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Null Controllability of Nonlinear Infinite Delay Systems with Distributed Delays in Control

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Sufficient conditions for the null controllability of nonlinear infinite delay systems with distributed delays in the control are developed. Namely, if the uncontrolled system is uniformly asymptotically stable and if the linear control system is proper, then conditions are obtained that imply the nonlinear infinite delay system is null controllable. © 1990 Academic Press, Inc.

1. INTRODUCTION

Hermes and LaSalle [5, p. 78] showed that if the linear system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \tag{1}$$

is proper and if

$$\dot{x}(t) = A(t)\,x(t) \tag{2}$$

is uniformly asymptotically stable then system (1) is null controllable. An analogous result was proved by Khambadkone [8] for the system

$$\dot{x}(t) = A(t)\,x(t) + \int_{-h}^{0} d_s H(t, s)\,u(t+s) \tag{3}$$

and by Chukwu [1] for the linear delay system

$$\begin{aligned} \dot{x}(t) &= L(t, x_t) + B(t)\,u(t) \\ x(t) &= \phi(t), \qquad t \text{ in } [-h, 0], \end{aligned} \tag{4}$$

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where $L(t, \phi)$ is continuous in t, linear in ϕ. Dauer [3] proved that (1) is null controllable iff the system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) + g(t, x)\,u(t) \tag{5}$$

is null controllable provided g is appropriately bounded. A similar result is given for the autonomous linear system with arbitrarily restrained controls. Chukwu [2] discussed the null controllability in function space of the nonlinear delay system

$$\dot{x}(t) = \int_{-h}^{0} d_s\eta(t, s)\,x(t+s) + B(t)\,u(t) + f(t, x_t, u) \tag{6}$$

with limited controls. In [9] Sinha developed sufficient conditions for the null controllability of the infinite delay system

$$\begin{aligned} \dot{x}(t) &= L(t, x_t) + B(t)\,u(t) + \int_{-\infty}^{0} A(\theta)\,x(t+\theta)\,d\theta + f(t, x(\cdot), u(\cdot)) \\ x(t) &= \phi(t) \qquad \text{for} \quad -\infty < t \leqslant 0, \end{aligned} \tag{7}$$

where $L(t, \phi)$ is continuous in t, linear in ϕ with constant delays $h_k \geqslant 0$ and is given by

$$L(t, \phi) = \sum_{k=0}^{N} A_k(t)\,\phi(-h_k). \tag{8}$$

In this paper, we obtain a similar result for the nonlinear infinite delay system

$$\begin{aligned} \dot{x}(t) &= L(t, x_t) + \int_{-h}^{0} d_\omega H(t, \omega)\,u(t+\omega) \\ &\quad + \int_{-\infty}^{0} A(\theta)\,x(t+\theta)\,d\theta + f(t, x(\cdot), u(\cdot)) \\ x(t) &= \phi(t), \qquad t \in (-\infty, 0]. \end{aligned} \tag{9}$$

The controls u are square integrable with values in the unit cube C^m:

$$C^m = \{u \colon u \in E^m,\ |u_i| \leqslant 1,\ i = 1, ..., m\}.$$

2. Preliminaries

In Eqs. (8) and (9), each A_k is a continuous $n \times n$ matrix function for $0 \leqslant h_k \leqslant h$, $A(\theta)$ is an $n \times n$ matrix whose elements are square integrable on $(-\infty, 0]$. The matrix function $H(t, \omega)$ is $n \times m$, continuous in t for fixed ω, and of bounded variation in ω on $[-h, 0]$ and f is an n-valued function. Let $\gamma \geqslant h \geqslant 0$ be given real numbers (γ may be $+\infty$), E^n be an n-dimensional linear vector space with norm $|\cdot|$. The function $\eta: [-\gamma, 0] \to (0, \infty)$ is Lebesgue integrable on $[-\gamma, 0]$, positive and non-decreasing on $[-\gamma, 0]$. Let $B = B([-\gamma, 0], E^n)$ be the Banach space of functions which are continuous on $[-\gamma, 0]$ and such that

$$|\phi| = \sup_{s \in [-\gamma, 0]} |\phi(s)| + \int_{-\gamma}^{0} \eta(\tau) \, |\phi(\tau)| \, d\tau < \infty.$$

For any $t \in R$, and any $x: [t-\gamma, t] \to E^n$, let $x_t: [-\gamma, 0] \to E^n$ be defined by $x_t(s) = x(t+s)$, $s \in [-\gamma, 0]$. The symbol d_ω in (9) denotes that the integral is in the Lebesgue–Stieltjes sense with respect to the variable ω.

Consider the linear homogeneous system

$$\dot{x}(t) = L(t, x_t) + \int_{-h}^{0} d_\omega H(t, \omega) \, u(t+\omega) \tag{10}$$

and

$$\dot{x}(t) = L(t, x_t) + \int_{-\infty}^{0} A(\theta) \, x(t+\theta) \, d\theta. \tag{11}$$

Here we study the controllability of (9) when it is assumed that the admissible controls have values in a compact convex subset P of E^m. To do this we introduce the notion of a proper control system for Eq. (10). Hale [4] obtained exponential estimates on the solutions of the linear Eq. (11).

Let X satisfy the equation

$$\frac{\partial X(t, s)}{\partial t} = L(t, X_t(\cdot, s)), \qquad t \geqslant s$$

$$X(t, s) = \begin{cases} 0 & s-h \leqslant t \leqslant s \\ I & t = s \end{cases},$$

where $X_t(\cdot, s)(\theta) = X(t+\theta, s)$, $-h \leqslant \theta \leqslant 0$. Then the solution of (9) is given by

$$\begin{aligned} x(t) = x(t; t_0, \phi) &+ \int_{t_0}^{t} X(t, s) \left(\int_{-h}^{0} d_\omega H(s, \omega) \, u(s+\omega) \right) ds \\ &+ \int_{t_0}^{t} X(t, s) \left(\int_{-\gamma}^{0} A(\theta) \, x(s+\theta) \, d\theta \right) ds \\ &+ \int_{t_0}^{t} X(t, s) \, f(s, x(\cdot), u(\cdot)) \, ds, \qquad t_0 \leqslant t \leqslant t_1 \end{aligned} \tag{12}$$

$$x(t) = \phi(t) \qquad \text{for} \quad t \in [t_0 - h, t_0]$$

with initial state $z(t_0)=(x(t_0), \phi, \eta)$ where $u_{t_0}=\eta$ and $x(t; t_0, \phi)$ is the solution of $\dot{x}(t)=L(t, x_t)$. Following Klamka [7], using the unsymmetric Fubini theorem, the solution can be written in the form

$$\begin{aligned}
x(t) = x(t; t_0, \phi) &+ \int_{-h}^{0} d_{H_\omega} \left(\int_{t_0+s}^{t+s} X(t, s-\omega)\, H(s-\omega, \omega)\, u(s)\, ds \right. \\
&\left. + \int_{t_0}^{t} \int_{-\gamma}^{0} X(t, s)\, A(\theta)\, x(s+\theta)\, d\theta\, ds + \int_{t_0}^{t} X(t, s)\, f(s, x(\cdot), u(\cdot)\, ds \right) \\
= x(t; t_0, \phi) &+ \int_{-h}^{0} d_{H_\omega} \int_{t_0+s}^{t_0} X(t, s-\omega)\, H(s-\omega, \omega)\, \eta(s)\, ds \\
&+ \int_{t_0}^{t} \left(\int_{-h}^{0} X(t, s-\omega)\, d_\omega H_t(s-\omega, \omega) \right) u(s)\, ds \\
&+ \int_{t_0}^{t} \int_{-\gamma}^{0} X(t, s)\, A(\theta)\, x(s+\theta)\, d\theta\, ds \\
&+ \int_{t_0}^{t} X(t, s)\, f(s, x(\cdot), u(\cdot))\, ds, \qquad (13)
\end{aligned}$$

where

$$H_t(s, \omega) = \begin{cases} H(s, \omega) & \text{for} \quad s \leqslant t \\ 0 & \text{for} \quad s > t. \end{cases}$$

Let

$$q(t, \eta) = \int_{-h}^{0} d_{H_\omega} \int_{t_0+s}^{t_0} X(t, s-\omega)\, H(s-\omega, \omega)\, \eta(s)\, ds$$

$$S(t, s) = \int_{-h}^{0} X(t, s-\omega)\, d_\omega H_t(s-\omega, \omega).$$

Then the reachable set of (10) is given by

$$R(t) = \left\{ \int_{t_0}^{t} S(t, s)\, u(s)\, ds : u \in L_2([t_0, t_1), C^m) \right\}. \qquad (14)$$

The controllability matrix of (10) at time t is

$$W(t_0, t) = \int_{t_0}^{t} S(t, s)\, S^*(t, s)\, ds, \qquad (15)$$

where the star denotes the matrix transpose.

DEFINITION 1 [9]. The system (10) is said to be *proper* in E^n on an interval $[t_0, t_1]$ if

$$c^*S(t_1, s) = 0 \qquad \text{a.e., } s \in [t_0, t_1],\ c \in E^n \text{ implies } c = 0.$$

If (10) is proper on $[t_0, t_0 + \alpha]$ for each $\alpha > 0$, we say the system is *proper at time* t_0. If (10) is proper on each interval $[t_0, t_1]$, $t_1 > t_0 \geqslant 0$, we say *the system is proper* in E^n.

DEFINITION 2. The system (9) is said to be *null controllable* if for each $\phi \in B([-\gamma, 0], E^n)$, there is a $t_1 \geqslant t_0$, $u \in L_2([t_0, t_1], P)$, P a compact convex subset of E^m, such that the solution $x(t, t_0, \phi, u)$ of (9) satisfies $x_{t_0}(t_0, \phi, u) = \phi$ and $x(t_1, t_0, \phi, u) = 0$.

3. MAIN RESULT

THEOREM. *Suppose that the constraint set U is an arbitrary compact subset of E^m, and that*

(i) *system* (11) *is uniformly asymptotically stable, so that the solution $x_t(t_0, \phi)$ satisfies $\|x_t(t_0, \phi)\| \leqslant Me^{-\alpha(t - t_0)}\|\phi\|$ for some $\alpha > 0$, $M > 0$,*

(ii) *the linear control system* (10) *is proper in E^n,*

(iii) *the continuous function f satisfies*

$$|f(t, x(\cdot), u(\cdot))| \leqslant \exp(-\beta t)\, \pi(x(\cdot), u(\cdot))$$
$$\text{for all } (t, x(\cdot), u(\cdot)) \in [t_0, \infty) \times B \times L_2,$$

$$\text{where } \int_{t_0}^{\infty} \pi(x(\cdot), u(\cdot))\, ds \leqslant K < \infty \text{ and } \beta - \alpha \geqslant 0.$$

Then (9) *is null controllable.*

Proof. Since (10) is proper in E^n, $W^{-1}(t_0, t_1)$ exists for each $t_1 > t_0$. Suppose the pair of functions x, u form a solution pair to the set of integral equations

$$u(t) = -S^*(t_1, t)\, W^{-1}(t_0, t_1) \Bigg[x(t_1; t_0, \phi) + q(t_1, \eta)$$
$$+ \int_{t_0}^{t_1} \int_{-\gamma}^{0} X(t_1, s)\, A(\theta)\, x(s + \theta)\, d\theta\, ds$$
$$+ \int_{t_0}^{t_1} X(t_1, s)\, f(s, x(\cdot), u(\cdot))\, ds \Bigg] \tag{16}$$

For some suitably chosen $t_1 \geq t \geq t_0$, $u(t) = \eta(t)$, $t \in [t_0 - h, t_0]$ and

$$\begin{aligned} x(t) &= x(t; t_0, \eta) + q(t, \eta) + \int_{t_0}^{t} S(t, s)\, u(s)\, ds \\ &\quad + \int_{t_0}^{t} \int_{-\gamma}^{0} A(\theta)\, x(t+\theta)\, d\theta\, ds + \int_{t_0}^{t} X(t, s)\, f(s, x(\cdot), u(\cdot))\, ds \\ x(t) &= \phi(t), \qquad t \in [t_0 - h, t_0]. \end{aligned} \tag{17}$$

Then u is square integrable on $[t_0 - h, t_1]$ and x is a solution of (9) corresponding to u with initial state $z(t_0) = (x(t_0), \phi, \eta)$, where $u_{t_0} = \eta$. Also,

$$\begin{aligned} x(t_1) &= x(t_1; t_0, \phi) + q(t_1, \eta) - \int_{t_0}^{t_1} S(t_1, s)\, S^*(t_1, s)\, W^{-1}(t_0, t_1) \\ &\quad \cdot \Bigg[x(t_1; t_0, \phi) + q(t_1, \eta) + \int_{t_0}^{t_1} X(t_1, s) \Bigg(f(s, x(\cdot), u(\cdot)) \\ &\quad + \int_{-\gamma}^{0} A(\theta)\, x(t+\theta)\, d\theta \Bigg) \Bigg] ds \\ &\quad + \int_{t_0}^{t_1} X(t_1, s) \left(f(s, x(\cdot), u(\cdot)) + \int_{-\gamma}^{0} A(\theta)\, x(t+\theta)\, d\theta \right) ds = 0. \end{aligned}$$

We now show that $u\colon [t_0, t_1] \to U$ is in the arbitrary compact constraint subset of E^m, that is $|u| \leq a$ for some constant $a > 0$. Since (11) is uniformly asymptotically stable and H is continuous in t and of bounded variation in ω, we have,

$$\begin{aligned} &|S^*(t_1, t)\, W^{-1}| \leq C_1 && \text{for some} \quad C_1 > 0 \\ &|x_t(t_0, \phi)| \leq C_2 \exp[-\alpha(t_1 - t_0)] && \text{for some} \quad C_2 > 0. \end{aligned}$$

Hence,

$$\begin{aligned} |u(t)| \leq C_1 \Bigg[& C_2 \exp[-\alpha(t_1 - t_0)] \\ & + \int_{t_0}^{t_1} M \exp[-\alpha(t_1 - s))\, \exp(-\beta s)\, \pi(x(\cdot), u(\cdot))\, ds \Bigg] \end{aligned}$$

and therefore

$$|u(t)| \leq C_1 [C_2 \exp[-\alpha(t_1 - t_0)] + KM \exp(-\alpha t_1)] \tag{18}$$

since $\beta - \alpha \geq 0$ and $s \geq t_0 \geq 0$. From (18) we see that t_1 can be chosen so large that $|u(t)| \leq a$, $t \in [t_0, t_1]$, proving that u is an admissible control for this choice of t_1. It remains to prove the existence of a solution pair of the integral Eqs. (16) and (17). Let B be the Banach space of all functions

$$(x, u): [t_0-h, t_1] \times [t_0-h, t_1] \to E^n \times E^m,$$

where $x \in B([t_0-h, t_1], E^n)$; $u \in L_2([t_0-h, t_1], E^m)$ with the norm defined by

$$\|(x, u)\| = \|x\|_2 + \|u\|_2$$

where

$$\|x\|_2 = \left[\int_{t_0-h}^{t_1} |x(s)|^2 \, ds\right]^{1/2}$$

$$\|u\|_2 = \left[\int_{t_0-h}^{t_1} |u(s)|^2 \, ds\right]^{1/2}.$$

Define the operator $T: B \to B$ by $T(x, u) = (y, v)$ where

$$v(t) = -S^*(t_1, t)\, W^{-1}(t_0, t_1)\Big[x(t_1; t_0, \phi) + q(t_1, \eta) + \int_{t_0}^{t_1}\int_{-\gamma}^{0} X(t_1, s)\, A(\theta)\, x(s+\theta)\, d\theta\, ds + \int_{t_0}^{t_1} X(t_1, s)\, f(s, x(\cdot), u(\cdot))\, ds\Big] \quad \text{for} \quad t \in J \equiv [t_0, t_1] \tag{19}$$

and

$$v(t) = \eta(t) \qquad \text{for} \quad t \in [t_0-\gamma, t_0];$$

$$y(t) = x(t; t_0, \phi) + q(t, \eta) + \int_{t_0}^{t} S(t, s)\, v(s)\, ds + \int_{t_0}^{t}\int_{-\gamma}^{0} X(t, s)\, A(\theta)\, x(t+\theta)\, d\theta\, ds + \int_{t_0}^{t} X(t, s)\, f(s, x(\cdot), u(\cdot))\, ds \qquad \text{for} \quad t \in J \tag{20}$$

and

$$y(t) = \phi(t) \qquad \text{for} \quad t \in [t_0-\gamma, t_0].$$

From (18) it is clear that $|v(t)| \leqslant a$, $t \in J$ and also $v: [t_0-h, t_0] \to U$ we have $|v(t)| \leqslant a$. Hence $\|v\|_2 \leqslant a(t_1+h-t_0)^{1/2} = \beta_0$. Next

$$|y(t)| \leqslant C_2 \exp[-\alpha(t-t_0)] + C_3 + C_4 \int_{t_0}^{t} |v(s)|\, ds + KM \exp(-\alpha t_1),$$

where $C_3 = \sup |q(t, \eta)|$ and $C_4 = \sup |S(t, s)|$. Since $\alpha > 0$, $t \geqslant t_0 \geqslant 0$ we deduce that

$$|y(t)| \leqslant C_2 + C_3 + C_4 a(t_1 - t_0) + KM \equiv \beta, \qquad t \in J$$

and

$$|y(t)| \leqslant \sup |\phi(t)| \equiv \delta, \qquad t \in [t_0 - h, t_0].$$

Hence if $\lambda = \max\{\beta, \delta\}$ then

$$\|y\|_2 \leqslant \lambda(t_1 + h - t_0)^{1/2} \equiv \beta_1.$$

Let $r = \max\{\beta_0, \beta_1\}$. Then if we let

$$Q(r) = \{(x, u) \in B\colon \|x\|_2 \leqslant r,\ \|u\|_2 \leqslant r\}.$$

We have proven that $T\colon Q(r) \to Q(r)$. Since $Q(r)$ is closed, bounded and convex, by Riez's theorem [6] it is relatively compact under T, then the Schauder theorem implies that T has a fixed point $(x, u) \in Q(r)$. This fixed point (x, u) of T is a solution pair of the set of integral Eqs. (19) and (20). Hence the system (9) is Euclidean null controllable.

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