölder’s inequality is available; see, for instance, [22].

**Lemma 2.3.** Let $\Phi: [0,\infty) \to [0,\infty)$ and $\Psi: [0,\infty) \to [0,\infty)$ be Young’s pair in the sense that

1. $\Phi$ and $\Psi$ are convex;
2. $\Phi': [0,\infty) \to [0,\infty)$ and $\Psi': [0,\infty) \to [0,\infty)$ are inverse to each other;

...
If \( f \in L^\Phi(\mathbb{R}^n) \) and \( g \in L^\Psi(\mathbb{R}^n) \), then \( fg \in L^1(\mathbb{R}^n) \) and we have
\[
\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^\Phi(\mathbb{R}^n)} \|g\|_{L^\Psi(\mathbb{R}^n)}.
\]

3. PROOF OF THE MAIN THEOREM

3.1. Simplifying Notations. For simplicity, we will use the following notations:

1. \( I = (1, \ldots, 1), \quad n = (n_1, \ldots, n_k) \in \mathbb{N}^k; \)
2. For \( j \in \{1, \ldots, k\}, \ x \in \mathbb{R}^{n_j}, \) and \( s > 0, \) we write
   \[
   B_j(x, s) = \{y \in \mathbb{R}^{n_j} : |y - x| < s\};
   \]
3. For \( j \in \{1, \ldots, k\}, \ \xi' \in S^{n_j-1}, \) and \( s > 0, \) we write
   \[
   C_{i(j)}(\xi') = \{y' \in S^{n_j-1} : |y' \cdot \xi'| \leq s^{-\frac{j}{2}}\};
   \]
4. For \( s = (s_1, \ldots, s_k) \in (\mathbb{R}_+)^k \) and \( y = (y_1, \ldots, y_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}, \) we write
   \[
   sy = (s_1y_1, \ldots, s_ky_k),
   \]
   \[
   s^{-1}y = (s_1^{-1}y_1, \ldots, s_k^{-1}y_k),
   \]
   \[
   s^m = \prod_{j=1}^k s_j^{m_j}, \quad m = (m_1, \ldots, m_k) \in \mathbb{N}^k,
   \]
   \[
   \tilde{\Omega}_s(y) = \frac{\Omega(y)}{|y_1|^{n_1-1} \cdots |y_k|^{n_k-1}} \chi_{B_1(0, s_1) \times \cdots \times B_k(0, s_k)}(y).
   \]

3.2. A Reduction. We can write
\[
F_s f(x) = \tilde{\Omega}_s \ast f(x).
\]
From the homogeneity of \( \Omega, \) we see
\[
\tilde{\Omega}_s(y) = s^1 [s^n]^{-1} \tilde{\Omega}_1(s^{-1}y).
\]
Thus, we obtain the following formula for the Fourier multiplier of \( F_s: \)
\[
\tilde{\Omega}_s(\xi) = s^1 \tilde{\Omega}_1(s\xi).
\]
Plancherel’s theorem allows us to write
\[ \|\tilde{\mu}_\Omega f\|_{L^2}^2 = \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^k} |F_s f(x)|^2 \frac{ds}{(s^1)^3} \, dx \]
\[ = \int_{(\mathbb{R}^n)^k} \|\tilde{\alpha}_s f\|_{L^2((s^1)^3)}^2 \, ds \]
\[ = \int_{(\mathbb{R}^n)^k} \|\tilde{\alpha}_s f\|_{L^2((s^1)^3)}^2 \, ds \]
\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \tilde{\Omega}_1(s\xi) \right|^2 \frac{ds}{s^1} |f(\xi)|^2 \, d\xi \]
\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \tilde{\Omega}_1(s\xi) \right|^2 \frac{ds}{s^1} |f(\xi)|^2 \, d\xi \]
\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \tilde{\Omega}_1(s\xi) \right|^2 \frac{ds}{s^1} |f(\xi)|^2 \, d\xi. \]

Therefore, it suffices to verify the uniform boundedness of
\[ I(\xi') \equiv \int_{(\mathbb{R}^n)^k} \left| \tilde{\Omega}_1(s\xi') \right|^2 \frac{ds}{s^1} = \int_{(\mathbb{R}^n)^k} \left| \tilde{\Omega}_1(s_1 \xi_1', \ldots, s_k \xi_k') \right|^2 \frac{ds_1 \cdots ds_k}{s_1 \cdots s_k} \]
in \( \xi' = (\xi_1', \ldots, \xi_k') \in S^{n-1} \times \cdots \times S^{n-1}. \)

3.3. Estimates on \( I(\xi'). \) We let
\[ R_0 = (0, 1), \quad R_1 = (1, \infty), \]
and write
\[ I(\xi') \equiv \sum_{\alpha_1, \ldots, \alpha_k = 0, 1} I_{\alpha_1, \ldots, \alpha_k}(\xi'), \]
with
\[ I_{\alpha_1, \ldots, \alpha_k}(\xi') = \int_{R_{\alpha_1} \times \cdots \times R_{\alpha_k}} \left| \tilde{\Omega}_1(s\xi') \right|^2 \frac{ds}{s^1}. \]

It suffices to verify the uniform boundedness of \( I_{\alpha_1, \ldots, \alpha_k}(\xi') \) in \( \xi' \in S^{n-1} \times \cdots \times S^{n-1}, \) for each choice of \( \alpha_1, \ldots, \alpha_k = 0, 1. \) Write
\[ J = \{1, \ldots, k\}, \]
and let
\[ J_0 = \{j \in \{1, \ldots, k\} : \alpha_j = 0\} \]
\[ J_1 = \{j \in \{1, \ldots, k\} : \alpha_j = 1\}. \]
By the cancellation property and homogeneity condition, one can see
\[
\left| \hat{\Omega}_l(s\xi') \right| = \left| \int_{\prod_{l \in J} B_{1}(0,1) \cap \left[ \prod_{l \in J} e^{-2\pi i l \xi_l y_l - 1} \right] \right|
\prod_{l \in J} e^{-2\pi i l \xi_l y_l - 1} \Omega_l(y_1, \ldots, y_k) \prod_{l \in J} dy_l \right| 
\leq \int_{\prod_{l \in J} B_{1}(0,1) \cap \left[ \prod_{l \in J} e^{-2\pi i l \xi_l y_l - 1} \right] \right| \left| \Omega_l(y_1, \ldots, y_k) \prod_{l \in J} dy_l \right| \prod_{l \in J} dy_l.
\]

Switching to polar coordinates, we obtain
\[
\left| \hat{\Omega}_l(s\xi') \right| \leq \prod_{s \in J_0} s_r \int_{\prod_{l \in J_0} [s_{\xi_l} y_l, \xi_l]} \left| \sin \pi s_{\xi_l} y_l \cdot s_{\xi_l} \xi_l \right| \Omega_l(y_1, \ldots, y_k) \prod_{l \in J} d\sigma(y'_l).
\]
Observe that for each \( l \in J_1, \)
\[
\left| \sin \pi s_{\xi_l} y'_l \cdot s_{\xi_l} \xi'_l \right| \leq s_{\xi_l}^{-1} + \chi_{\mathcal{C}_\nu(\xi)}(y'_l)
\]
holds.

Therefore, it suffices to estimate integrals of the form
\[
\tilde{T}_{j_0, j_1^{(1)}, j_1^{(2)}}(s\xi') = \int_{\prod_{l \in J_0} R_{l_0}} \int_{\prod_{l \in J_1^{(1)}} R_l} \mathcal{F}(\tilde{\xi}) \left[ \prod_{l \in J_1^{(2)}} \frac{d\tilde{s}_l}{\tilde{s}_l^2} \right] \prod_{l \in J_0} s_{\xi_l} d\sigma,
\]
with
\[
\mathcal{F}(\tilde{\xi}) = \int_{\prod_{l \in J_1^{(2)}} R_l} \left[ \prod_{l \in J_1^{(2)}} \chi_{\mathcal{C}_\nu(\xi)}(y'_l) \right] \Omega_l(y'_1, \ldots, y'_k) \prod_{l \in J_1^{(2)}} \frac{d\tilde{s}_l}{\tilde{s}_l},
\]
where \( \{J_{0}, J_1^{(1)}, J_1^{(2)}\} \) is any partition of \( J \). An application of Minkowski’s inequality on integrals gives us
\[
\mathcal{F}(\tilde{\xi})^2 \leq \int_{\prod_{l \in J_1^{(2)}} R_l} \left[ \prod_{l \in J_1^{(2)}} \left( \log \frac{1}{|y'_l|} \right) \right] \Omega_l(y'_1, \ldots, y'_k) \prod_{l \in J_1^{(2)}} d\sigma(y'_l).
\]
We have the following lemma:

**Lemma 3.1.** Let \( K \geq 1 \) be an integer, \( n_1, \ldots, n_k \geq 2 \) be integers, \( \xi_1 \in S^{n_1-1}, \ldots, \xi_K \in S^{n_K-1} \), and \( \Phi, \Psi : (0, \infty) \to (0, \infty) \) be a Young’s pair with
\[
\Phi(u) = u \left( \log^+ u \right)^{\frac{K}{2}}, \quad \text{for large } u > 0.
\]
Then
\[
\prod_{j=1}^{K} \left( \log \frac{1}{|y'_j \cdot \xi'_j|} \right)^{\frac{1}{2}} \in L^\Psi(S^{n-1} \times \cdots \times S^{n_k-1}),
\]
and moreover there exists a constant \(C(K)\) such that
\[
\left\| \prod_{j=1}^{K} \left( \log \frac{1}{|y'_j \cdot \xi'_j|} \right)^{\frac{1}{2}} \right\|_{L^\Psi(S^{n-1} \times \cdots \times S^{n_k-1})} \leq C
\]
holds uniformly in \(\xi'_1, \ldots, \xi'_K \in S^{n-1} \times \cdots \times S^{n_k-1}\).

**Proof of Lemma 3.1.** Observe that there exists a constant \(C\) such that for large \(u > 0\)
\[
\Psi(u) \lesssim \exp\left( Cu^{\frac{k}{2}} \right)
\]
holds. For \(\lambda \geq 1\), we have
\[
\Psi\left( \lambda^{-1} \left( \prod_{j=1}^{K} \log \frac{1}{|y'_j \cdot \xi'_j|} \right)^{\frac{1}{2}} \right)
\lesssim \exp\left[ C \lambda^{-\frac{k}{2}} \left( \prod_{j=1}^{K} \log \frac{1}{|y'_j \cdot \xi'_j|} \right)^{\frac{1}{2}} \right] \lesssim \exp\left[ CK^{-1} \lambda^{-\frac{k}{2}} \sum_{j=1}^{K} \log \frac{1}{|y'_j \cdot \xi'_j|} \right]
\lesssim \prod_{j=1}^{K} \exp\left[ CK^{-1} \lambda^{-\frac{k}{2}} \log \frac{1}{|y'_j \cdot \xi'_j|} \right] = \prod_{j=1}^{K} |y'_j \cdot \xi'_j|^{-CK^{-1} \lambda^{-\frac{k}{2}}}
\]
from the geometric-arithmetic mean inequality and (3.1).

Uniform integrability of \(\Psi(\lambda^{-1} \prod_{j=1}^{K} \log(1/|y'_j \cdot \xi'_j|))\) on \(S^{n-1} \times \cdots \times S^{n_k-1}\) is clear and the lemma is proved.

Applying Hölder’s inequality (Lemma 2.3) followed by Lemma 3.1, we obtain a uniform estimate on \(\mathcal{F}(\hat{s})\).

Therefore, we obtain the uniform boundedness of \(I(\xi')\) and the proof of the theorem is now complete.

**Remark 3.2.** In view of the sharpness of Walsh’s result, one can see that the size condition on \(\Omega|S^{n-1} \times \cdots \times S^{n_k-1}\) is sharp.
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