JOURNAL OF DIFFERENTIAL EQUATIONS 13, 567–587 (1973)

Periodic Solutions of Hereditary Differential Systems*

G. B. GUSTAFSON AND KLAUS SCHMITT

Department of Mathematics, University of Utah, Salt Lake City, Utah 84112

Received August 24, 1972

1. INTRODUCTION

The purpose of this paper is to establish the existence of T-periodic solutions for the hereditary differential systems

$$x'(t) = -F(Ux_t) + f(t), \tag{1.1}$$
$$x'(t) = -A(t)\, Ux_t(0) + f(t), \tag{1.2}$$
$$x'(t) = -F(t, Ux_t) + f(t), \tag{1.3}$$
$$x'(t) = -F(t, Ux_t) + g(t, x_t, U^*x_t), \tag{1.4}$$
$$x''(t) = h(t, x(t), x(t - d(t, x(t)))) \tag{1.5}$$

(where U^* is an operator of the same type as U), which are listed here in increasing order of complexity. The function F will be linear in the variable $u = Ux_t \in C([-h, 0] \to R^n)$, T-periodic in t, whenever it is present and continuous. The forcing term $f(t)$ is continuous and T-periodic, while the terms g, h, d are T-periodic in t, continuous in their variables, and generally nonlinear. The notation x_t is the same as in the monograph of Hale [4].

In Section 2 we prove a degree-theoretic structure theorem for hereditary systems slightly more general than (1.1)–(1.4) and apply these results in subsequent sections to (1.1)–(1.4).

The simple case $x'(t) = -AUx_t(0) + f(t)$ is covered by (1.1) and is treated in Section 3. In particular, we are able to treat the scalar equation

$$x'(t) = -ax(t - r + \mu k(t, x(t))) + f(t),$$

investigated by Stephan [11]; the equation arises from an unsolved perturbation problem in biological modelling theory (see Cooke [2]). We show that the Lipschitz condition imposed by Stephan [11] can be dropped, the term $\mu k(t, x(t))$ can be more complicated, and we extend the range of known T-periodic solutions from $0 < aT < \pi/2$ to $0 < aT < 2(3)^{1/2}$.

* This research was supported by the U. S. Army under grant number ARO-D-31-124-72-G56.

The nonautonomous cases (1.2) and (1.3) are considered in Section 4. In particular, we can treat the equation

$$x'(t) = -a(t)\,x(t - r + k(t, x_t, \mu)) + f(t) \qquad \text{with} \qquad a(t) > 0$$

(see Corollary 4.5), and similar delay-differential systems (see Corollary 4.4).

The case of a nonlinear perturbation (1.4) presents a number of distinctive problems, and we indicate in Section 5 several ways to overcome these problems and obtain periodicity theorems. For example it is shown how to obtain 1-periodic solutions for the nonlinear equation

$$x'(t) = -\alpha x(t - 1 + \mu \sin 2\pi t)[1 + x(t)] + b \sin 2\pi t,$$

with $0 < \alpha < 2(3)^{1/2}$ and $|b|$ small enough (see Example 5.5).

In Section 6 we consider systems (1.5) and provide a simple geometric condition for the existence of T-periodic solutions. This equation is not necessarily a special case of Eqs. (1.1)–(1.4), since the right side need not be linear, and the term $d(t, x)$ is not required to be bounded.

We next define the *heredity operator* U. We shall consider only the simplest case, since the essential features of the calculations to be done will be sufficiently transparent for such operators. For extensions to more complicated operators we refer to Section 7.

Let $i = 1,..., n$ and let $k_i(t, x, \mu)$ be continuous on

$$R \times C([-h, 0] \to R^n) \times [-\mu_0, \mu_0],$$

with values in R, where $\mu_0 > 0$, $h > 0$. Let $r_i \in [0, h]$ $(1 \leqslant i \leqslant n)$. We define the coordinate operators

$$U^i: R \times C(R \to R^n) \to C([-h, 0] \to R^n)$$

by

$$U^i(t, x)(\theta) = x_i(t + \theta - r_i + k_i(t, x_t, \mu)), \quad -h \leqslant \theta \leqslant 0, \quad |\mu| \leqslant \mu_0,$$

and put

$$U(t, x) = (U^1(t, x),..., U^n(t, x)).$$

In order to avoid complications in notation we do not explicitly include the dependence on the parameter μ in our notation. Further, we abbreviate

$$U(t, x) \equiv Ux_t\,.$$

Concerning the dependence of k on the parameter μ, we assume that there exist continuous functions $\ell_i : [0, \infty) \to R$, $\ell_i(0) = 0$, such that

$$|k_i(t, x, \mu)| \leqslant \ell_i(|\mu|),$$

and we put $\ell(r) = (\ell_1(r),..., \ell_n(r))$.

2. Preliminaries

In the sections to follow we make use of several formulas from the theory of vector Fourier series, which we cite here for the reader's convenience. In addition, we give two structure theorems for periodic solutions of hereditary systems. The theorems are modeled after the functional results of Strygin [12]. Enough technical problems occur to justify including a complete proof of the main theorem. The results could also have been obtained by a modified Cesari method (see Mawhin [7]), though this approach seems more complicated in our opinion.

The calculations in vector Fourier series will occur in the space $L^2([0, T] \to R^n)$ with inner product

$$x \cdot y = \int_0^T x(t) \cdot y(t)\, dt$$

and norm $\| x \|_{L^2} = (x \cdot x)^{1/2}$. If $x(t): R \to R^n$ is a T-periodic function of class C^1, then $x(t)$ has the Fourier series

$$x(t) = \frac{1}{T} \int_0^T x(s)\, ds + \sum_{k=1}^{\infty} (a_k \cos \omega_k t + b_k \sin \omega_k t), \tag{2.1}$$

where $\omega_k = 2k\pi/T$, and the derivative $\dot{x}$ satisfies

$$\frac{2}{T} \| \dot{x} \|_{L^2}^2 = \sum_{k=1}^{\infty} (| \omega_k a_k |^2 + | \omega_k b_k |^2). \tag{2.2}$$

Using (2.1), (2.2) and the C-B-S inequality we obtain

$$\left| x(t) - \frac{1}{T} \int_0^T x(s)\, ds \right| \leqslant \left(\sum_{k=1}^{\infty} \omega_k^{-2} \right)^{1/2} \left(\frac{2}{T} \| \dot{x} \|_{L^2}^2 \right)^{1/2}, \tag{2.3}$$

the sum on the right side in (2.3) can be evaluated:

$$\sum_{k=1}^{\infty} \omega_k^{-2} = T^2/24. \tag{2.4}$$

Combining (2.3) and (2.4) gives

$$\left| x(t) - \frac{1}{T} \int_0^T x(s)\, ds \right| \leqslant (T/12)^{1/2} \| \dot{x} \|_{L^2}. \tag{2.5}$$

The following structure theorems for periodic solutions of hereditary differential systems will be used repeatedly in the sequel.

THEOREM 2.1. *Let $\mathcal{O}$ be a bounded open convex set in R^n, $G = C([-h, 0] \to \mathcal{O})$, and assume*

(i) *$f(t, u, \epsilon) : R \times \bar{G} \times [0, \epsilon_0] \to R^n$ is continuous, bounded, and T-periodic in t, $T \geqslant h$.*

(ii) *The field $\mathcal{F} : \bar{\mathcal{O}} \to R^n$, defined by*

$$\mathcal{F}(u_0) = -\frac{1}{T}\int_0^T f(t, u_0, 0)\, dt, \qquad u_0 \in \bar{\mathcal{O}},$$

is nonzero on $\partial\mathcal{O}$ and has Brouwer degree $d(\mathcal{F}, \mathcal{O}, 0) \neq 0$.

Then the hereditary system

$$x'(t) = \epsilon f(t, Ux_t, \epsilon) \tag{2.6}$$

has a T-periodic solution x with values in $\mathcal{O}$, for all small $\epsilon > 0$.

THEOREM 2.2. *Let ϵ_0 be a positive number and let the hypotheses of Theorem 2.1 hold. Further assume that all possible T-periodic solutions $x(t)$ of (2.6) for $0 < \epsilon \leqslant \epsilon_0$ satisfy $x(t) \notin \partial\mathcal{O}$, $0 \leqslant t \leqslant T$. Then (2.6) has a T-periodic solution for every ϵ, $0 < \epsilon \leqslant \epsilon_0$, with values in $\mathcal{O}$.*

Proof of Theorem 2.1. Let $\mathcal{O}_1$ be an open set such that $\bar{\mathcal{O}}_1 \subset \mathcal{O}$ and such that for some $\delta > 0$, a δ-neighborhood of $\bar{\mathcal{O}}_1$ also belongs to $\mathcal{O}$, and the field in (ii) does not vanish on $\bar{\mathcal{O}} - \mathcal{O}_1$. Put

$$G_0 = \{x \in C([0, T] \to R^n) : x(t) \in \mathcal{O}_1, |x(0) - x(T)| < \delta\}.$$

Define the imbedding operator $\mathcal{L}$ as follows:

$$\mathcal{L} : [0, 1] \times C([0, T] \to R^n) \to C(R \to R^n)$$

$\mathcal{L}(\lambda, x) = y$ if and only if

$$y(t) = \lambda\bar{y}(t) + (1 - \lambda)\bar{y}(T),$$

where $\bar{y}(t)$ is the $4T$-periodic function given by

$$\bar{y}(t) = \begin{cases} \bar{x}(t) & , \quad -T \leqslant t \leqslant T \\ \bar{x}(2T - t), & \quad T \leqslant t \leqslant 3T \end{cases}$$

and

$$\bar{x}(t) = \begin{cases} x(t), & 0 \leqslant t \leqslant T \\ x(t + T) - x(T) + x(0), & \quad -T \leqslant t \leqslant 0. \end{cases}$$

The operator $\mathscr{L}$ is continuous, with the usual norm in $[0, 1] \times C([0, T] \to R^n)$ and the topology of uniform convergence in the range space $C(R \to R^n)$.

Define $a(t, \lambda) = \lambda t + (1 - \lambda)T$, $0 \leqslant t \leqslant T$, $0 \leqslant \lambda \leqslant 1$ and consider the field $H: \bar{G}_0 \times [0, 1] \times [0, \epsilon_0] \to C([0, T] \to R^n)$ defined by

$$H(x, \lambda, \epsilon)(t) = x(t) - x(T) - \epsilon \int_0^{a(t,\lambda)} f(s, U\mathscr{L}(\lambda, x)_s, \epsilon)\, ds. \tag{2.7}$$

The system (2.6) will have a T-periodic solution $y = \mathscr{L}(1, x)$, $x \in \mathcal{O}_1$, provided the field $H(x, 1, \epsilon)$ vanishes at x, as one can easily verify. Therefore, we show that for $\epsilon > 0$ sufficiently small, the field (2.7) does not vanish on ∂G_0 for $0 \leqslant \lambda \leqslant 1$, that each $H(\cdot, \lambda, \epsilon)$ $(0 \leqslant \lambda \leqslant 1)$ is a completely continuous perturbation of the identity in $C([0, T] \to R^n)$, and $d(H(\cdot, 0, \epsilon), G_0, 0) \neq 0$. Then the result follows from homotopy invariance of Leray–Schauder degree [10].

The set $\{U\mathscr{L}(\lambda, x)_t : 0 \leqslant t \leqslant T, \lambda \in [0, 1]\}$ will belong to G if $x \in \bar{G}_0$, because $\mathscr{L}(\lambda, x)(s)$ is within δ of a value of x. The Arzela–Ascoli theorem applies because of (i) to show $H - I$ compact. We use the continuity of $\mathscr{L}$ and the definition of U to verify continuity of $H - I$; hence, $H - I$ is completely continuous.

If the assertion of H being singularity-free on ∂G_0 $0 \leqslant \lambda \leqslant 1$ fails, then we can select a sequence of ϵ's and x's and λ's which make (2.7) equal to zero, with $\epsilon \to 0$. We put $t = T$ in (2.7), use the compactness of $H - I$, and find that the field in (ii) vanishes at some point of ∂G_0, which is impossible by the choice of G_0.

Let us put $\lambda = 0$ in (2.7). Then $\mathscr{L}(0, x) = x(T)$, $U\mathscr{L}(0, x)_s \equiv x(T)$ and $H(\cdot, 0, \epsilon) - I$ is finite-dimensional. For $\epsilon > 0$ sufficiently small, its degree is defined and coincides with Brouwer degree of the field in (ii), because of continuity in the parameter ϵ, and the definition of degree. Therefore, $d(H(\cdot, 0, \epsilon), G_0, 0) \neq 0$ for all small $\epsilon > 0$, and the proof is complete.

Proof of Theorem 2.2. There does not exist a sequence of T-periodic solutions of (2.6) $(0 < \epsilon \leqslant \epsilon_0)$ with values in $\mathcal{O}$ whose values cluster on $\partial\mathcal{O}$, because any such sequence is precompact and T-periodic solutions of (2.6) $(0 < \epsilon \leqslant \epsilon_0)$ satisfy $x(t) \notin \partial\mathcal{O}$, while the field in (ii) being nonzero on $\partial\mathcal{O}$ eliminates $\epsilon \to 0$.

Therefore, we may replace the set $\mathcal{O}_1$ in the proof of Theorem 2.1 by a slightly larger set, so that all possible T-periodic solutions of (2.6) for $0 < \epsilon \leqslant \epsilon_0$ satisfy $x(t) \notin \bar{\mathcal{O}} - \mathcal{O}_1$. The field $H(x, 1, \epsilon)$ cannot vanish on ∂G_0, so the result follows by homotopy invariance of Leray–Schauder degree applied to the parameter ϵ. In fact, $d(H(\cdot, 1, \epsilon), G_0, 0) = d(\mathscr{F}, \mathcal{O}, 0)$, $0 < \epsilon \leqslant \epsilon_0$.

THEOREM 2.3. *Let $\mathcal{O}$, G, f, $\mathcal{F}$ satisfy the conditions of Theorem* 2.1, *and suppose that the possible T-periodic solutions of*

$$x'(t) = \epsilon f(t, Ux_t, \epsilon), \qquad (|\mu| \leqslant \mu_0) \quad (0 < \epsilon \leqslant 1)$$

satisfy $x(t) \notin \partial\mathcal{O}$. Let N_μ be the set of T-periodic solutions of $x'(t) = f(t, Ux_t, 1)$ with values in $\mathcal{O}$. Then $\rho(N_\mu, N_0) \equiv \inf\{\| x - y \| : x \in N_\mu, y \in N_0\} \to 0$ as $|\mu| \to 0$.

Proof. The union of the sets N_μ is a precompact family in $C([0, T] \to \mathcal{O})$. If $\{\mu_n\} \to 0$, $x_n \in N_{\mu_n}$, and $\{x_n\}$ converges, then its limit is in N_0. The conclusion follows.

3. AUTONOMOUS SYSTEMS

Let $F(u)$ be a continuous linear functional on $C([-h, 0] \to R^n)$, with values in R^n, $f(t)$ a continuous T-periodic R^n-valued function, and consider the hereditary nonhomogeneous differential system

$$x'(t) = -F(Ux_t) + f(t). \tag{3.1}$$

Write the functional $F(u)$ in integral form via the Riesz theorem:

$$F(u) = \int_{-h}^{0} d\eta(\theta)\, u(\theta), \tag{3.2}$$

where the matrix $\eta(\theta)$ is of bounded variation.

The purpose of this section is to establish the existence of T-periodic solutions of (3.1) for small $|\mu|$. The hypotheses are given in terms of the matrix $\eta(\theta)$ in the representation (3.2).

We single out the particular case $F(u) = Au(0)$ where A is an $n \times n$ matrix. In this case, $\eta(\theta) = 0$ for $-h \leqslant \theta < 0$, $\eta(0) = A$.

LEMMA 3.1. *Let $A \equiv \int_{-h}^{0} d\eta(\theta)$ be invertible, and assume that for $|\mu| \leqslant \mu_0$ the following inequality is valid.*

$$E(\mu) \equiv 1 - \| A^{-1} \| \| F \|^2 \, |\ell(|\mu|)| - T \| F \| / (12)^{1/2} > 0.$$

Put

$$D(\mu) \equiv \| A^{-1} \| \| F \| \, |\ell(|\mu|)| + \| A^{-1} \| + T/(12)^{1/2}.$$

Then for every $\mu \in [-\mu_0, \mu_0]$ every T-periodic solution $x(t)$ of the equation,

$$x'(t) = \epsilon[-F(Ux_t) + f(t)] \qquad (0 < \epsilon \leqslant 1),$$

satisfies the inequality

$$\| x \| \leqslant [D(\mu)/E(\mu)] \, \| f \|.$$

Proof. Let $x(t)$ be a possible T-periodic solution, and integrate the equation to obtain

$$\int_0^T F(Ux_t)\, dt = \int_0^T f(t)\, dt.$$

We apply the standard interchange theorem for Stieltjes integrals to get

$$\frac{1}{T} \int_{-h}^{0} d\eta(\theta) \int_0^T [Ux_t](\theta)\, dt = \frac{1}{T} \int_0^T f(t)\, dt. \tag{3.3}$$

Let V be the heredity operator U when $\mu = 0$.

Let us estimate the integral

$$\frac{1}{T} \int_{-h}^{0} d\eta(\theta) \int_0^T [Ux_t - Vx_t](\theta)\, dt, \tag{3.4}$$

for such a solution. We have

$$\begin{aligned} | [U_i x_t - V_i x_t](\theta) | &= \left| \int_0^1 x'(t - r_i + \theta + s k_i(t, x_t, \mu))\, k_i(t, x_t, \mu)\, ds \right| \\ &\leqslant \ell_i(|\mu|) \, [\max_{0 \leqslant \tau \leqslant T} | F(Ux_\tau)| + \| f \|]. \end{aligned}$$

On the other hand,

$$\max_{0 \leqslant \tau \leqslant T} | F(Ux_\tau) | \leqslant \| F \| \, \| x \|,$$

with

$$\| F \| = \int_{-h}^{0} | d\eta(\theta) |.$$

Therefore, the integral in (3.4) is bounded in norm by

$$K = \| F \|^2 \, \| x \| \, | \ell(|\mu|)| + \| F \| \, \| f \| \, | \ell(|\mu|)|. \tag{3.5}$$

Next, let us calculate the integral

$$\int_{-h}^{0} d\eta(\theta) \int_0^T [Vx_t](\theta)\, dt.$$

One gets, by a change of variable and periodicity, the relation

$$\int_0^T [V_i x_t](\theta)\, dt = \int_0^T x_i(t - r_i + \theta)\, dt = \int_0^T x_i(s)\, ds,$$

so it follows that

$$\int_{-h}^{0} d\eta(\theta) \int_{0}^{T} [Vx_t](\theta)\, dt = [\eta(0) - \eta(-h)] \int_{0}^{T} x(t)\, dt. \tag{3.6}$$

The matrix $A \equiv \eta(0) - \eta(-h)$ is invertible, by hypothesis. We compute a bound on $|x(t)|$ via the inequality

$$|x(t)| \leqslant \left| \frac{1}{T} \int_{0}^{T} x(s)\, ds \right| + \left| x(t) - \frac{1}{T} \int_{0}^{T} x(s)\, ds \right|. \tag{3.7}$$

The first term in (3.7) can be estimated by (3.3)–(3.6):

$$\begin{aligned} \left| \frac{1}{T} \int_{0}^{T} x(t)\, dt \right| &\leqslant \| A^{-1} \| \left| \frac{1}{T} \int_{0}^{T} Ax(t)\, dt \right| \\ &\leqslant \| A^{-1} \| \left(K + \left| \frac{1}{T} \int_{0}^{T} f(t)\, dt \right| \right) \\ &\leqslant \| A^{-1} \| \, [K + \| f \|]. \end{aligned} \tag{3.8}$$

Since $\| x' \|_{L^2} \leqslant (T)^{1/2} \| F \| \| x \| + (T)^{1/2} \| f \|$ (via the differential equation) one can use inequalities (2.5) and (3.8) in (3.7) to get

$$|x(t)| \leqslant \| A^{-1} \| \, [K + \| f \|] + T[\| F \| \| x \| + \| f \|]/(12)^{1/2}.$$

A rearrangement of terms completes the proof.

THEOREM 3.2. *Let the following conditions be met*: (a) $F(u)$ *has the representation* (3.2);

(b) $A = \int_{-h}^{0} d\eta(\theta)$ *is invertible, and*

(c) $E(\mu) \equiv 1 - \| A^{-1} \| \| F \|^2 \, | \ell(|\mu|) | - T \| F \| / (12)^{1/2} > 0$ *for* $-\mu_0 \leqslant \mu \leqslant \mu_0$.

Then the equation $x'(t) = -F(Ux_t) + f(t)$ *has at least one T-periodic solution for* $|\mu| \leqslant \mu_0$.

Proof. By Lemma 3.1, all possible T-periodic solutions of the auxiliary equation are bounded by some number $R > 0$. We choose $R > 0$ even larger, if needed, to insure that

$$-F(u_0) + \frac{1}{T} \int_{0}^{T} f(t)\, dt \neq 0,$$

for $|u_0| = R$, $u_0 \in R^n$. This can be done, because

$$F(u_0) = Au_0,$$

and A is invertible. The Brouwer degree of the mapping

$$u_0 \to -\frac{1}{T}\int_0^T [-F(u_0) + f(t)]\, dt$$

at 0 relative to $|u_0| < R$ is precisely the sign of the determinant of A, hence, nonzero. An application of Theorem 2.2 completes the proof.

COROLLARY 3.3. *If* $T\|F\| < 2(3)^{1/2}$ *and* (a) *and* (b) *hold, then* $x'(t) = -F(Ux_t) + f(t)$ *has T-periodic solutions for all small* $\mu > 0$.

COROLLARY 3.4. *If* $0 < |a|\, T < 2(3)^{1/2}$, $k(t, x)$ *is a bounded continuous function, T-periodic in t, then*

$$x'(t) = -ax(t - r + \mu k(t, x(t))) + f(t) \tag{3.9}$$

has T-periodic solutions for all small $\mu > 0$, *for every T-periodic continuous f.*

Remark. Equation (3.9) is the subject of study in [11]. There it is proved that for $a > 0$ and $ar < \pi/2$ (3.9) will have a T-periodic solution for all $\mu > 0$ small enough, for every T-periodic, continuous f, $T \geqslant r$. Corollary 3.4 extends the result in [11] to the range

$$(\pi/2) \leqslant |a|\, r \leqslant |a|\, T < 2(3)^{1/2}.$$

COROLLARY 3.5. *The periodic solutions whose existence follows from Theorem* 3.2 *tend, as* $|\mu| \to 0$, *to the periodic solutions of the equation*

$$x'(t) = -F(Vx_t) + f(t).$$

Proof. Apply Theorem 2.3.

EXAMPLE 3.6. The equation,

$$x'(t) = -x(t - 1 + \mu \sin 2\pi t) + \sin 2\pi t, \tag{3.10}$$

has 1-periodic solutions for $0 \leqslant |\mu| < 1 - (3^{1/2}/6)$.

Remark. It follows from the work of Hale [3] that (3.10) has 1-periodic solutions for $0 \leqslant \mu < \pi/2 - 1$. Example 3.6 shows that for a somewhat larger range of values of μ the existence of 1-periodic solutions still follows. However, for the range $\pi/2 - 1 \leqslant \mu < 1 - (3^{1/2}/6)$ the uniqueness and asymptotic stability obtained by Hale [3] cannot be proved using our methods.

4. Nonautonomous Systems

Let $F(t, u)$ be a continuous functional on $(-\infty, \infty) \times C([-h, 0] \to R^n)$ with values in R^n, $F(t+T, u) \equiv F(t, u)$, $f(t)$ a continuous T-periodic R^n-valued function, and consider the hereditary nonhomogeneous differential system

$$x'(t) = -F(t, Ux_t) + f(t). \tag{4.1}$$

The purpose of this section is to establish the existence of T-periodic solutions of Eq. (4.1) for $|\mu|$ sufficiently small.

Let us apply Riesz' theorem in order to write $F(t, u)$ in the integral form

$$F(t, u) = \int_{-h}^{0} d\eta(t, \theta)\, u(\theta), \tag{4.2}$$

where $\eta(t, \theta)$ is an $n \times n$ matrix of bounded variation, $\eta(t, -h) \equiv 0$. The hypotheses will depend on $\eta(t, \theta)$. We single out the special case $F(t, u) = A(t)\, u(0)$ with $A(t)$ an $n \times n$ T-periodic continuous matrix, then use results for this special case to obtain results for (4.1). In this case, $\eta(t, \theta) = 0$ $(-h \leqslant \theta < 0)$ and $\eta(t, 0) = A(t)$.

We establish the following lemma for the special case when $F(t, u) \equiv A(t)\, u(0)$ in Eq. (4.1).

Lemma 4.1. *Let $A(t)$ be T-periodic, of class C^1, with $A(T)$ invertible. Assume that the following inequality is valid for $|\mu| \leqslant \mu_0$:*

$$E_1(\mu) \equiv 1 - \| A(T)^{-1} \| \, [\| A(\cdot)\|^2 \, | \ell(|\mu|)| + T \| A'(\cdot)\|/2] - T \| A(\cdot)\|/(12)^{1/2} > 0.$$

Put

$$D_1(\mu) = \| A(T)^{-1} \| \, [1 + \| A(\cdot)\| \, | \ell(|\mu|)|] + T/(12)^{1/2}.$$

Then for every $\mu \in [-\mu_0, \mu_0]$ every T-periodic solution $x(t)$ of the auxiliary equation,

$$x'(t) = \epsilon[-A(t)\, Ux_t(0) + f(t)] \qquad (0 < \epsilon \leqslant 1),$$

satisfies the inequality

$$\| x \| \leqslant [D_1(\mu)/E_1(\mu)] \, \| f \|.$$

Proof. Let $x(t)$ be a possible T-periodic solution. Integrate the equation on $[0, T]$ to obtain

$$\frac{1}{T} \int_0^T A(t)\, Vx_t(0)\, dt = \frac{1}{T} \int_0^T f(t)\, dt + \frac{1}{T} \int_0^T A(t)[Vx_t - Ux_t](0)\, dt. \tag{4.3}$$

The integral on the left side is subjected to an integration by parts to obtain

$$\int_0^T A(t)\, Vx_t(0)\, dt = A(T) \int_0^T x(s)\, ds - \int_0^T A(t) \int_0^t Vx_s(0)\, ds\, dt, \quad (4.4)$$

because

$$\int_0^T Vx_t(0)\, dt = \int_0^T x(s)\, ds,$$

as shown previously. Therefore, the invertibility of $A(T)$ and relation (4.4) give the inequality

$$\left| \frac{1}{T} \int_0^T x(s)\, ds \right| \leqslant \| A(T)^{-1} \| \left| \frac{1}{T} \int_0^T A(t)\, Vx_t(0)\, dt \right| + T \| A(T)^{-1} \| \, \| A'(\cdot) \| \, \| x \| /2. \quad (4.5)$$

The second term on the right in relation (4.3) can be estimated by the methods already exploited in the autonomous case, and we arrive at the bound

$$K \equiv [\| A(\cdot) \|^2 \| x \| + \| A(\cdot) \| \, \| f \|] \, | \ell(| \mu |) |; \quad (4.6)$$

hence, by (4.5) we have

$$\left| \frac{1}{T} \int_0^T x(s)\, ds \right| \leqslant \| A(T)^{-1} \| \, [\| f \| + K] + T \| A(T)^{-1} \| \, \| A'(\cdot) \| \, \| x \| /2. \quad (4.7)$$

As in the autonomous case, one can easily verify, using (4.7) and the differential equation, that

$$| x(t) | \leqslant \| A(T) \|^{-1} [\| f \| + K + T \| A'(\cdot) \| \, \| x \| /2] + T[\| A(\cdot) \| \, \| x \| + \| f \|]/(12)^{1/2}. \quad (4.8)$$

A rearrangement of terms completes the proof.

With the aid of Lemma 4.1, we now extend the bounds obtained for the special case of a matrix to system (4.1).

LEMMA 4.2. *Let $\eta(t, 0)$ be of class C^1, $\eta(T, 0)$ invertible, $\eta(t + T, \theta) \equiv \eta(t, \theta)$, and put*

$$M = \sup_{0 \leqslant t \leqslant T} \left(\int_{-h}^0 \| \eta(t, \theta) \|^2 \, d\theta \right)^{1/2},$$

$$||| F ||| = \sup \{ \| F(t, \cdot) \| : t \in [0, T] \},$$

$$E_1(\mu) \equiv 1 - \| \eta(T, 0)^{-1} \| \, [\| \eta(\cdot, 0) \|^2 \, | \ell(| \mu |) | + T \| \eta'(\cdot, 0) \| /2] - T \| \eta(\cdot, 0) \| /(12)^{1/2},$$

$$D_1(\mu) \equiv \| \eta(T, 0)^{-1} \| \, [1 + \| \eta(\cdot, 0) \| \, | \ell(| \mu |) |] + T/(12)^{1/2},$$

$$E_2(\mu) \equiv E_1(\mu) - D_1(\mu)[||| F |||^2 + M(T)^{1/2} ||| F |||],$$

$$D_2(\mu) = D_1(\mu)[||| F ||| \, | \ell(| \mu |) | + M(T)^{1/2}].$$

If $E_2(\mu) > 0$ *for* $|\mu| \leqslant \mu_0$, *then all possible T-periodic solutions of the auxiliary equation,*

$$x'(t) = \epsilon[-F(t, Ux_t) + f(t)] \qquad (0 < \epsilon \leqslant 1),$$

satisfy the inequality

$$\|x\| \leqslant [D_2(\mu)/E_2(\mu)] \|f\| \qquad (|\mu| \leqslant \mu_0).$$

Proof. Let $x(t)$ be a possible T-periodic solution. Then $x(t)$ is a T-periodic solution of the equation

$$x'(t) = \epsilon[-A(t)\, Vx_t(0) + f_1(t)], \tag{4.9}$$

where $A(t) = \eta(t, 0)$ and

$$f_1(t) = \int_{-h}^{0} d\eta(t, \theta)[Vx_t - Ux_t](\theta) + \int_{-h}^{0} \eta(t, \theta)[Vx_t]'(\theta)\, d\theta. \tag{4.10}$$

Let us apply the Lemma already proved for Eqs. (4.9) to obtain

$$E_1(\mu) \|x\| \leqslant D_1(\mu) \|f_1\|. \tag{4.11}$$

The estimates already obtained in previous proofs give

$$\left| \int_{-h}^{0} d\eta(t, \theta)[Vx_t - Ux_t](\theta) \right| \leqslant |||F|||\, [|||F|||\, \|x\| + \|f\|]\, |\ell(|\mu|)|,$$

and since $T \geqslant h$ the C-B-S inequality and the differential equation give

$$\left| \int_{-h}^{0} \eta(t, \theta)[Vx_t]'(\theta)\, d\theta \right| \leqslant M(T)^{1/2}\, [|||F|||\, \|x\| + \|f\|].$$

The function $f_1(t)$ will, therefore, satisfy the inequality

$$\|f_1\| \leqslant [|||F|||^2 \|x\| + |||F|||\, \|f\|]\, |\ell(|\mu|)| + M(T)^{1/2}\, [|||F|||\, \|x\| + \|f\|]. \tag{4.12}$$

The two inequalities (4.11) and (4.12) imply the conclusion of the lemma, and the proof is complete.

THEOREM 4.3. *Let the following conditions hold*: (a) $F(t, u)$ *has the representation* (4.2);

(b) $\eta(t, 0)$ *is of class* C^1, $\eta(T, 0)$ *and* $\int_0^T \eta(t, 0)\, dt$ *are invertible*; *and*

(c) *The function* $E_2(\mu)$ *of Lemma* 4.2 *is positive for* $|\mu| \leqslant \mu_0$.

Then the system $x'(t) = -F(t, Ux_t) + f(t)$ *has at least one T-periodic solution* $(|\mu| \leqslant \mu_0)$, *for every T-periodic continuous* f.

Proof. The proof is identical with that of Theorem 3.2, by virtue of Lemma 4.2, except that we must calculate the degree of the mapping

$$u_0 \to -\frac{1}{T}\int_0^T F(t, u_0)\,dt + \frac{1}{T}\int_0^T f(t)\,dt,$$

$|u_0| < R$, $R > 0$ sufficiently large. One has

$$\begin{aligned}\int_0^T F(t, u_0)\,dt &= \left[\int_0^T \int_{-h}^0 d\eta(t, \theta)\,dt\right] u_0 \\ &= \left[\int_0^T \eta(t, 0)\,dt\right] u_0;\end{aligned}$$

hence, the degree of the mapping is nonzero by (b), completing the proof.

COROLLARY 4.4. *If $A(t)$ is a continuously differentiable T-periodic $n \times n$ matrix, $A(T)$ and $\int_0^T A(t)\,dt$ are invertible, and $E_2(\mu) > 0$ ($|\mu| \leqslant \mu_0$), then the system $x'(t) = -A(t)\,Ux_t(0) + f(t)$ has at least one T-periodic solution ($|\mu| \leqslant \mu_0$), for every T-periodic continuous f.*

COROLLARY 4.5. *If $a(t) \neq 0$ is T-periodic, continuously differentiable, $k(t, x)$ is a bounded continuous function, $k(t + T, x) \equiv k(t, x)$, and $T\|a\| < 2(3)^{1/2}$, then whenever $0 < r \leqslant h \leqslant T$ the scalar equation,*

$$x'(t) = -a(t)\,x(t - r + \mu k(t, x(t))) + f(t),$$

has T-periodic solutions for small $|\mu|$, for every T-periodic continuous f.

COROLLARY 4.6. *The periodic solutions given by Theorem 4.3 tend, as $\mu \to 0$, to the periodic solutions of the equation $x'(t) = -F(t, Vx_t) + f(t)$.*

5. SYSTEMS SUBJECT TO NONLINEAR FORCES

The inequalities developed in Sections 3 and 4 supply a simple vehicle for obtaining periodicity theorems for the forced nonautonomous hereditary system

$$x'(t) + F(t, Ux_t) = g(t, x_t, U^*x_t), \tag{5.1}$$

where U^* is an operator of the same kind as U, with or without parameter dependence, and g is a continuous R^n-valued functional on

$$R \times C([-h, 0] \to R^n) \times C([-h, 0] \to R^n),$$

$g(t, u, v) \equiv g(t + T, u, v)$.

The idea is to consider a possible T-periodic solution $x(t)$ of (5.1), put $f(t) = g(t, x_t, U^*x_t)$, and apply the inequalities of Section 3 and 4 to the systems (3.1) and (4.1). We may then put hypotheses on g to insure that $x(t)$ belongs to a certain bounded open set in R^n, add Brouwer degree hypotheses, and obtain periodicity theorems by arguments involving Theorems 2.1, 2.2, Schauder's theorem, or Leray–Schauder degree theory. Two illustrations will be given.

THEOREM 5.1. *Let the following conditions hold*:

(a) $F(t, u)\colon R \times C([-h, 0] \to R^n) \to R^n$ *is continuous*;

(b) *the conditions of Lemma* 4.2 *are met*; *and*

(c) $g(t, u, v)\colon R \times C([-h, 0] \to R^n) \times C([-h, 0] \to R^n) \to R^n$ *is continuous, and for some* $a \geqslant 0$, $b \geqslant 0$, $\alpha \in [0, 1)$,

$$| g(t, x_t, U^*x_t)| \leqslant a \| x \|^\alpha + b \qquad (0 \leqslant t \leqslant T),$$

for every T-periodic continuous $x(t)\colon R \to R^n$; *and*

(d) *the mapping* $u_0 \to 1/T \int_0^T [g(t, u_0, u_0) - F(t, u_0)]\, dt$ *does not vanish for* $| u_0 | \geqslant R_0$, $u_0 \in R^n$, *and its Brouwer degree at* 0 *relative to* $| u_0 | < R_0$ *is nonzero*.

Then system (5.1) *has at least one T-periodic solution for* $| \mu | \leqslant \mu_0$.

COROLLARY 5.2. *The theorem is valid for* $\alpha = 1$, *provided*

$$E_2(\mu) - aD_2(\mu) > 0 \qquad (| \mu | \leqslant \mu_0).$$

Proof. Let $R_1 > 0$ be the largest positive solution of the inequality $E_2(\mu) R_1 \leqslant D_2(\mu)[aR_1^\alpha + b]$. Then by Lemma 4.2 and (c) it follows that all possible T-periodic solutions of the auxiliary equation,

$$x'(t) = \epsilon[-F(t, Ux_t) + g(t, x_t, U^*x_t)] \qquad (0 < \epsilon \leqslant 1), \tag{5.2}$$

are bounded by R_1.

Choose $R > R_1$ and $R > R_0$. We complete the proof by applying (d) and Theorem 2.2.

THEOREM 5.3. *Let* (a), (b) *of Theorem* 5.1 *hold, and in addition assume*:

(c′) $G(t, u, v)$ *is continuous. Also there exists a polynomial* $P(\cdot)$ *and a number* $r > 0$ *such that*

$$D_2(\mu)\, P(r) < E_2(\mu) r, \qquad | \mu | \leqslant \mu_0$$

and such that

$$| G(t, x_t, U^* x_t)| \leqslant P(r)$$

for all continuous T-periodic functions $x: R \to R^n$ *satisfying* $\| x \| \leqslant r$ *and*

(d′) *The mapping* $u_0 \to 1/T \int_0^T [g(t, u_0, u_0) - F(t, u_0)]\, dt$ *is nonzero for* $| u_0 | = r$, $u_0 \in R^n$, *and has nonzero Brouwer degree at* 0 *relative to* $| u_0 | < r$.

Then system (5.1) *has at least one T-periodic solution for* $| \mu | \leqslant \mu_0$.

Proof. By Lemma 4.2 and (c′) it follows that a possible T-periodic solution of system (5.2) must satisfy $E_2(\mu) | x(t)| \leqslant D_2(\mu) P(r)$; hence, $| x(t)| \neq r$ by (c′). Because of (d′), we may apply Theorem 2.2 to complete the proof.

EXAMPLE 5.4. The scalar equation,

$$x'(t) = -\alpha x(t - 1 + \mu \sin 2\pi t) + ax(t) + b \sin 2\pi t,$$

will have 1-periodic solutions for all small $| \mu |$, provided

$$1 - | \alpha | - | a | [1/| \alpha | + T/(12)^{1/2}] > 0,$$

by Corollary 5.2.

EXAMPLE 5.5. The scalar equation,

$$x'(t) = -\alpha x(t - 1 + \mu \sin 2\pi t)[1 + x(t)] + b \sin 2\pi t,$$

has 1-periodic solutions for all small $| \mu |$, provided $0 < | \alpha | < 2(3)^{1/2}$, $| b | < b_0 \equiv | \alpha | [((12)^{1/2} - | \alpha |)/2(12)^{1/2} + | \alpha |)]^2$.

Indeed, under the assumptions made, any number r with $1 > r > \frac{1}{2}((12)^{1/2} - | \alpha |)/((12)^{1/2} + | \alpha |) - (b_0/| \alpha | - 4 | b |/| \alpha |)^{1/2}$ will be a solution of the inequality,

$$r^2 - \left(\frac{(12)^{1/2} - | \alpha |}{(12)^{1/2} + | \alpha |}\right) r + | b |/| \alpha | < 0,$$

and, hence, for $| \mu | \leqslant \mu_0$ and μ_0 sufficiently small, the numbers $E(\mu)$, $D(\mu)$ of Lemma 3.1 will satisfy $D_2(\mu) P(r) < E_2(\mu) r$, with $P(r) = | \alpha | r^2 + | b |$.

In order to apply Theorem 5.3 to the example, we must calculate $d(H, | x | < r, 0)$ for the mapping $Hx = \alpha x(1 + x)$. By definition, $d(H, | x | < r, 0) = \text{sign}[\alpha] \neq 0$, because $0 < r < 1$.

Remark. The term $b \sin 2\pi t$ can be replaced by a continuous 1-periodic function $f(t)$, of small norm, to obtain a similar result with $\int_0^T f(t)\, dt$ not necessarily zero.

6. Systems of Second Order Hereditary Equations

Let $h\colon R \times R^n \times R^n \to R^n$ and $d\colon R \times R^n \to R$ be continuous. Consider the hereditary system

$$x''(t) = h(t, x(t), x(t - d(t, x(t)))). \tag{6.1}$$

We assume that h and d are T-periodic in t.

The purpose of this section is to establish the following result.

THEOREM 6.1. *Let $\Omega \subseteq R^n$ be a bounded open convex set containing* 0, *and assume there exists a function*

$$\mathcal{N}\colon \partial\Omega \to R^n - \{0\},$$

such that

(i) $\mathcal{N}(x) \cdot x > 0$, $x \in \partial\Omega$.

(ii) *For every $x \in \partial\Omega$ and $y \in \bar{\Omega}$, $\mathcal{N}(x) \cdot h(t, x, y) > 0$ for $0 \leqslant t \leqslant T$.*

(iii) $\bar{\Omega} \subseteq \{ y : \mathcal{N}(x) \cdot (y - x) \leqslant 0\}$ *for all $x \in \partial\Omega$.*

Then Eq. (6.1) has a T-periodic solution $x(t)$ with values in Ω.

Remarks. We note that h only needs to be defined on $R \times \bar{\Omega} \times \bar{\Omega} \to R^n$ and that the term d need not be bounded. Theorem 6.1 provides a simple geometric condition for the existence of T-periodic solutions of (6.1). The hypotheses are motivated by results for ordinary differential equations obtained by Bebernes and Schmitt [1], Knobloch [6], Schmitt [8], and results for linear delay-differential equations in Schmitt [9].

We prove Theorem 6.1 by applying Theorem 2.2 to a hereditary differential system which is a modification of (6.1) and has the property that T-periodic solutions of it with range contained in Ω are T-periodic solutions of (6.1).

For every $x \notin \bar{\Omega}$ let $\bar{x}$ be the positive constant multiple of x on $\partial\Omega$, and define

$$h_1(t, x, y) = \begin{cases} h(t, x, y), & x \in \bar{\Omega} \\ h(t, \bar{x}, y), & x \notin \bar{\Omega}, \end{cases}$$

further let

$$H(t, x, y) = \begin{cases} h_1(t, x, y), & y \in \bar{\Omega} \\ h_1(t, x, \bar{y}), & y \notin \bar{\Omega} \end{cases}$$

and

$$k(t, x) = d(t, \bar{x}).$$

We now consider the modified equation

$$x''(t) = \epsilon H(t, x(t), x(t - k(t, x(t)))), \qquad 0 < \epsilon \leqslant 1. \tag{6.2}$$

LEMMA 6.2. *If $x(t)$ is a T-periodic solution of* (6.2) *with* $\epsilon = 1$ *and* $x(t) \in \bar{\Omega}$, $0 \leqslant t \leqslant T$, *then* $x(t)$ *is a T-periodic solution of* (6.1).

By virtue of Lemma 6.2, the proof of Theorem 6.1 is established by showing the existence of a T-periodic solution of (6.2) for $\epsilon = 1$ whose range is contained in Ω.

LEMMA 6.3. *For* $0 < \epsilon \leqslant 1$, *all possible T-periodic solutions of* (6.2) *have range in* Ω.

Proof. Put

$$\Omega_\lambda = \{x \notin \bar{\Omega} : | x - \bar{x} | < \lambda | x |\} \cup \bar{\Omega}, \qquad 0 < \lambda < 1,$$
$$\Omega_0 = \Omega.$$

Then each Ω_λ, $0 \leqslant \lambda < 1$, is a bounded open convex set containing 0 and $\bigcup_{\lambda \geqslant 0} \Omega_\lambda = R^n$. Let $x(t)$ be a T-periodic solution of (6.2) and assume that its range is not contained in Ω. Then there is a $\lambda \geqslant 0$ such that $x(t) \in \partial\Omega_\lambda$ for some t and, hence, a maximal $\lambda \geqslant 0$ and a $t_0 \in [0, T]$ such that $x(t_0) \in \partial\Omega_\lambda$ and $x(t) \in \bar{\Omega}_\lambda$ for all other values of t. In particular $x(t - k(t, x(t))) \in \bar{\Omega}_\lambda$, $0 \leqslant t \leqslant T$.

Letting $x_0 = x(t_0)$, $y_0 = x(t_0 - k(t, x_0))$ we obtain

$$\mathcal{N}(\bar{x}_0) \cdot \epsilon H(t_0, x_0, y_0) > 0.$$

Let us write

$$x(t_0 + \delta) = x(t_0) + x'(t_0)\delta + \int_0^1 x''(t_0 + s\delta) \frac{\delta^2}{2}\, ds, \tag{6.3}$$

and choose $\delta > 0$ small enough so that

$$\mathcal{N}(\bar{x}_0) \cdot \epsilon H(t, x(t), x(t - k(t, x(t)))) > 0, \qquad t_0 \leqslant t \leqslant t_0 + \delta. \tag{6.4}$$

We use the convexity of Ω_λ and the equation relating x to $\bar{x}$ to verify the inclusion $\bar{\Omega}_\lambda \subseteq \{ y : \mathcal{N}(\bar{x}_0) \cdot (y - x_0) \leqslant 0\}$. Hence, $\mathcal{N}(\bar{x}_0)$ is an outer normal to Ω_λ at x_0. Considering difference quotients, therefore, gives

$$\mathcal{N}(\bar{x}_0) \cdot x'(t_0) = 0. \tag{6.5}$$

It now follows from (6.3) to (6.5) that

$$\mathcal{N}(\bar{x}_0) \cdot [x(t_0 + \delta) - x_0] > 0,$$

and, thus, that $x(t_0 + \delta) \notin \bar{\Omega}_\lambda$ for δ sufficiently small, which contradicts the choice of λ.

Consider now the $2n$-dimensional system

$$\begin{cases} y_1' = \epsilon y_2 \\ y_2' = \epsilon H(t, y_1(t), y_1(t - k(t, y_1(t)))) \end{cases} \qquad 0 < \epsilon \leqslant 1. \tag{6.6}$$

LEMMA 6.4. *All possible T-periodic solutions of* (6.6) *satisfy $y_1(t) \in \Omega$ and $|y_2(t)| < N$, $0 \leqslant t \leqslant T$, where N is a constant independent of ϵ.*

Proof. Let $(y_1(t), y_2(t))$ be a T-periodic solution of (6.6). Then $y_1(t)$ is a T-periodic solution of (6.2) with ϵ replaced by ϵ^2, and, hence, has values in Ω by Lemma 6.3. Since y_1 is T-periodic and $y_1' = \epsilon y_2$, it follows that each component of y_2 vanishes at some point in $[0, T]$. Thus, the second equation in (6.6) implies that the components of y_2 are bounded independent of ϵ.

LEMMA 6.5. *Let $\mathcal{O} = \{(y_1, y_2) : y_1 \in \Omega, |y_2| < N\}$, where N is as in Lemma* 6.4. *Then the Brouwer degree $d(\mathscr{F}, \mathcal{O}, 0)$ of the mapping*

$$\mathscr{F} : \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \to - \begin{bmatrix} y_2 \\ \dfrac{1}{T} \displaystyle\int_0^T H(t, y_1, y_1)\, dt \end{bmatrix}$$

is nonzero.

Proof. Let D be the $2n \times 2n$ matrix

$$D = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix, and consider the mapping $\mathscr{F}_1 = D\mathscr{F}$. It suffices to show that $d(\mathscr{F}_1, \mathcal{O}, 0) \neq 0$. By the product theorem for Brouwer degree [10],

$$d(\mathscr{F}_1, \mathcal{O}, 0) = d\left(\frac{1}{T} \int_0^T H(t, y_1, y_1)\, dt, \Omega, 0\right). \tag{6.7}$$

To compute the degree on the right side of (6.7), we observe that the field,

$$\Phi(y_1, \lambda) = \lambda y_1 + (1 - \lambda) \frac{1}{T} \int_0^T H(t, y_1, y_1)\, dt \qquad (0 \leqslant \lambda \leqslant 1),$$

is zero free on $\partial\Omega$ by (i) and (ii) of Theorem 6.1 and, hence, is a homotopy, which by the homotopy invariance theorem [10] implies that

$$d\left(\frac{1}{T} \int_0^T H(t, y_1, y_1), \Omega, 0\right) = 1.$$

Proof of Theorem 6.1. We first observe that the system (6.6) is of the type (2.6) and satisfies (i) of Theorem 2.1 since k is bounded. It follows then from Lemma 6.5 that (ii) of Theorem 2.1 holds and from Lemma 6.4 that Theorem 2.2 may be applied to Eq. (6.6) ($\epsilon_0 = 1$). Thus, (6.6) has a T-periodic solution for $\epsilon = 1$; hence, (6.2) has a T-periodic solution $x(t)$ with range in Ω. Theorem 6.1, therefore, follows from Lemma 6.2.

Remark. One may easily check that the open bounded convex region Ω of Theorem 6.1 may be replaced by an open bounded convex region which does not necessarily contain 0 as an interior point.

COROLLARY 6.6. *Let x and h be scalars, and let there exist constants α and β, $\alpha < \beta$, such that*

$$h(t, \alpha, \alpha) < 0 < h(t, \beta, \beta), \qquad 0 \leqslant t \leqslant T. \tag{6.8}$$

Further let $h(t, x, y)$ be nonincreasing in y for fixed (t, x). Then the equation

$$x''(t) = h(t, x(t), x(t - d(t, x(t)))) \tag{6.9}$$

has a T-periodic solution $x(t)$ with $\alpha < x(t) < \beta$.

Proof. It suffices to observe that Ω may be chosen to be the open interval (α, β). (6.8) together with the monotonicity of h with respect to y then imply that all conditions of Theorem 6.1 hold.

Remark. Corollary 6.6 provides an extension of Theorem 2 of [9] to nonlinear equations. Similar results for systems patterned after the results in [1], [6] and [8] may also be obtained.

7. EXTENSIONS, REMARKS, UNSOLVED PROBLEMS

The results of Section 2–6 will extend to more general heredity operators with virtually no change in the proofs. In particular, we could also have considered equations with terms of the form

$$x(t - r + k(t, \tilde{U}x_t, \mu)),$$

where $\tilde{U}$ is itself a heredity operator. Therefore, there is a natural extension of the results of Sections 2–6 to finite iterates of the simple heredity operators; the reader can easily supply the statements.

Another direction could also be pursued, namely to formulate the results in terms of the hereditary systems introduced by Jones [5]. We leave this task to the reader, because we cannot justify the space in view of the cases already treated.

The parameter μ which appears throughout can be an element of an arbitrary normed linear space, without change in the proofs. We had hoped to be able to treat the equation

$$S'(t) = -r(t)\, S(t) \left[I_0 + S_0 - S\left(t + \frac{mr(t)\, S(t)}{\rho(t) S'(t)}\right)\right] + f(t),$$

with the aid of Theorem 2.1 (see Cooke [2, p. 173]) and the correct interpretation of μ. We were unsuccessful, and the periodicity question remains unsolved.

It should be apparent how to weaken the differentiability assumption on $A(t)$ and $\eta(t, 0)$ in Section 4 to a bounded variation requirement. It seems unlikely that such a hypothesis is necessary, and we leave unsolved the problem of determining whether or not continuity of $A(t)$, $\eta(t, 0)$ is sufficient.

An interesting problem occurs if we allow the functions $k_i(t, x, \mu)$ to be unbounded but still bounded on bounded sets. Our progress in this direction is limited to the results of Section 6, but we still feel that more can be done. We remark that a theorem similar to Theorem 6.1 can be formulated for first order systems of the kind considered in Theorem 2.2. This task is left to the reader.

References

1. J. W. Bebernes and K. Schmitt, Periodic boundary value problems for systems of second order differential equations, *J. Differential Equations*, **13** (1973), 32–47.
2. K. L. Cooke, Functional differential systems: some models and perturbation problems, *in* "Differential Equations and Dynamical Systems" (J. K. Hale and J. P. LaSalle, Eds.), pp. 167–183, Academic Press, New York, 1967.
3. J. Hale, Periodic and almost periodic solutions of functional–difference equations, *Arch. Rational Mech. Anal.* **15** (1964), 289–304.
4. J. Hale, "Functional Differential Equations," Springer, New York, 1971.
5. G. S. Jones, Hereditary dependence in dynamics, *in* "Differential Equations and Dynamical Systems" (J. K. Hale and J. P. LaSalle, Eds.), pp. 189–198, Academic Press, New York, 1967.
6. H. W. Knobloch, On the existence of periodic solutions of second order vector differential equations, *J. Differential Equations* **9** (1971), 67–85.
7. J. Mawhin, Periodic solutions of nonlinear functional differential equations, *J. Differential Equations* **10** (1971), 240–261.
8. K. Schmitt, Periodic solutions of systems of second order differential equations, *J. Differential Equations* **11** (1972), 180–192.
9. K. Schmitt, Periodic solutions of linear second order differential equations with deviating argument, *Proc. AMS* **26** (1970), 282–285.
10. J. T. Schwartz, "Nonlinear Functional Analysis," Gordon and Breach Science Publishers, New York, 1969.

11. B. H. Stephan, On the existence of periodic solutions of $z'(t) = -az(t - r + \mu k(t, z(t))) + F(t)$, *J. Differential Equations* **6** (1969), 408–419.
12. V. V. Strygin, A theorem concerning the existence of periodic solutions of systems of differential equations with delayed arguments, *Math. Notes Acad. Sci. USSR* **8** (1970), 600–602.