NORTH-HOLLAND

On the Problem of Consistent Marking of a Graph

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Dedicated to Professor John Maybee on the occasion of his 65th birthday.

Submitted by J. Richard Lundgren

ABSTRACT

A marked graph is a graph in which each vertex is given a sign + or −. We call such a graph consistent if every cycle has an even number of − signs. Consistent marked graphs arise in the study of communication networks and social networks. We discuss the problem of characterizing graphs that can be consistently marked using at least one − sign, reduce the problem to blocks, and solve it for blocks whose longest cycle has length at most 5.

Consider a *marked graph*, a graph in which each vertex is given a sign, + or −. We call a marked graph *consistent* if every cycle[1] has an even number of − signs. In this paper, we discuss the problem of characterizing those graphs that can be consistently marked using at least one − sign, reduce the problem to blocks, and solve it for blocks whose longest cycle has length at most 5.

Consistent marked graphs were introduced by Bieneke and Harary (1978a), and the analogous concept for marked digraphs was introduced by Bieneke and Harary (1978b). Suppose a marked graph is thought of as

[1] In this paper, paths and cycles have no repeated vertices. We adopt the graph-theoretical terminology of Bondy and Murty (1976).

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a model of a communication network. Suppose binary messages are sent through that network, with vertices having sign $-$ reversing the messages and vertices having sign $+$ leaving them intact. Then a consistent marked graph has an important consistency property: If a message is sent from x to y through two different paths, y will receive the same message no matter which path is followed. The notion of consistency also arises in the study of social networks, for example where the vertices represent people who always lie or who always tell the truth. [See Bieneke and Harary (1978b) and Harary (1983).] The notion of consistency of a marked graph has proven useful in the theory of balance of graphs with signs on edges, *signed graphs.* Specifically, by setting up a correspondence between marked graphs and *balanced signed graphs,* signed graphs where every cycle has an even number of $-$ edges, Harary and Kabell (1980, 1981) were able to describe an efficient algorithm for determining if a given signed graph is balanced. [Balanced signed graphs have been widely studied and have a variety of interesting applications in sociology, psychology, political science, economics, energy modeling, and discrete optimization. See Johnsen (1989) and Roberts (1989) for references.] The correspondence between marked graphs and balanced signed graphs has also been useful in solving the problem of counting balanced signed graphs (Harary and Kabell, 1981); the problems of enumerating both balanced signed graphs and marked graphs are also studied by Harary, Palmer, Robinson, and Schwenk (1977).

The problem of characterizing consistent marked digraphs was solved by Beineke and Harary (1978b). Rao (1984) obtained an early characterization of consistent marked graphs and also gave a polynomial algorithm for recognizing them. Other characterizations were given by Acharya (1983, 1984). The recent paper by Hoede (1992) characterizes consistent marked graphs in terms of fundamental cycles of a cycle basis and observes that the characterization gives rise to a polynomial algorithm for determining whether a marked graph is consistent that seems simpler than that of Rao.

Given an unmarked graph or digraph, it can always be marked in a consistent way, by giving all vertices a $+$ sign. However, this cannot always be done if some $-$ sign must be used. Consider for example the complete graph K_4. We call a graph or digraph *markable* if its vertices can be given signs in a consistent way with at least one $-$ sign. Beineke and Harary (1978b) solve the problem of characterizing markable digraphs. However, the same problem for graphs is still unsolved. It is the purpose of this paper to present this problem and some results about it. Specifically, we describe all the markable blocks with no cycle of length greater than 5.

LEMMA 1 (Beineke and Harary, 1978a). *In a consistent marking of a graph, no pair of vertices of opposite signs can be joined by three pairwise* *internally vertex-disjoint paths.*

Proof. Suppose vertex a is signed $-$, vertex b is signed $+$, and P_1, P_2, P_3 are three paths from a to b intersecting only at a and b. By consistency of the marking, either P_1 or P_2, say P_1, has an odd number of internal vertices of sign $-$, and the other has an even number of such vertices. Then, by considering P_1 and P_3, we conclude that P_3 also has an even number of internal vertices of sign $-$. It follows that P_2 and P_3 form a cycle with an odd number of $-$ signs, which is a contradiction. ∎

We say that a connected graph G is *n-connected* if the minimum number of vertices whose deletion disconnects G or results in a graph with one vertex is at least n. By Lemma 1, we have the following result.

PROPOSITION 1. *If a graph G is 3-connected, then G is markable if and only if it is bipartite.*

Proof. By Menger's theorem [see for example Bondy and Murty (1976)], for each $x \neq y$ from $V(G)$, there are three internally vertex-disjoint paths from x to y. Thus, by Lemma 1, every pair x and y must have the same sign, and this sign must be $-$. It follows that G is markable if and only if it has no odd cycles. ∎

It follows from Proposition 1 that K_n is markable if and only if $n < 4$.

By virtue of Proposition 1, we may concentrate on graphs that are not 3-connected. Note that a graph is 2-connected if and only if it is a block consisting of more than one edge, where a *block* is a connected graph with more than one vertex and no cutpoints. A block in a graph is a maximal subgraph that is a block. A graph is markable if and only if every connected component is markable. Moreover, a graph G is markable if every block of G is markable. This is trivial if G is a block and in general follows by induction on the number of blocks. For, suppose that G is not a block, let v be a cutpoint, let $H_1, H_2, \dots, H_p$ be the connected components of $G - v$, and let G_i be the subgraph generated by vertices of H_i and v. Since all blocks of G_i are blocks of G, the number of blocks of G_i is less than the number of blocks of G, and by the inductive hypothesis we can find for each G_i a consistent marking with a $-$. If all of these markings have the same sign at v, then we combine them to get a consistent marking of G with a $-$ sign. Otherwise, we find some G_i in which v gets a $+$ sign in its consistent marking and there is a vertex w in G_i which gets a $-$ sign. Then we use this marking of G_i and mark all G_j, $j \neq i$, with all $+$ signs. This gives a consistent marking of G with a $-$ on w. Thus, if all blocks are

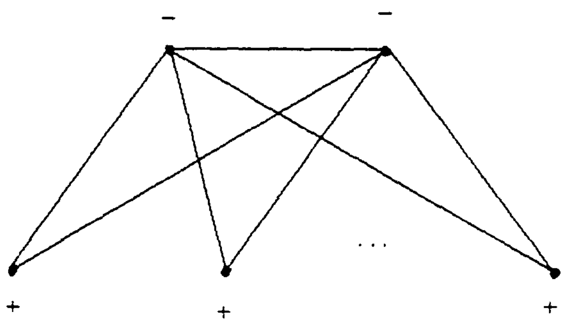

FIG. 1. $K(2,n)+e_2$ and a consistent marking with a $-$.

markable, then so is G.

The converse is false. Consider two K_4's attached at one vertex, and remove an edge incident to that vertex from one of the K_4's. Then we get a graph that is markable but has a nonmarkable block K_4.

In this paper, we consider markability of blocks. We leave open the question of markablility of connected graphs that are not blocks.

In particular, we shall describe all of the markable blocks whose longest cycle has length at most 5. To give this description, let us use the notation $K(m,n)$ for the complete bipartite graph with classes of sizes m and n. Then we define $K(2,n)+e_2$ to be the graph obtained from $K(2,n)$ by adding an edge between the two vertices in the class of two vertices. $K(2,n)+e_2$ is shown in Figure 1. We define the graph $J(n,p)$ as follows. Start with a 4-cycle whose vertices in order are a,b,c,d. Add p vertices adjacent to a and d, and n vertices adjacent to a and c. The graph $J(n,p)$ is shown in Figure 2. Also, we define the graph $L(m,n)$ as follows. We start with a 5-cycle whose vertices in order are a,b,c,d,e. We add m vertices adjacent to a and c, and n vertices adjacent to c and e. The graph $L(m,n)$ is shown in Figure 3.

THEOREM 1. *Suppose that G is a block with no cycle of length greater than* 5. *Then G is markable if and only if G is K_2, K_3, $K(2,n)$, $n \geq 2$, $K(2,n)+e_2$, $n \geq 2$, $J(n,p)$, $n \geq 0$, $p \geq 1$, or $L(m,n)$, $m,n \geq 0$.*

To prove Theorem 1, we note that, trivially, K_2 and K_3 are; markable; that $K(2,n)$, $n \geq 2$, is markable can be seen by putting a $-$ on all vertices in one class and a $+$ on all vertices in the other class; that $K(2,n)+e_2$, $n \geq 2$, $J(n,p)$, $n \geq 0$, $p \geq 1$, and $L(m,n)$, $m,n \geq 0$, are markable is shown in Figures 1, 2, and 3, respectively. To prove the converse, we introduce a series of very simple lemmas which are nevertheless worth stating because

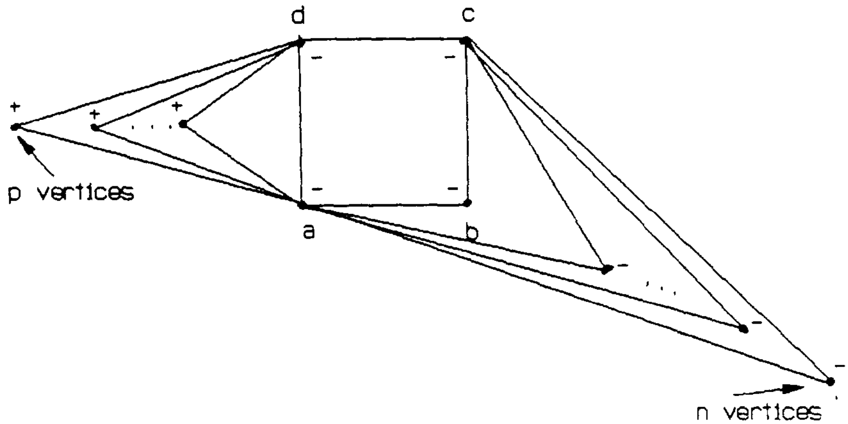

FIG. 2. $J(n,p)$ and a consistent marking with a $-$.

we use them so often.

Suppose that C is a cycle of graph G and u, v are vertices of C. A *u, v-handle* is a path from u to v in G that does not use any vertices of C other than u and v.

LEMMA 2. *Suppose that G is a block, C is a cycle in G, and x is a vertex not in C. Then there are $u \neq v$ in C such that x is an internal vertex of a u, v-handle.*

Proof. By Menger's theorem, there are two internally vertex-disjoint paths P and Q from x to C. Let P' be the part of P from x to the first

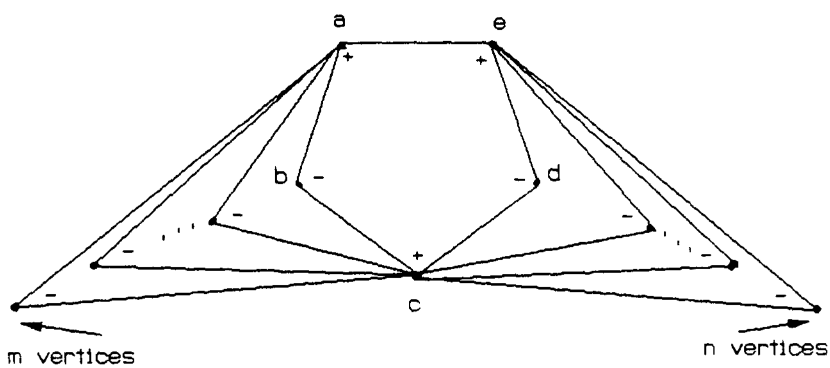

FIG. 3. $L(m,n)$ and a consistent marking with a $-$.

vertex a on P that is on C, and Q' be the part of Q from x to the first vertex b on Q that is on C. If $a \neq b$, then we let $u = a$, $v = b$, and form the handle from the two paths P' and Q'. If $a = b$, then since $G - a$ is connected, there is a path R from x to a vertex w of C different from a and such that R has no other vertices of C. Since x is on both R and $P' \cup Q'$, there is a last vertex y of R that is on both R and $P' \cup Q'$. Without loss of generality, suppose that y is on Q'. Now let S be the path that uses Q' from x to y and then R from y to w. We take $u = a$, $v = w$, and use the two internally vertex-disjoint paths P' and S to make the handle. ■

If C is a cycle of a marked graph G and u, v are two vertices of C, then $d_C(u, v)$ will denote the shortest distance along the cycle between u and v. We let c be the length of a longest cycle of G having a $-$ sign, if there is such a cycle.

LEMMA 3. *If a cycle C in a consistently marked block G has length c and has a $-$ sign, then if $d_C(u, v) \leq 2$, any u, v-handle has length at most $d_C(u, v)$.*

Proof. Consider a path along C between u and v and having distance $d_C(u, v)$, and let $C(u, v)$ be the other path between u and v along C. Since $d_C(u, v) \leq 2$ and C has at least two $-$ signs, it follows that $C(u, v)$ contains a $-$ sign. Hence, if a u, v-handle has length $> d_C(u, v)$, the handle plus $C(u, v)$ is a cycle of length $> c$ and having a $-$ sign. ■

LEMMA 4. *If a cycle C in a consistently marked block G has length c and has a $-$ sign, then no x outside C is adjacent to two adjacent vertices in C.*

Proof. This follows directly from Lemma 3. ■

LEMMA 5. *If a cycle C in a consistently marked block G has length $c \leq 5$ and has a $-$ sign, then no two vertices x and y outside C can be adjacent.*

Proof. By Lemma 2, x is an internal vertex of a u, v-handle for $u \neq v$. Since $c \leq 5$, $d_C(u, v) \leq 2$. By Lemma 3, the u, v-handle has length $\leq d_C(u, v) \leq 2$. Hence, x is adjacent to u and v. Similarly, y is adjacent to u' and v', $u' \neq v'$, on C. Then either $u' \neq u$ or $v' \neq u$, say without loss of generality $u' \neq u$. Now if x is adjacent to y, then u, x, y, u' is a u, u'-handle of length $> 2 \geq d_C(u, u')$, which contradicts Lemma 3. ■

We are now ready to complete the proof of Theorem 1. If G has no cycles, then G is K_2. Suppose that G has a cycle and that G has a consistent marking using a $-$ sign. Since G is 2-connected, every vertex is on a cycle. Hence, we may find a cycle C with a $-$ sign and having length c. We consider the three possibilities that $c = 3$, 4, or 5.

Suppose that $c = 3$. Then for all $u \neq v$ in C, $d_C(u, v) = 1$. Hence, by Lemma 3 there can be no u, v-handle. It follows by Lemma 2 that there can be no vertices outside C, and hence that G is K_3.

Suppose next that $c = 4$. Let the vertices in order around C be a, b, c, d. By Lemma 5, any two vertices outside C are nonadjacent. Suppose that x is not in C. Then by Lemma 2, x is an internal vertex of a u, v-handle, and by Lemma 3 this handle has length at most $d_C(u, v) \leq 2$. Hence, x is adjacent to u and v, and by Lemma 4, u is not adjacent to v. It follows that x is adjacent to exactly two vertices of C, u and v, else x is adjacent to two adjacent vertices of C. If x is adjacent to a and c, and y to b and d, then a, x, c, d, y, b, a is a cycle of length 6, which is impossible. Thus, either all vertices outside C are adjacent to a and c or all are adjacent to b and d.

Since K_4 is not markable, the subgraph $G[C]$ generated by vertices of C is either a cycle or a cycle with one chord. We consider both of these possibilities. Suppose first that $G[C]$ is a 4-cycle. Then since either all vertices outside C are adjacent to a and c or all are adjacent to b and d, and since no two such x can be adjacent, it follows that G is $K(2, n)$, $n \geq 2$.

Suppose next that $G[C]$ is a 4-cycle with a chord, say b to d. If all x outside C are adjacent to a and c, we get a 5-cycle with a $-$ sign, namely d, a, x, c, b, d. This contradicts the hypothesis $c = 4$. Thus, all x outside C are adjacent to b and d. Then, since all such x's have to be nonadjacent, we conclude that G is $K(2, n) + e_2$, $n \geq 2$.

Finally, suppose that $c = 5$. Let the vertices in order around C be a, b, c, d, e. As in the case $c = 4$, we conclude that no two x outside of C can be adjacent and that every such x is adjacent to exactly two, nonadjacent vertices of C.

Let $G[C]$ be the subgraph of G generated by vertices of C. We argue that $G[C]$ cannot have two or more chords. If there are two or more chords, then by symmetry either $\{a, c\}$ and $\{a, d\}$ are chords or $\{a, d\}$ and $\{b, e\}$ are chords. In the former case, by Lemma 1, a and c have the same sign, and a and d have the same sign. Hence, since these three vertices form a triangle, that sign must be $+$. It follows that b and e must both be $+$, else there are triangles with an odd number of $-$ signs. Hence, all five vertices of C get sign $+$. But by hypothesis, C has a $-$ sign. We have reached a contradiction. Suppose that C has chords $\{a, d\}$ and $\{b, e\}$. Lemma 1 allows us to conclude that a and d have the same sign and that b and e have the same sign. Since a, b, e and a, d, e are triangles, all four vertices have the same sign, and hence all five vertices do. The sign must be $+$. Again, this is a contradiction. Hence, $G[C]$ has at most one chord.

If there is any x outside of C, then we may suppose without loss of generality that some x outside of C is adjacent to a and c in C. Then we cannot have y outside of C adjacent to b and d, for otherwise a, x, c, d, y, b, a is a cycle of length 6. Similarly, we cannot have y outside of C adjacent to b and e. It follows that any other y outside C that is not adjacent to a and c is adjacent to c and e or to a and d. By the same reasoning as above, we cannot have both y outside C adjacent to c and e and z outside C adjacent to a and d. Thus, without loss of generality, there are at most two types of vertices outside of C, those adjacent to a and c (type 1) and those adjacent to c and e (type 2).

We know that $G[C]$ is either a 5-cycle or a 5-cycle with one chord. In the former case, G is $L(m,n)$, $m,n \geq 0$. Consider the latter case. We need only consider the possible chords $\{c,e\}$, $\{b,d\}$, or $\{b,e\}$, since the case of chord $\{a,c\}$ is analogous to that of chord $\{c,e\}$ and the case of chord $\{a,d\}$ is analogous to that of chord $\{b,e\}$. If the chord is $\{c,e\}$, then G is $J(n,p)$, $n \geq 0$, $p \geq 1$.

Suppose next that the chord is $\{b,d\}$. If x is a type-1 vertex outside of C then a, e, d, b, c, x, a is a cycle of length 6, which is impossible. If y is a type-2 vertex outside of C, then e, a, b, d, c, y, e is a cycle of length 6, which is impossible. Hence, we conclude that $G = G[C]$, which is $J(0,1)$.

The remaining case is when the chord is $\{b,e\}$. If there is a type-1 vertex x not in C, then a, x, c, d, e, b, a is a cycle of length 6, which is impossible. Hence, there are only type-2 vertices not in C. It follows that G is $J(n,1)$, $n \geq 0$. This concludes the proof of Theorem 1.

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