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Chart description for genus-two Lefschetz fibrations and a theorem on their stabilization

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ABSTRACT

Chart descriptions are a graphic method to describe monodromy representations of various topological objects. Here we introduce a chart description for genus-two Lefschetz fibrations, and show that any genus-two Lefschetz fibration can be stabilized by fiber-sum with certain basic Lefschetz fibrations.

1. Introduction

Chart descriptions were originally introduced in order to describe 2-dimensional braids in [5,6] (cf. [7]). In [9], a chart description for genus-one Lefschetz fibrations was introduced and an elementary proof of Matsumoto's classification theorem was given. At the third JAMEX meeting in Oaxaca, Mexico, 2004, the author generalized it to a method describing any monodromy representation [8]. Here we introduce a chart description for genus-two Lefschetz fibrations, and show that any genus-two Lefschetz fibration can be stabilized by fiber-sum with certain basic Lefschetz fibrations. Our result was partially announced at 'The Second East Asia School of Knots and Related Topics in Geometric Topology' in Dalian, China, 2005.

2. Lefschetz fibrations

Let M and B be compact, connected, and oriented smooth 4-manifold and 2-manifold, respectively. Let $f : M \to B$ be a smooth map with $\partial M = f^{-1}(\partial B)$. A critical point p is called a *Lefschetz singular point* of *positive type* (or of *negative type*, respectively) if there exist local complex coordinates z_1, z_2 around p and a local complex coordinate ξ around $f(p)$ such that f is locally written as $\xi = f(z_1, z_2) = z_1 z_2$ (or $\bar{z}_1 z_2$, resp.). We call f a (smooth or differentiable) *Lefschetz fibration* if all critical points are Lefschetz singular points and if there exists exactly one critical point in the preimage of each critical value.

A *general fiber* is the preimage of a regular value of f. A *singular fiber* of *positive type* (or *negative type*, resp.) is the preimage of a critical value which contains a Lefschetz singular point of positive type (or negative type, resp.). A singular fiber is obtained by shrinking a simple loop, called a vanishing cycle, on a general fiber. In this paper we assume that a Lefschetz fibration is 'relatively minimal', i.e., all vanishing cycles are essential loops. We say that a singular fiber is of *type* I or of *type* II if the vanishing cycle is a non-separating loop or a separating loop, respectively.

A singular fiber is of *type* I^+ if it is of type I and of positive type. Similarly *type* I^-, *type* II^+ *type* II^- are defined. We denote by $n^+_{\mathrm{I}}(f)$, $n^-_{\mathrm{I}}(f)$, $n^+_{\mathrm{II}}(f)$, and $n^-_{\mathrm{II}}(f)$, the numbers of singular fibers of f of type I^+, I^-, II^+, and II^-, respectively. A Lefschetz fibration is called *irreducible* if every singular fiber is of type I, i.e., $n^+_{\mathrm{II}}(f) = n^-_{\mathrm{II}}(f) = 0$. A Lefschetz fibration is called *chiral* or *symplectic* if every singular fiber is of positive type, i.e., $n^-_{\mathrm{I}}(f) = n^-_{\mathrm{II}}(f) = 0$.

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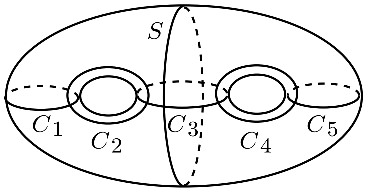

Fig. 1. Loops.

Let $f : M \to B$ be a Lefschetz fibration, and $\Delta = \{q_1, \ldots, q_n\}$ the set of critical values. Let $\rho : \pi_1(B \setminus \Delta, q_0) \to MC$ be the monodromy representation of f, where q_0 is a base point of $B \setminus \Delta$ and MC is the mapping class group of the fiber $f^{-1}(q_0)$. Consider a Hurwitz arc system for Δ, say $\mathcal{A} = (A_1, \ldots, A_n)$; each A_i is an embedded arc in B connecting q_0 and a point of Δ such that $A_i \cap A_j = \{q_0\}$ for $i \neq j$, and they appear in this order around q_0. When B is a 2-sphere or a 2-disk, the system $\mathcal{A}$ determines a system of generators of $\pi_1(B \setminus \Delta, q_0)$, say $(a_1, \ldots, a_n)$. We call $(\rho(a_1), \ldots, \rho(a_n))$ a *Hurwitz system* of f. For details on Lefschetz fibrations and Hurwitz systems, refer to [1,4,10–17], etc.

3. Main result

Let ζ_i $(i = 1, \ldots, 5)$ be positive Dehn twists along the loops C_i $(i = 1, \ldots, 5)$ illustrated in Fig. 1. The mapping class group MC of a genus-two Riemann surface is generated by $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5$, and the following relations are defining relations (cf. [3]).

$$\zeta_i\zeta_j = \zeta_j\zeta_i \quad \text{if } |i-j| \geqslant 2, \tag{1}$$

$$\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1} \quad \text{for } i = 1, \ldots, 4, \tag{2}$$

$$\iota^2 = 1 \quad \text{where } \iota = \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1, \tag{3}$$

$$(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5)^6 = 1, \tag{4}$$

$$\iota\zeta_i = \zeta_i\iota \quad \text{for } i = 1, \ldots, 5. \tag{5}$$

Let σ be a positive Dehn twist along the loop S illustrated in Fig. 1. Then $\sigma = (\zeta_1\zeta_2)^6$.

If $(g_1, \ldots, g_n)$ is a Hurwitz system of a genus-two Lefschetz fibration, then each g_j is a conjugate of ζ_i or ζ_i^{-1}, or a conjugate of σ or σ^{-1}.

Now we define basic Lefschetz fibrations.

Definition 1. ([1,2,13,17]) *Basic Lefschetz fibrations*, f_0, f_1, f_2, f_1' and f_2', are Lefschetz fibrations over S^2 whose Hurwitz systems are

(1) $W_0 = (T)^2$ where $T = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1)$,
(2) $W_1 = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^6$,
(3) $W_2 = (\sigma, (\zeta_3, \zeta_4, \zeta_5, \zeta_2, \zeta_3, \zeta_4, \zeta_1, \zeta_2, \zeta_3)^2, T)$,
(4) $W_1' = (\zeta_1, \zeta_1^{-1})$,
(5) $W_2' = (\sigma, \sigma^{-1})$,

respectively.

For example, f_0 has 20 singular fibers, which are of type I^+. Thus f_0 is chiral and irreducible.

LF	# of sing. fib.				Chiral	Irreducible
	n_{I}^+	n_{I}^-	n_{II}^+	n_{II}^-		
f_0	20	0	0	0	○	○
f_1	30	0	0	0	○	○
f_2	28	0	1	0	○	×
f_1'	1	1	0	0	×	○
f_2'	0	0	1	1	×	×

For two Lefschetz fibrations f and f' over S^2, we denote by $f \# f'$ the fiber-sum of f and f'. By $\#mf$ for a positive integer m, we mean the fiber-sum of m copies of f.

Theorem 2. *Let f be a genus-two Lefschetz fibration over S^2. Suppose that $n_{\mathrm{II}}^+(f) \geqslant n_{\mathrm{II}}^-(f)$. Then*

(1) $\mathcal{E}(f) := n_{\mathrm{I}}^+(f) - n_{\mathrm{I}}^-(f) - 28(n_{\mathrm{II}}^+(f) - n_{\mathrm{II}}^-(f))$ *is a multiple of* 10.

(*Hence we can define the parity,* $\epsilon(f) \in \{0, 1\}$, *by* $\epsilon(f) \equiv \mathcal{E}(f)/10 \bmod 2$.)

(2) *There exists a positive integer m_0 such that for any integer $m \geqslant m_0$,*

$$f \# m f_0 \cong \#(a+m) f_0 \# b f_1 \# c f_2 \# d f_1' \# e f_2'$$

for some non-negative integers a, b, c, d and e.

(3) *In* (2), *it holds that $c = n_{\mathrm{II}}^+(f) - n_{\mathrm{II}}^-(f)$, $d = n_{\mathrm{I}}^-(f)$ and $e = n_{\mathrm{II}}^-(f)$. Although a and b are not determined uniquely, we can take $b = \epsilon(f) \in \{0, 1\}$, and then $a = (\mathcal{E}(f) - 30\epsilon(f))/20$.*

(4) *If $n_{\mathrm{II}}^-(f) = 0$, then we may take m_0 in* (2) *to be $n_{\mathrm{I}}^-(f) + 2n_{\mathrm{II}}^+(f) + 1$.*

Remark 3. If f is chiral and irreducible, then $n_{\mathrm{I}}^-(f) = n_{\mathrm{II}}^+(f) = n_{\mathrm{II}}^-(f) = 0$ and by (4) we may assume $m_0 = 1$. Thus, we have

$$f \# f_0 \cong \#(a+1) f_0 \# b f_1.$$

This is due to B. Siebert and G. Tian [17]. Our proof concerning the assertion (4) of Theorem 2 is based on their result. In Section 7, we observe that Theorem 2 except the assertion (4) can be proved without Siebert and Tian's result.

If f is chiral, then $n_{\mathrm{I}}^-(f) = n_{\mathrm{II}}^-(f) = 0$. By Theorem 2, we have

$$f \# m f_0 \cong \#(a+m) f_0 \# b f_1 \# c f_2.$$

This is due to D. Auroux [1]. Here m is any integer with $m \geqslant 2n_{\mathrm{II}}^+(f) + 1$.

4. Chart description

In this section we introduce a chart description for genus-two Lefschetz fibrations. We use the terminologies on chart description in [8]. For simplicity's sake, we only consider genus-two Lefschetz fibrations over B such that ∂B is empty or connected, and if ∂B is not empty, we assume that the monodromy along ∂B is trivial. Unless otherwise stated, genus-two Lefschetz fibrations over B are assumed to be so.

Definition 4. ([7–9]) A *chart* in B is a finite graph Γ in B (possibly being empty or having *hoops* that are closed edges without vertices) whose edges are labeled with an element of $\{1, 2, 3, 4, 5, \sigma\}$, and oriented so that the following conditions are satisfied (see Fig. 2):

(1) The degree of each vertex is 1, 4, 6, 20, 30, 22 or 13.
(2) For a degree-1 vertex, the adjacent edge is oriented outward or inward.
(3) For a degree-4 vertex, two edges in each diagonal position have the same label and are oriented coherently; and the labels i and j of the diagonals are in $\{1, \dots, 5\}$ with $|i - j| > 1$.
(4) For a degree-6 vertex, the six edges are alternately labeled i and j in $\{1, \dots, 5\}$ with $|i - j| = 1$; and three consecutive edges are oriented outward while the other three are oriented inward.
(5) For a degree-20 vertex, the edges are labeled with $(1, 2, 3, 4, 5, 5, 4, 3, 2, 1)^2$; and all edges are oriented outward or all edges are oriented inward.
(6) For a degree-30 vertex, the edges are labeled with $(1, 2, 3, 4, 5)^6$ in a counterclockwise direction (or clockwise direction, resp.); and all edges are oriented outward (or inward, resp.).
(7) For a degree-22 vertex, the edges are labeled with $(1, 2, 3, 4, 5, 5, 4, 3, 2, 1, i)^2$ in a counterclockwise direction where $i \in \{1, \dots, 5\}$; and the first 11 edges are oriented outward and the latter ones are oriented inward.
(8) For a degree-13 vertex, the edges are labeled with $((1, 2)^6, \sigma)$ in a counterclockwise direction (or clockwise direction, resp.); and the edges with labels 1 and 2 are oriented outward (or inward, resp.), and the edge with label σ is oriented inward (or outward, resp.).
(9) $\Gamma \cap \partial B = \emptyset$.
(10) Γ misses the base point $q_0 \in B$.

Remark 5. When we would treat genus-two Lefschetz fibrations over B with $\partial B \neq \emptyset$ such that the monodromies along ∂B are not trivial, the condition (9) should be removed. See [8].

We call a degree-1 vertex a *black vertex*. We say that a chart is *chiral* if every black vertex has an adjacent edge oriented outward. We say that a chart is *irreducible* if there exist no edges with label σ.

For a chart Γ, let Δ_Γ be the set of black vertices. A chart Γ determines a homomorphism $\pi_1(B \setminus \Delta_\Gamma, q_0) \to MC$ as in [8]. By Theorem 5 of [8], we have the following theorem.

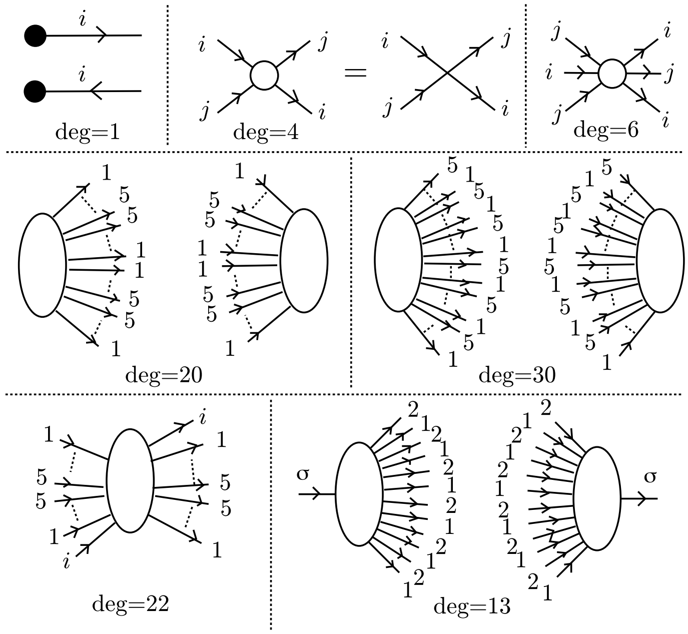

Fig. 2. Vertices of a chart.

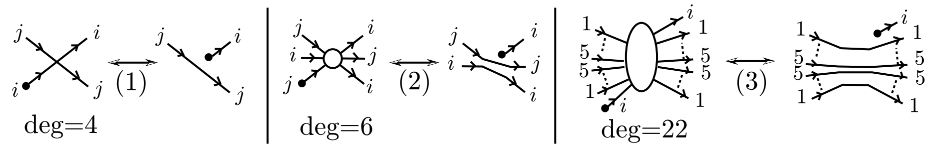

Fig. 3. Some chart moves.

Theorem 6. *Let f be a genus-two Lefschetz fibration over B, and let ρ be the monodromy representation. Then there is a chart Γ in B such that the monodromy representation ρ equals the homomorphism ρ_Γ determined by Γ.*

A chart Γ as in Theorem 6 is called a *chart description* of f or a *chart describing* f. A chart Γ in D^2 is also regarded as a chart in S^2 in the trivial way.

We introduce some local moves on chart descriptions.

(C1) For a chart Γ, suppose that there exists a chart Γ' and an embedded 2-disk, say E, in B such that (i) ∂E intersects with Γ and Γ' transversely (or do not intersect with them) avoiding their vertices, (ii) Γ and Γ' have no black vertices in E, and (iii) Γ and Γ' are identical outside of E. Then we say that Γ' is obtained from Γ by a C1-move.

(C2) For a chart, suppose that there is an edge e joining a degree-4 vertex and a black vertex. Remove the edge e as in Fig. 3(1). We call this local move a C2-move.

(C3) For a chart, suppose that there is an edge e joining a degree-6 vertex and a black vertex. Suppose that e is neither the middle of three edges oriented outward nor the middle of the three edges oriented inward. Then, remove the edge as in Fig. 3(2). We call this local move a C3-move.

(C4) In a chart, suppose that there is an edge e joining a degree-22 vertex and a black vertex. Suppose that e is one of the two edges labeled i in Fig. 2. Then, remove the edge as in Fig. 3(3). We call this local move a C4-move.

When $\partial B \neq \emptyset$ and the base point q_0 is in ∂B, we introduce another move.

(C5) Suppose that $\partial B \neq \emptyset$ and $q_0 \in \partial B$. Let Γ' be a chart that is the union of a chart Γ and some hoops which are parallel to and sufficiently near ∂B. Then we say that Γ' is obtained from Γ by a C5-move.

Definition 7. (1) *Chart moves* are C1-moves, C2-moves, C3-moves, C4-moves and their inverse moves.

(2) Two charts in B are said to be *chart move equivalent* (with respect to the base point q_0) if they are related by a finite sequence of chart moves and ambient isotopies of B rel q_0, where we assume that chart moves are applied in embedded 2-disks in B missing q_0.

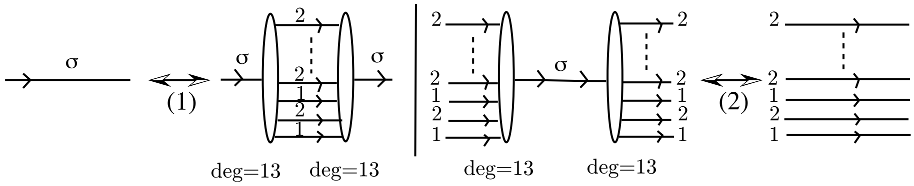

Fig. 4. Chart moves.

(3) Two charts in B are said to be *chart move equivalent up to conjugation* (with respect to the base point q_0) if they are related by a finite sequence of chart moves, C5-moves and ambient isotopies of B rel q_0. (It is not necessary to assume that chart moves are applied in embedded 2-disks in B missing q_0.)

We say that two monodromy representations $\rho : \pi_1(B \setminus \Delta, q_0) \to MC$ and $\rho' : \pi_1(B \setminus \Delta', q_0) \to MC$ are *equivalent* if there is a diffeomorphism $h : (B, q_0) \to (B, q_0)$ which is isotopic to the identity map rel q_0 such that $h(\Delta) = \Delta'$ and $\rho = \rho' \circ h_*$, where $h_* : \pi_1(B \setminus \Delta, q_0) \to \pi_1(B \setminus \Delta', q_0)$ is the induced isomorphism.

We say that two monodromy representations $\rho : \pi_1(B \setminus \Delta, q_0) \to MC$ and $\rho' : \pi_1(B \setminus \Delta', q_0) \to MC$ are *equivalent up to conjugation* if there is an inner-automorphism of MC, say t, and there is a diffeomorphism $h : (B, q_0) \to (B, q_0)$ which is isotopic to the identity map rel q_0 such that $h(\Delta) = \Delta'$ and $\rho = t \circ \rho' \circ h_*$.

C1-moves in this paper are called *chart moves of type W* in Definition 7 of [8]. C2-moves, C3-moves, C4-moves, C5-moves are not given explicitly in [8]. However, as shown in Figs. 22 and 23 of [8], C2-moves and C3-moves are equivalent to some local moves called *chart moves of transition* in Definition 14 of [8]. C4-moves are also equivalent to chart moves of transition in the sense of [8]. Thus, as stated in Section 8 of [8], we see that if two charts are chart move equivalent in our sense (Definition 7(2)) then the monodromy representations determined by them are equivalent. C5-moves are equivalent to *chart moves of conjugacy* in (3) and (4) of Fig. 17 of [8]. Again as in Section 7 of [8], we see that if two charts are chart move equivalent up to conjugation (Definition 7(3)) then the monodromy representations determined by them are equivalent up to conjugation.

Thus we have the following.

Theorem 8. *For two charts in B, if they are chart move equivalent* (*or chart move equivalent up to conjugation, resp.*) *then the monodromy representations determined by them are equivalent* (*or equivalent up to conjugation, resp.*)*, and hence the Lefschetz fibrations described by them are isomorphic.*

Remark 9. By Theorem 16 of [8], we see that two charts determine equivalent monodromy representations if and only if they are related by C1-moves (chart move of type W), chart moves of transition, and ambient isotopies of B rel q_0. It is unknown to the author whether all chart moves of transition are consequence of our chart moves.

We say that a black vertex of a chart Γ is of *type* I^+, *type* I^-, *type* II^+ or *type* II^- if the adjacent edge is labeled in $\{1, \dots, 5\}$ and oriented outward, if the adjacent edge is labeled in $\{1, \dots, 5\}$ and oriented inward, if the adjacent edge is labeled σ and oriented outward, or if the adjacent edge is labeled σ and oriented inward, respectively.

When Γ is a chart description of a genus-two Lefschetz fibration $f : M \to B$, black vertices correspond to critical values of f, and the types of the vertices are the same with the types of the singular fibers over the corresponding critical values. For a chart Γ, we denote by $n_{\mathrm{I}}^+(\Gamma)$, $n_{\mathrm{I}}^-(\Gamma)$, $n_{\mathrm{II}}^+(\Gamma)$, and $n_{\mathrm{II}}^-(\Gamma)$, the numbers of black vertices of type I^+, type I^-, type II^+ and type II^-, respectively. They are equal to $n_{\mathrm{I}}^+(f)$, $n_{\mathrm{I}}^-(f)$, $n_{\mathrm{II}}^+(f)$, and $n_{\mathrm{II}}^-(f)$, respectively.

If a chart Γ is irreducible, then it is obvious that $n_{\mathrm{II}}^+(\Gamma) = n_{\mathrm{II}}^-(\Gamma) = 0$. The converse is not true. However we have the following.

Lemma 10. *Every chart Γ with $n_{\mathrm{II}}^+(\Gamma) = n_{\mathrm{II}}^-(\Gamma) = 0$ is chart move equivalent to an irreducible chart.*

Proof. We can replace every hoop labeled σ into 12 parallel hoops with labels 1 or 2 by a chart move depicted in Fig. 4(1) followed by one in Fig. 4(2). Every edge labeled σ whose endpoints are degree-13 vertices is also removed by the latter move. □

Proposition 11. *A chiral* (*or irreducible, resp.*) *genus-two Lefschetz fibration has a chart description which is chiral* (*or irreducible, resp.*)*.*

Proof. If f is chiral, local monodromies around the critical values are all positive Dehn twists. By the definition of a chart description, the adjacent edges of the black vertices are oriented outward. Thus any chart description of f is chiral. If f is irreducible, local monodromies around the critical values are Dehn twists along non-separating simple loops, which are

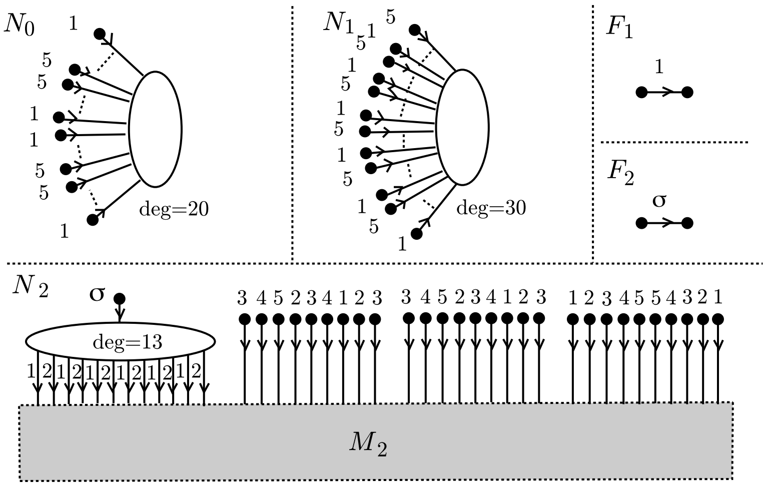

Fig. 5. Charts N_0, N_1, N_2, F_1 and F_2 describing f_0, f_1, f_2, f_1' and f_2'.

conjugates of $\zeta_1, \dots, \zeta_5$ and their inverses. Thus any chart description Γ of f satisfies $n_{\mathrm{II}}^+(\Gamma) = n_{\mathrm{II}}^-(\Gamma) = 0$. By Lemma 10, it changes to an irreducible one. $\square$

In Fig. 5, we show charts N_0, N_1, N_2, F_1 and F_2 describing f_0, f_1, f_2, f_1' and f_2'. We call N_0 a (positive) *nucleon of degree*-20 and N_1 a (positive) *nucleon of degree*-30. The region named M_2 is an arbitrary chart consisting of edges with labels in $\{1, \dots, 5\}$ and vertices whose degrees are in $\{4, 6, 20, 22\}$. (There exists such a chart M_2, Lemma 14.) A *free edge* means a chart consisting two black vertices and a single edge connecting them. F_1 and F_2 are free edges.

Let Γ and Γ' be charts in $B = D^2$. Divide D^2 into 2-disks D_1^2 and D_2^2 by a properly embedded arc in D^2. Put a small copy of Γ in D_1^2 and a small copy of Γ' in D_2^2. We have a new chart in $D^2 = D_1^2 \cup D_2^2$. We call it the *product* of Γ and Γ' and denote it by $\Gamma \oplus \Gamma'$. We say that Γ is a *factor* of $\Gamma \oplus \Gamma'$. The chart $\Gamma \oplus \Gamma'$ is a chart description of the fiber sum $f \# f'$ of the Lefschetz fibrations f and f' described by Γ and Γ'. We denote by $n\Gamma$ the product $\Gamma \oplus \cdots \oplus \Gamma$ of n copies of Γ. (When $B = D^2$, the fiber sum $f \# f'$ of f and f' over B is defined by using the boundary connected sum of the base spaces.)

Theorem 12. *Let Γ be a chart in $B = D^2$. Suppose that $n_{\mathrm{II}}^+(\Gamma) \geqslant n_{\mathrm{II}}^-(\Gamma)$. Then there exists a positive integer m_0 such that for any integer $m \geqslant m_0$, the chart $\Gamma \oplus mN_0$ is chart move equivalent to*

$$\Gamma' \oplus \big(n_{\mathrm{II}}^+(\Gamma) - n_{\mathrm{II}}^-(\Gamma)\big)N_2 \oplus n_{\mathrm{I}}^-(\Gamma)F_1 \oplus n_{\mathrm{II}}^-(\Gamma)F_2$$

for some chart Γ' with $n_{\mathrm{I}}^-(\Gamma') = n_{\mathrm{II}}^+(\Gamma') = n_{\mathrm{II}}^-(\Gamma') = 0$ such that Γ' has N_0 as a factor. Moreover if $n_{\mathrm{II}}^-(\Gamma) = 0$, we may take m_0 to be $n_{\mathrm{I}}^-(\Gamma) + 2n_{\mathrm{II}}^+(\Gamma) + 1$.

We prove Theorem 12 in Section 5.

Corollary 13. *Let f be a genus-two Lefschetz fibration over $B = D^2$ (or S^2) with $n_{\mathrm{II}}^+(f) \geqslant n_{\mathrm{II}}^-(f)$. Then there exists a positive integer m_0 such that for any integer $m \geqslant m_0$, the fiber sum $f \# mf_0$ is equivalent to*

$$f' \# \big(n_{\mathrm{II}}^+(f) - n_{\mathrm{II}}^-(f)\big) f_2 \# n_{\mathrm{I}}^-(f) f_1' \# n_{\mathrm{II}}^-(f) f_2'$$

for some chiral and irreducible genus-two Lefschetz fibration f' over $B = D^2$ (or S^2) such that the monodromy representation of f' is transitive. Moreover if $n_{\mathrm{II}}^-(f) = 0$, we may take m_0 to be $n_{\mathrm{I}}^-(f) + 2n_{\mathrm{II}}^+(f) + 1$.

Lemma 14. *There is a chart satisfying the condition of M_2.*

Proof. See Fig. 6 where M_3 and M_4 are charts depicted in Figs. 7 and 8. $\square$

5. Proof of Theorem 12

Definition 15. A chart Γ in a 2-disk is *nomadic with respect to* a chart Γ_0 in B if for any two regions of the complement $B \setminus \Gamma_0$, say R_1 and R_2, the chart Γ_0 together with a small copy of Γ in R_1 is chart move equivalent to the chart Γ_0 together with a small copy of Γ in R_2. A chart Γ in a 2-disk is *nomadic* if it is nomadic with respect to every chart.

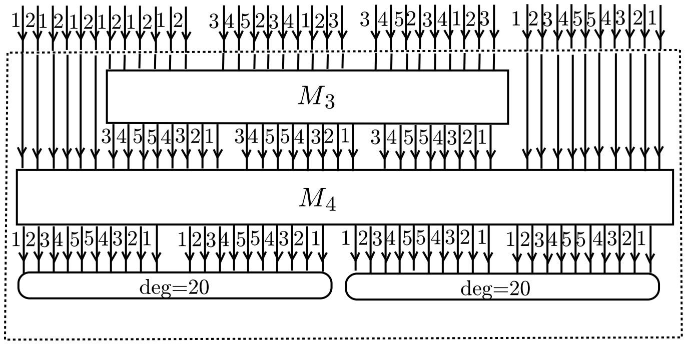

Fig. 6. Chart M_2.

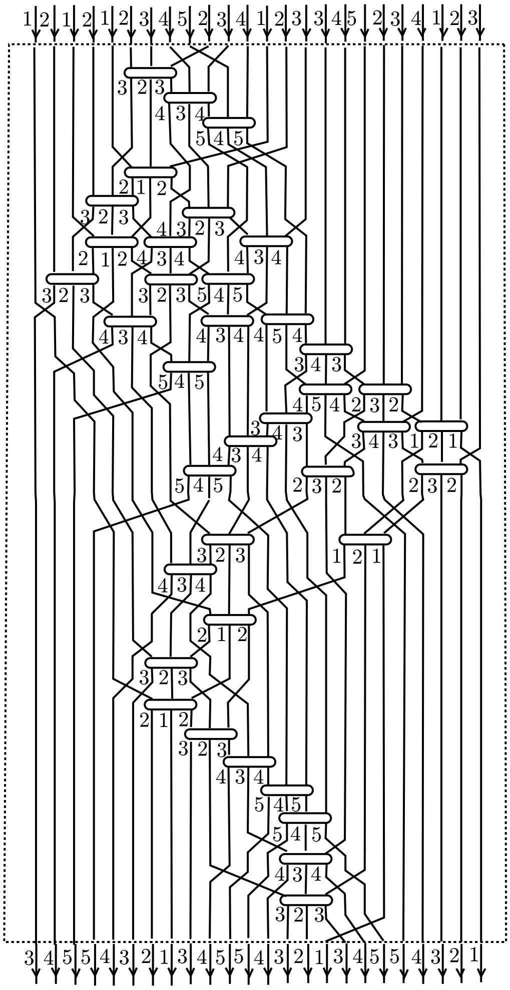

Fig. 7. Chart M_3.

Lemma 16. *Let D be a 2-disk and B a compact, connected and oriented surface.*

(1) *Let Γ be a chart in D. If there is a 2-disk U in D such that $\Gamma \cap U$ is as in Fig. 9(1), then Γ is nomadic.*
(2) *Let Γ_0 be a chart in B. If there is a 2-disk U in B such that $\Gamma_0 \cap U$ is as in Fig. 9(1), then any chart Γ in a 2-disk is nomadic with respect to Γ_0.*

Proof. (1) First we consider a special case where Γ is as in Fig. 9(2). Let Γ_0 be any chart in B, and put a small copy of Γ in a region of $B \setminus \Gamma_0$. As shown in Fig. 11, it can pass through any edge of Γ_0 which is labeled in $\{1, \ldots, 5\}$. For an

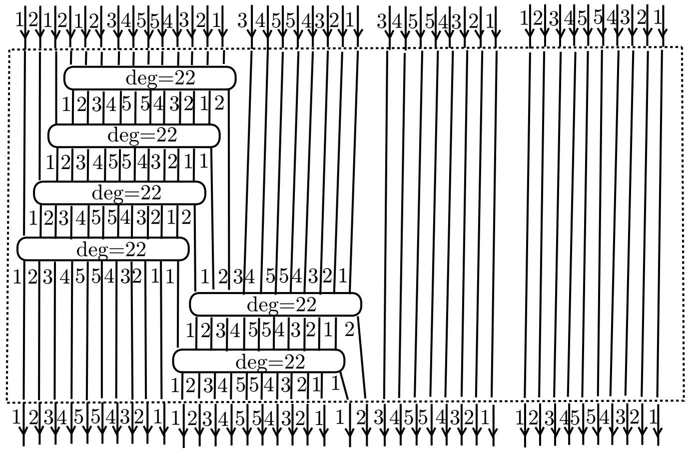

Fig. 8. Chart M_4.

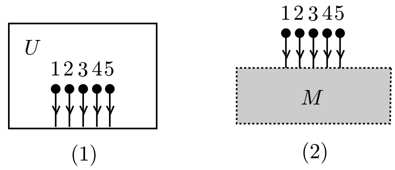

Fig. 9. Nomadic charts.

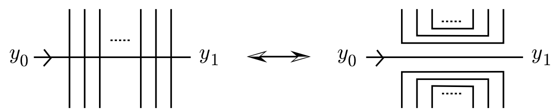

Fig. 10. Chart moves.

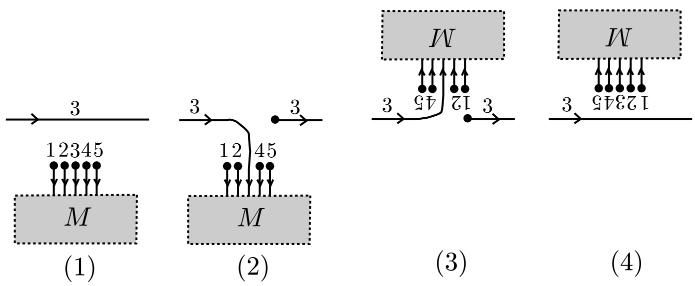

Fig. 11. Chart moves.

edge labeled σ, apply a chart move as in Fig. 4(1), let Γ pass through the 12 edges with labels 1 and 2, and recover the edge labeled σ by the move in Fig. 4. Thus we see that Γ is nomadic. Now we consider a general case. Take a point y_0 in the region U and a point y_1 in the boundary ∂D. Consider a simple path $\eta : [0,1] \to D$ connecting y_0 and y_1 such that η intersects Γ transversely. Let w be the intersection word of η with respect to Γ (see [7,8]). Let Γ' be a chart obtained from Γ by adding some hoops surrounding Γ such that the intersection word w' of η with respect to Γ' is $w \cdot w^{-1}$. Applying a chart move in a neighborhood of η as in Fig. 10, we have a chart Γ'' such that it coincides with Γ' outside of the neighborhood of η and the path η misses Γ''. So Γ'' is as in Fig. 9(2). Note that Γ'' is chart move equivalent to Γ, since one can add or remove any hoop surrounding it by chart moves as in Fig. 12. Since Γ'' is nomadic as shown in the previous case, we see that Γ is nomadic.

Now we prove (2). Let U be a region such that $\Gamma_0 \cap U$ is as in Fig. 9(1). It is sufficient to show that any chart Γ put in a region of $B \setminus \Gamma_0$ can be moved into U. As shown in Fig. 13, Γ can pass through any edge of Γ_0 by getting a surrounding hoop. When Γ arrives in U, it is surrounded some hoops, which can be removed by use of the edges of Γ_0 in U as in Fig. 12. □

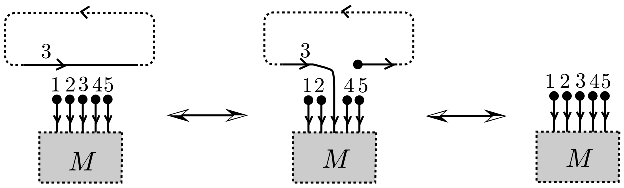

Fig. 12. Chart moves.

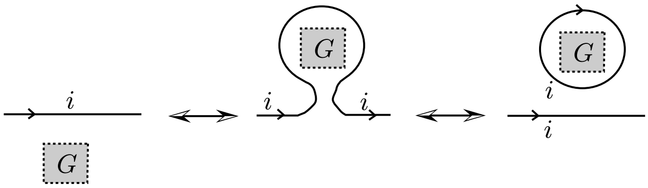

Fig. 13. Chart moves.

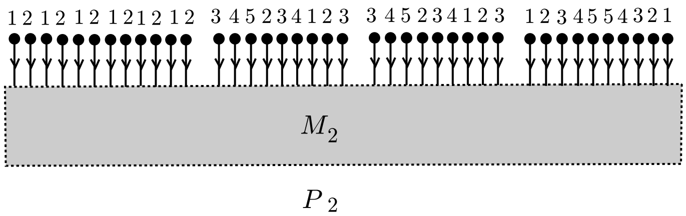

Fig. 14. Chart P_2, which is equivalent to $2N_0$.

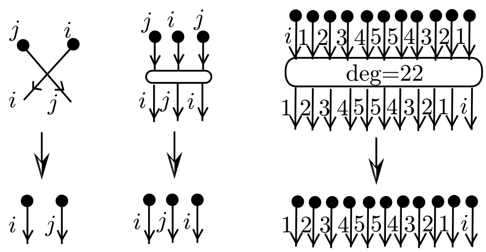

Fig. 15. Chart moves.

Lemma 17. *Let P_2 be a chart depicted in Fig.* 14*, where M_2 is the chart depicted in Fig.* 6*. It is chart move equivalent to* $2N_0$*.*

Proof. Applying chart moves depicted in Fig. 15 to the chart P_2, we have $2N_0$. □

Now we prove Theorem 12.

Proof of Theorem 12. First we consider a case where Γ is a chart with $n_{\mathrm{II}}^-(\Gamma) = 0$. It suffices to show that $\Gamma \oplus (n_{\mathrm{I}}^-(\Gamma) + 2n_{\mathrm{II}}^+(\Gamma) + 1)N_0$ is chart move equivalent to

$$\Gamma' \oplus n_{\mathrm{II}}^+(\Gamma)N_2 \oplus n_{\mathrm{I}}^-(\Gamma)F_1$$

for some chart Γ' with $n_{\mathrm{I}}^-(\Gamma') = n_{\mathrm{II}}^+(\Gamma') = n_{\mathrm{II}}^-(\Gamma') = 0$ such that Γ' has N_0 as a factor. By Lemma 16, N_0 is nomadic. Thus we can move N_0 freely up to chart move equivalence. For each black vertex of type I^-, move a chart N_0 near the vertex and apply a chart move as in Fig. 16 to make a free edge. Move the free edge toward the boundary of B by the chart move as in Fig. 13. Since there is at least one N_0 near ∂B, the hoops surrounding the free edge can be removed (Fig. 12), and we may also assume that the label of the free edge is 1 (Lemma 18.24 of [7]). Thus we can change $\Gamma \oplus (n_{\mathrm{I}}^-(\Gamma) + 2n_{\mathrm{II}}^+(\Gamma) + 1)\,N_0$ so that all black vertices of type I^- are endpoints of F_1's near ∂B. We still have $2n_{\mathrm{II}}^+(\Gamma) + 1$ N_0's near ∂B. For each black vertex of type II^+, move a pair of N_0 near the vertex. Change the pair of N_0's to a chart P_2 in Fig. 14 (Lemma 17). The edge adjacent to the vertex of type II^+ is oriented outward and is labeled σ. Apply a chart move as in Fig. 17, and then apply

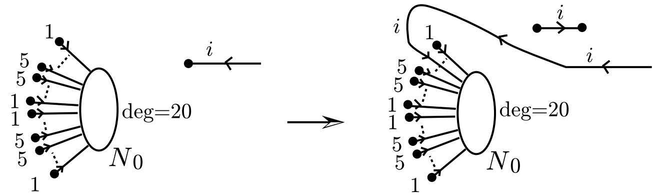

Fig. 16. Chart moves.

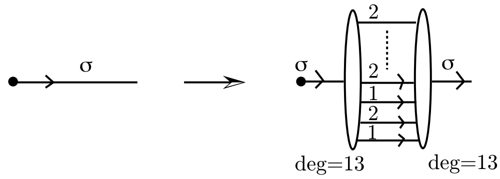

Fig. 17. Chart moves.

a chart move between the 12 edges there and the 12 edges of P_2 to get one N_2. Move the chart N_2 toward ∂B. (Note that N_2 is nomadic by Lemma 16.) Now all black vertices of type II^+ belong to N_2's near ∂B. We still have one N_0 near ∂B. Thus the chart is $\Gamma' \oplus n_{\mathrm{II}}^+(\Gamma)N_2 \oplus n_{\mathrm{I}}^-(\Gamma)F_1$ for a chart Γ' with $n_{\mathrm{I}}^-(\Gamma') = n_{\mathrm{II}}^+(\Gamma') = n_{\mathrm{II}}^-(\Gamma') = 0$ such that Γ' has N_0 as a factor.

We consider a case where Γ is a chart with $n_{\mathrm{II}}^+(\Gamma) \geqslant n_{\mathrm{II}}^-(\Gamma) > 0$. Let v be a black vertex of type II^-. Choose a black vertex v' of type II^+ and consider a simple path η from v to v'. If η intersects an edge labeled σ, then apply a chart move depicted in Fig. 4(1) and we assume that η intersects only edges with labels in $\{1, \ldots, 5\}$. For each intersection of η and the chart, we assert one N_0 and apply a chart move as in Fig. 11(2) so that η does not intersect the chart. Now move v along η toward v' and by a chart move we can make a free edge with label σ, that is F_2. Move this F_2 toward ∂B by moves as in Fig. 13. The hoops surrounding the free edge can be removed by adding one N_0 near ∂B as before. By this procedure, we can move all black vertices of type II^- near ∂B as endpoints of F_2's. The number of F_2's is $n_{\mathrm{II}}^-(\Gamma)$. There are $n_{\mathrm{II}}^+(\Gamma) - n_{\mathrm{II}}^-(\Gamma)$ black vertices of type II^+ in the chart, besides the endpoints of F_2's. For each black vertex of type II^+, that is not an endpoint of F_2, add a pair of N_0 to make P_2. As in the previous case, we can move the black vertex of type II^+ as an endpoint of N_2 near ∂B. The number of N_2's is $n_{\mathrm{II}}^+(\Gamma) - n_{\mathrm{II}}^-(\Gamma)$. As in the previous case, we move black vertices of type I^- as endpoints of F_1'as near ∂B. The number of F_1's is $n_{\mathrm{I}}^-(\Gamma)$. Thus we have a chart written as

$$\Gamma' \oplus \big(n_{\mathrm{II}}^+(\Gamma) - n_{\mathrm{II}}^-(\Gamma)\big)N_2 \oplus n_{\mathrm{I}}^-(\Gamma)F_1 \oplus n_{\mathrm{II}}^-(\Gamma)F_2$$

for some chart Γ' with $n_{\mathrm{I}}^-(\Gamma') = n_{\mathrm{II}}^+(\Gamma') = n_{\mathrm{II}}^-(\Gamma') = 0$ such that Γ' has N_0 as a factor. $\square$

6. Proof of Theorem 2

Proof of Theorem 2. By Theorem A of Siebert and Tian [17] (cf. [2]) a chiral and irreducible genus-two Lefschetz fibration f' with transitive monodromy representation is holomorphic and hence it is a fiber sum of some copies of f_0 and f_1. Therefore Corollary 13 implies the assertions (2) and (4) of Theorem 2, and the former part of (3). It is well known that $3f_0 \cong 2f_1$. Thus we can take b to be 0 or 1. We shall compare the number of singular fibers of each type of $f \# mf_0$ with that of $\#(a+m)f_0 \# bf_1 \# cf_2 \# df_1' \# ef_2'$. We have already used the information on the numbers of singular fibers of types I^-, II^+ and type II^- to determine c, d and e; $c = n_{\mathrm{II}}^+(f) - n_{\mathrm{II}}^-(f)$, $d = n_{\mathrm{I}}^-(f)$ and $e = n_{\mathrm{II}}^-(f)$. The number of singular fibers of type I^+ of $f \# mf_0$ is $n_{\mathrm{I}}^+(f) + 20m$, and that of $\#(a+m)f_0 \# bf_1 \# cf_2 \# df_1' \# ef_2'$ is $20(a+m) + 30b + 28c + d$. From this equality, we have

$$20a + 30b = n_{\mathrm{I}}^+(f) - n_{\mathrm{I}}^-(f) - 28\big(n_{\mathrm{II}}^+(f) - n_{\mathrm{II}}^-(f)\big).$$

Thus the right-hand side, which is $\mathcal{E}(f)$, is a multiple of 10. And we see that the parity of b equals to the parity $\epsilon(f)$. Therefore when we assume $b = \epsilon(f)$, we have $a = (\mathcal{E}(f) - 30\epsilon(f))/20$. $\square$

7. Concluding remark

In the proof of Theorem 2, we assumed the deep result due to Siebert and Tian [17] stating that any chiral and irreducible genus-two Lefschetz fibration f' over S^2 with transitive monodromy representation is holomorphic and it is a fiber sum of some copies of f_0 and f_1. If one does not need a lower bound m_0 given in (4) of Theorem 2, we can prove Theorem 2 without assuming Siebert and Tian's result.

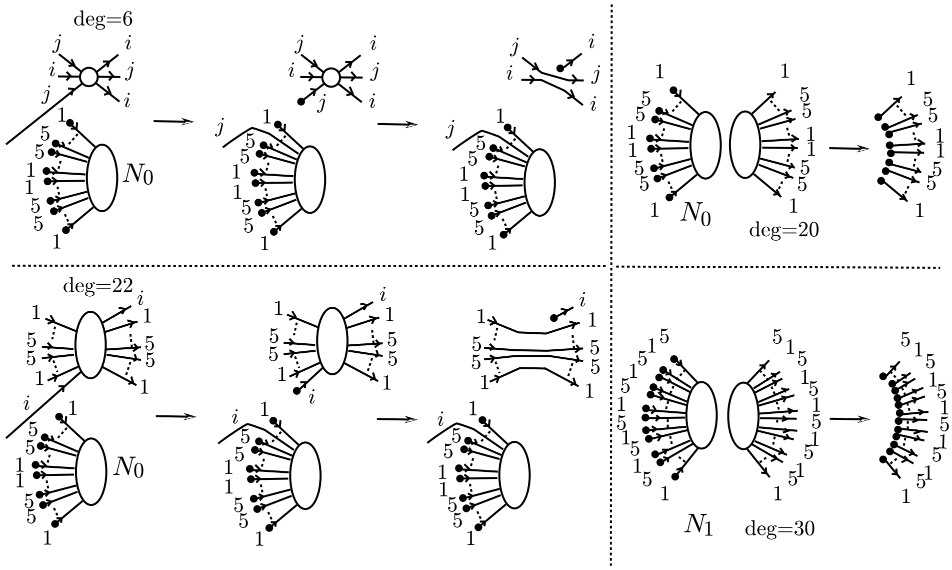

Fig. 18. Chart moves.

Proposition 18. *Let Γ be a chart description of a chiral and irreducible genus-two Lefschetz fibration over $B = D^2$ (or S^2). There exists a positive integer m such that $\Gamma \oplus mN_0$ is chart move equivalent to $(a+m)N_0 \oplus bN_1$ for some integers a and b.*

Proof. Since f is chiral and irreducible, we may assume that Γ is chiral and irreducible by Proposition 11. Adding some N_0's to the chart and applying chart moves shown in Fig. 18, we can remove all degree-6 vertices, degree-22 vertices, degree-20 vertices whose adjacent edges are oriented outward, and degree-30 vertices whose adjacent edges are oriented outward. (Since $3N_0$ is chart move equivalent to $2N_1$, we may add N_1's too.) Remove all hoops using an N_0 (Fig. 12). Now every edge is adjacent to a black vertex, a degree-4 vertex, a degree-20 vertex whose adjacent edges are oriented inward or a degree-30 vertex whose adjacent edges are oriented inward. Note that for a degree-4 vertex, the two incoming adjacent edges have black vertices at the other end. Thus by a chart move depicted in Fig. 15, we can remove the degree-4 vertex. Remove all degree-4 vertices this way. Now the chart is a union of some N_0's and N_1's. □

Now we have a corollary to Proposition 18.

Corollary 19. *Let f be a chiral and irreducible genus-two Lefschetz fibration over S^2. There exists a positive number m such that $f \# mf_0 \cong (a+m) f_0 \# bf_1$ for some integers a and b.*

Using this corollary, we have a proof of Theorem 2, except the assertion (4), without using Siebert and Tian's result.

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