Weak block diagonally dominant matrices, weak block H-matrix and their applications

Shu-huang Xiang \(^a\),*; Zhao-yong You \(^b\)

\(^a\) Department of Mathematics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, People's Republic of China

\(^b\) Research Center of Applied Mathematics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, People's Republic of China

Received 23 October 1997; received in revised form 5 April 1998; accepted 16 April 1998

Submitted by R.A. Brualdi

Abstract

Here we introduce more general definitions of weak block diagonally dominant matrix and weak block H-matrix which permit block triangular factorizations and extend the theory to the block diagonally dominant matrices and the block H-matrices. Furthermore, by the theory of weak block H-matrix, we prove that any partitioned block form of a pointwise H-matrix has a block triangular factorization. © 1998 Elsevier Science Inc. All rights reserved.

AMS classification: 15A18; 15A48; 15A57

Keywords: Block diagonally dominant matrix; Block H-matrix; Weak block diagonally dominant matrix; Weak block H-matrix; Generalized ultrametric matrix

1. Introduction

We shall consider a linear system \( Ax = b \) where \( A \) is an \( m \times m \) matrix, partitioned into block matrix form
where \( A_{ij} \) is of order \( m_i \times m_j, \ 1 \leq m_j \leq m, \ 1 \leq m_i \leq m \). In most practical applications, the matrices \( A_{ij} \) are sparse and many of the block matrices are zero.

Solving the linear system \( Ax = b \), we often approximate the solution with some sort of preconditioned iterative methods. For this, we usually construct a preconditioning matrix \( C \) – an incomplete factorization of \( A \). Varga et al. [11], and Manteuffel [7] gave some methods for \( A \) being a pointwise H-matrix. For a recent survey of these factorizations in the case where \( A \) is a block H-matrix, see [1,9]. But some partitioned strictly diagonally dominant matrices are not block H-matrices. Here we introduce a definition of \textit{weak block diagonally dominant matrix} different from the block diagonally dominant matrix defined by Feingold and Varga [4] and Robert [10]. We further present the class of \textit{weak block H-matrices} which permit a block triangular factorization and whose block Jacobi iteratives are convergent. Finally we prove that any partitioned block form of a pointwise H-matrix has a block triangular factorization, and discuss some applications in generalized ultrametric matrices.

In this paper, we confine ourselves to the vector norm \( \|x\|_x = \max_i |x_i| \) and the matrix norm \( \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \).

2. Block diagonally dominant matrices and block H-matrices

Let \( A \in \mathbb{C}^{n,n} \). Then its comparison matrix \( \mathcal{U}(A) = [b_{ij}] \) is defined by

\[
b_{ij} = \begin{cases} 
|a_{ij}|, & i = j, \\
-|a_{ij}|, & i \neq j.
\end{cases}
\]

\textbf{Definition 2.1.} \( A \) is said to be a nonsingular H-matrix if its comparison matrix is a nonsingular M-matrix. \( A \) is a singular H-matrix if \( \mathcal{U}(A) \) is a singular M-matrix.

For future reference, we will define the set \( \Omega_d \) and \( \Omega_h \):

\[
\Omega_d = \left\{ A \in \mathbb{C}^{n,n} \mid |a_{ii}| > \sum_{j \neq i} |a_{ij}|, \forall i \right\}
\]

i.e. the set of strictly diagonally dominant matrices;

\[
\Omega_h = \left\{ A \in \mathbb{C}^{n,n} \mid A \text{ is a nonsingular H-matrix} \right\}.
\]
Lemma 2.1. \( A \) is a nonsingular H-matrix if and only if there exists a positive vector \( v \) such that \( \mathcal{H}(A)v > 0 \).

This due to Fan [3].

Lemma 2.2 ([5]). Let \( M \) be an \( n \times n \) matrix and \( N \) be an \( n \times m \) matrix. If \( M \) is a strictly diagonally dominant matrix, then

\[
\|M^{-1}N\|_\infty \leq \max_i \left\{ \frac{\sum_{j=1}^{m} |n_{ij}|}{|m_{ii}| - \sum_{j \neq i} |m_{ij}|} \right\}.
\]

**Proof.** Suppose that \( M \) is an \( n \times n \) strictly diagonally dominant matrix and let the absolute row sum \( \gamma_{i_0} \) of row \( i_0 \) be the most of all absolute row sums of the matrix \( M^{-1}N \). Select a vector \( x \) with \( |x_i| = 1 \) (\( i = 1, 2, \ldots, n \)) satisfying that the \( i_0 \)th element of \( M^{-1}Nx \) is \( \gamma_{i_0} = \|M^{-1}N\|_\infty \). Define \( y = M^{-1}Nx \), then

\[
Mx = Ny \quad \text{and} \quad m_{i_0,i_0}x_{i_0} + \sum_{j \neq i_0} m_{i_0,j}y_j = \sum_{j=1}^{m} n_{i_0,j}x_j.
\]

Using the assumptions of \( \gamma_{i_0} \) and \( x \), we get

\[
|m_{i_0,i_0}x_{i_0}| - \sum_{j \neq i_0} |m_{i_0,j}y_j| \leq |m_{i_0,i_0}x_{i_0}| - \sum_{j \neq i_0} |m_{i_0,j}y_j| \leq \sum_{j=1}^{m} |n_{i_0,j}|,
\]

and

\[
\gamma_{i_0} = \|M^{-1}N\|_\infty \leq \frac{\sum_{j=1}^{m} |n_{i_0,j}|}{|m_{i_0,i_0}| - \sum_{j \neq i_0} |m_{i_0,j}|} \leq \max_i \left\{ \frac{\sum_{j=1}^{m} |n_{ij}|}{|m_{ii}| - \sum_{j \neq i} |m_{ij}|} \right\}.
\]

Let \( C_{n,n}^m \) denote the set of all matrices in \( C_{m,n}^{m,n} \) which are of form (1.1) relative to some given block partitioning \( \pi \) (we will only consider \( \pi \) for which the diagonal blocks are square matrices). Let \( A = [A_{ij}] \) and \( A_d \) (\( i = 1, 2, \ldots, n \)) be nonsingular. Then its block comparison matrix \( \mathcal{H}_b(A) = [b_{ij}] \) is defined by

\[
b_{ij} = \begin{cases} \|A_{ij}^{-1}\|_{\infty}^{-1}, & i = j, \\ -\|A_{ij}\|_{\infty}, & i \neq j. \end{cases}
\]

We can reformulate the definitions of block diagonally dominant matrix due to Fenigold and Varga [4] as follows:

**Definition 2.2** ([4]). \( A \) is called a strictly block diagonally dominant matrix if its block comparison matrix \( \mathcal{H}_b(A) \) exists and is strictly diagonally dominant.

Now we define the set \( \Omega_D \) as
\( \Omega_D = \{ A \in \mathbb{C}^{m,n} | \mathcal{U}_b(A) \text{ is a strictly diagonally dominant matrix} \} \).

\( D = [D_{ij}] \in \mathbb{C}^{m,n} \) is said to be block diagonal if \( D_{ij} = 0 \) for all \( i \neq j \).

**Definition 2.3 ([9])**. \( A \) is said to be a nonsingular block H-matrix if there exist nonsingular block diagonal matrices \( D, E \) and \( D, E \in \mathbb{C}^{m,n} \) such that \( \mathcal{U}_b(DAE) \) is a nonsingular M-matrix. We define

\[ \Omega_H = \{ A \in \mathbb{C}^{m,n} | A \text{ is a nonsingular block H-matrix} \} \].

**Lemma 2.3 ([9])**. If \( A \in \Omega_H \), then there exist nonsingular block diagonal matrices \( D \) and \( E \) such that \( DAE \) is strictly block diagonally dominant.

3. Weak block diagonally dominant matrices

**Definition 3.1**. \( A \in \mathbb{C}^{m,n} \) is said to be weak block diagonally dominant if \( D = \text{diag}(A_{11}, A_{22}, \ldots, A_{nn}) \) is nonsingular and \( D^{-1}A \) is a strictly diagonally dominant matrix.

**Theorem 3.1**. \( A \in \mathbb{C}^{m,n} \) is weak block diagonally dominant if and only if every vector \( x \in \mathbb{C}^m \) with \( x \neq 0 \), \( x^T = (x_1^T, x_2^T, \ldots, x_n^T), x_i \in \mathbb{C}^m \) satisfies that for every \( i \), \( \|x_i\|_{\infty} < \|x\|_{\infty} \) whenever \( \sum_{j=1}^n A_{ij}x_j = 0 \).

Before proving Theorem 3.1, firstly we prove the lemmas as follows.

**Lemma 3.1**. Suppose that \( A \in \mathbb{C}^{m,n} \) satisfies that every vector \( x \in \mathbb{C}^m \) with \( x \neq 0 \), \( x^T = (x_1^T, x_2^T, \ldots, x_n^T), x_i \in \mathbb{C}^m \) satisfies that for every \( i \), \( \|x_i\|_{\infty} < \|x\|_{\infty} \) whenever \( \sum_{j=1}^n A_{ij}x_j = 0 \). Then \( A_{ii} \) are nonsingular \( (i = 1, 2, \ldots, n) \) and \( A \) is nonsingular.

**Proof**. If for some \( i_0, A_{i_0,i_0} \) is singular, then there is a vector \( x_{i_0} \in \mathbb{C}^{m_{i_0}}, x_{i_0} \neq 0 \) such that

\( A_{i_0,i_0}x_{i_0} = 0 \).

Define \( x^T = (0, 0, \ldots, 0, x_{i_0}^T, 0, \ldots, 0) \), then

\[ \sum_{j=1}^n A_{i_0,j}x_j = 0. \]

Using the assumptions we can get

\[ \|x_{i_0}\|_{\infty} < \|x\|_{\infty} = \max_j \|x_j\|_{\infty} = \|x_{i_0}\|_{\infty}. \]
This is a contradiction. So $A_{ii}$ is nonsingular. If $A$ is singular, then there exists a $x^T = (x_1^T, x_2^T, \ldots, x_n^T)$ with $x \neq 0$ such that

$$\sum_{j=1}^{n} A_{ij}x_j = 0, \quad i = 1, 2, \ldots, n.$$  

Using the same method, we have that $\|x\|_\infty < \|x\|_\infty$. This is a contradiction. So $A$ is nonsingular.

Lemma 3.2. Suppose $A \in \mathbb{C}^{m \times n}$ then $A$ is a weak block diagonally dominant matrix if and only if

$$\|I - D^{-1}A\|_\infty < 1.$$  

Proof. By Definition 3.1, it is obvious.

Proof of Theorem 3.1. (i) Sufficiency. Suppose $A \in \mathbb{C}^{m \times n}$. Let the absolute row sum $\gamma_{i0}$ of row $i_0$ be the most of all absolute row sums of the matrix $I - D^{-1}A$. Select a vector $x$ with $|x_i| = 1 (i = 1, 2, \ldots, m)$ satisfying

$$\sum_{j=1}^{m} C_{i0,j}x_j = \gamma_{i0},$$  

where $C = I - D^{-1}A$. Partition $x$ according to $\pi$ and let $y = (I - D^{-1}A)x$, then

$$A_{ii}y_i = -\sum_{j \neq i} A_{ij}x_j, \quad i = 1, 2, \ldots, n.$$  

For vector $x(i)^T = (x_1^T, \ldots, x_{i-1}^T, y_i^T, x_{i+1}^T, \ldots, x_n^T)$ we have

$$\|y_i\|_\infty < \|x(i)\|_\infty = \max_{j \neq i} \|x_j\|_\infty \leq \|x\|_\infty = 1, \quad i = 1, 2, \ldots, n.$$  

That is to say, $\|I - D^{-1}A\|_\infty < 1$. By Lemma 3.2, $A$ is a weak block diagonally dominant matrix.

(ii) Necessity. Suppose that $A \in \mathbb{C}^{m \times n}$ is weak block diagonally dominant then $\|I - D^{-1}A\|_\infty < 1$. Assume that $x \in \mathbb{C}^m$ with $x \neq 0$ is a vector such that

$$A_{ii}x_i = -\sum_{j \neq i} A_{ij}x_j \quad \text{for some } i.$$  

Then

$$\|x\|_\infty = \left\| \sum_{j \neq i} A_{ii}^{-1}A_{ij}x_j \right\|_\infty \leq \left\| \sum_{j \neq i} A_{ii}^{-1}A_{ij} \right\|_\infty |x_j| \|x\|_\infty \leq \left\| \sum_{j \neq i} A_{ii}^{-1}A_{ij}e_j \right\|_\infty \|x\|_\infty,$$  

where $|A| = (|a_{ij}|)$, $e_j^T = (1, 1, \ldots, 1) \in \mathbb{C}^n$. By $\|\sum_{j \neq i} |A^{-1}_{ii}| A_{ij}| e_j\|_\infty < \|I - D^{-1}A\|_\infty < 1$, we have

$$\|x\|_\infty \leq \left\| \sum_{j \neq i} |A^{-1}_{ii}| A_{ij}| e_j \right\|_\infty \|x\|_\infty < \|x\|_\infty.$$

\[\square\]

**Corollary 3.1.** If $A$ is a weak block diagonally dominant matrix then $A$ is nonsingular.

**Theorem 3.2.** If $A \in \mathbb{C}^{n,n}$ is strictly block diagonally dominant, then $A$ is weak block diagonally dominant.

**Proof.** For $x \in \mathbb{C}^n$ and $x \neq 0$, if for some $i$, the following equality holds

$$\sum_{j=1}^n A_{ij}x_j = 0,$$

we can get

$$\|x\|_\infty = \left\| - \sum_{j \neq i} |A^{-1}_{ii}| A_{ij}| x_j \right\|_\infty \leq \|A^{-1}_{ii}\|_\infty \left( \sum_{j \neq i} \|A_{ij}\|_\infty \right) \|x\|_\infty.$$

Since $A$ is strictly block diagonally dominant, $1/\|A^{-1}_{ii}\|_\infty > \sum_{j \neq i} \|A_{ij}\|_\infty$, that is,

$$\|A^{-1}_{ii}\|_\infty \sum_{j \neq i} \|A_{ij}\|_\infty < 1.$$

So

$$\|x\|_\infty < \|x\|_\infty.$$

By Theorem 3.1, $A$ is a weak block diagonally dominant matrix. \(\square\)

**Corollary 3.2.** If $A$ is strictly block diagonally dominant, then $D^{-1}A$ is strictly diagonally dominant where $D = \text{diag}(A_{11}, A_{22}, \ldots, A_{nn})$.

**Corollary 3.3.** If $A$ is a strictly diagonally dominant matrix and is partitioned into form (1.1), then $A$ is a weak block diagonally dominant matrix.

**Proof.** Suppose that $A$ is an $m \times m$ strictly diagonally dominant matrix partitioned into form (1.1). Then for every $i (i = 1, 2, \ldots, m)$, $A$ satisfies

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$
Let
\[
D = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
0 & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{nn}
\end{pmatrix}, \quad B = A - D,
\]

\[A_{kk} = (a^{(k)}_{st}), \quad 1 \leq s, \ t \leq m_k, \ k = 1, 2, \ldots, n.\]

Now we estimate \(\|D^{-1}B\|_\infty\). Since every \(i\) can be written as \(i = m_1 + m_2 + \cdots + m_{k-1} + l, \ 0 < l \leq m_k, \ m_0 = 0\), by Lemma 2.2, we get
\[
\|D^{-1}B\|_\infty \leq \max_{i} \frac{\sum_{j=1}^{m} |b_{ij}|}{|a^{(k)}_{ii}| - \sum_{j\neq i} |a^{(k)}_{ij}|}.
\]

Since \(\sum_{j=1}^{m} |b_{ij}| + \sum_{j\neq i} |a^{(k)}_{ij}| = \sum_{j\neq i} |a_{ij}| < |a_{ii}| = |a^{(k)}_{ii}|\), we get
\[
\max_{i} \frac{\sum_{j=1}^{m} |b_{ij}|}{|a^{(k)}_{ii}| - \sum_{j\neq i} |a^{(k)}_{ij}|} < 1, \quad i = 1, 2, \ldots, m.
\]

So \(\|D^{-1}B\|_\infty = \|I - D^{-1}A\|_\infty < 1\). And by Lemma 3.2, \(A\) partitioned into form (1.1) is a weak block diagonally dominant matrix. \(\square\)

**Corollary 3.4.** If \(A\) is weak block diagonally dominant, then the block Jacobi iterative matrix of \(A\) is convergent.

**Proof.** According to Lemma 3.2, if we write \(A = D - B\), where \(D = \text{diag}(A_{11}, A_{22}, \ldots, A_{nn})\), then \(\|D^{-1}B\|_\infty = \|I - D^{-1}A\|_\infty < 1\), so
\[
\rho(D^{-1}B) \leq \|D^{-1}B\|_\infty < 1.
\]

Hence the Jacobi iterative matrix of \(A\) is convergent. \(\square\)

In [3], Robert defined the matrix \(N(A) = [b_{ij}]\) where
\[
b_{ij} = \begin{cases} 
1, & i = j, \\
-\|A^{-1}_{ii}A_{ij}\|_\infty, & i \neq j.
\end{cases}
\]

If \(N(A)\) is a strictly diagonally dominant matrix then \(A\) is called Robert block diagonally dominant matrix. For Robert block diagonally dominant matrices we can get the same results as for strictly block diagonally dominant matrices.
4. Weak block H-matrices

Definition 4.1. A ∈ C^{m,n} is called weak block H-matrix if there are nonsingular block diagonal matrices D and E such that DAE is a weak block diagonally dominant matrix. We define

Ω_w = \{ A ∈ C^{m,n} | A is a weak block H-matrix \}.

Theorem 4.1. If A is a nonsingular H-matrix and is partitioned into the form (1.1), then A ∈ Ω_w.

Proof. Since A is a nonsingular H-matrix, there exists a diagonal matrix D such that AD is a strictly diagonally dominant matrix. By Corollary 3.3 and Definition 3.1, A is a weak block H-matrix. □

Theorem 4.2. If A ∈ Ω_w, then A has a block triangular factorization.

Proof. First, we prove that if A is weak block diagonally dominant partitioned into form (1.1), then A has a block triangular factorization.

Since A is a weak block diagonally dominant matrix, D^{-1}A is a strictly diagonally dominant matrix where D = diag(A_{11}, A_{22}, \ldots, A_{nn}). Then the block principal submatrix C_k of D^{-1}A formed by block rows and block columns numbers 1, 2, \ldots, k in form (1.1) is strictly diagonally dominant and nonsingular (k = 1, 2, \ldots, n). According to Faddeev [2] (Chapter 1, Section 1.13, p. 24), D^{-1}A has a block triangular factorization. So A has a block triangular factorization.

If A ∈ Ω_w, then there are nonsingular block diagonal matrices D and E such that DAE is weak block diagonally dominant. So DAE has a block triangular factorization. That is to say, A has a block triangular factorization. □

Note. By Theorem 4.2, we can construct an incomplete block factorization for a weak block H-matrix by the same way as for block H-matrices (see [9,1]).

Theorem 4.3. Ω_H ⊂ Ω_w and Ω_H ≠ Ω_w.

Proof. Ω_H ⊂ Ω_w follows from Lemma 2.3 and Theorem 3.2. and Definition 4.1. Let
A = \begin{pmatrix}
1 & 0 & 1 - \varepsilon & 0 & 0 \\
0 & 1 & 0 & 1 - \varepsilon & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 - \varepsilon & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 - \varepsilon & 0 & 0 \\
0 & 0 & 0 & 1 - \varepsilon & 0 \\
\end{pmatrix}, \text{ where } 0 < \varepsilon < 1.

By Polman [9], A is strictly diagonally dominant and A \not\in \Omega_\Omega. But, by Corollary 3.3, A is a weak block diagonally dominant matrix and A \in \Omega_\Omega. \quad \square

Due to Corollary 3.4 and Definition 3.1, it is easy to verify the following theorem.

**Theorem 4.4.** A \in \Omega_\Omega, the Jacobi iterative matrix of A is convergent.

We consider the applications of weak block H-matrices in generalized ultrametric matrices. McDonald et al. [6], Nabben and Varga [8] introduced the new class of generalized ultrametric matrices in studying inverse M-matrices problem.

**Definition 4.2.** A matrix A = [a_{ij}] \in \mathbb{C}^{n,n} is called a generalized ultrametric matrix if:

(i) A has nonnegative entries,

(ii) a_{ij} \geq \min\{a_{ik}, a_{kj}\} for all i, j, k \in N = \{1, 2, \ldots, n\};

(iii) a_{ii} = \max\{a_{ik}, a_{ki}\} for all i \in N;

(iv) each triple \(q,s,t\) in \(N^3\) can be reordered as a triple \(i,j,k\) such that

(iv.i) \(a_{jk} = a_{ik}\) and \(a_{kj} = a_{ki}\),

(iv.ii) \(\max\{a_{ij}, a_{ji}\} \geq \max\{a_{ik}, a_{ki}\}\),

where, if \(n = 1\), (iii) is interpreted as \(a_{11} \geq 0\). A matrix A is called a strictly generalized ultrametric matrix if the above conditions hold with strict inequality in (iii).

Let A be a strictly generalized ultrametric matrix and \(\tau(A) := \min\{a_{ij}: i,j \in N\}\), \(\omega(A) = \min\{a_{ji}: a_{ij} = \tau(A)\}\), \(\delta(A) = \omega(A) - \tau(A)\), \(\mu(A) = \max\{a_{ij}\}\), then there exists a permutate matrix P such

\[\bar{A} := P^T A P = \begin{bmatrix}
\begin{array}{cc}
\mathbf{C} & \mathbf{0} \\
\delta(A) \xi_{n-r} \xi_r & \mathbf{D}
\end{array}
\end{bmatrix}
+ \tau(A) \xi_n \xi_n^T = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}.
\]
$C \in C^{r \times r}$ and $D \in C^{n-r \times n-r}$ are also strictly generalized ultrametric matrices and $A^{-1}, C^{-1}, D^{-1}$ are strictly diagonally dominant $M$-matrices, where

$$A_{11} = C + \tau(A) \xi_r \xi_r^T, \quad A_{22} = D + \tau(A) \xi_{n-r} \xi_{n-r}^T,$$

$$\xi_r = (1, 1, \ldots, 1)^T \in C^r, \quad \xi_{n-r} = (1, 1, \ldots, 1)^T \in C^{n-r},$$

$\xi_n = (1, 1, \ldots, 1)^T \in C^n$ (see [8]).

**Theorem 4.5.** Suppose $\bar{A}$ as above, then $\bar{A}$ is a weak block $H$-matrix.

**Proof.** By Sherman–Morrison formula, we have that

$$(C + \tau(A) \xi_r \xi_r^T)^{-1} = C^{-1} - \frac{\tau(A) C^{-1} \xi_r \xi_r^T C^{-1}}{1 + \tau(A) \xi_r^T C^{-1} \xi_r}$$

is an $M$-matrix, and

$$A_{21} A_{11}^{-1} = \omega(A) \xi_n \xi_r^T \left( C^{-1} - \frac{\tau(A) C^{-1} \xi_r \xi_r^T C^{-1}}{1 + \tau(A) \xi_r^T C^{-1} \xi_r} \right)$$

$$= \frac{\omega(A) \xi_n \xi_r^T C^{-1}}{1 + \tau(A) \xi_r^T C^{-1} \xi_r}.$$

$$A_{21} A_{11}^{-1} \xi_r = \frac{\omega(A) \xi_r^T C^{-1} \xi_r}{1 + \tau(A) \xi_r^T C^{-1} \xi_r} \xi_n.$$

Since $C^{-1}$ is a strictly diagonally dominant $M$-matrix, $\omega(A) \xi_n \xi_r^T C^{-1}$ and $A_{21} A_{11}^{-1}$ are nonnegative matrices. According to Theorem 3.5 in [8] (pp. 387, 388),

$$(\omega(A) - \tau(A)) \xi_r^T C^{-1} \xi_r = \delta(A) \xi_r^T C^{-1} \xi_r < 1$$

so we get

$$\frac{\omega(A) \xi_r^T C^{-1} \xi_r}{1 + \tau(A) \xi_r^T C^{-1} \xi_r} < 1 \quad \text{and} \quad \|A_{21} A_{11}^{-1} \xi_r\|_\infty < 1.$$

Since $A_{21} A_{11}^{-1}$ is nonnegative, we have that $\|A_{21} A_{11}^{-1}\|_\infty < 1$.

By the same way, we have that $\|A_{12} A_{22}^{-1}\|_\infty < 1$. So

$$\bar{A} \begin{bmatrix} A_{11}^{-1} \\ A_{22}^{-1} \end{bmatrix}$$

is a row strictly diagonally dominant matrix and by Corollary 3.3 and Definition 4.1, $\bar{A}$ is a weak block $H$-matrix. $\Box$
For example, let
\[
A = \begin{bmatrix}
2 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 & 1 \\
& & & & \\
1 & 1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 2
\end{bmatrix}
\]
\[
AD^{-1} = \begin{bmatrix}
1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
& & & & \\
\frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 1 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & 1
\end{bmatrix}
\]
is a row strictly diagonally dominant matrix. So $A$ is a weak block diagonally dominant matrix. But $A$ is not a block diagonally dominant matrix, since its block comparison matrix
\[
\mathcal{U}_b(A) = \begin{bmatrix}
1 & 3 \\
2 & \frac{5}{4}
\end{bmatrix}
\]
is not a strictly diagonally dominant matrix. In this case, $A$ is also a Robert block diagonally dominant matrix.

According to Theorem 4.5, we get the following corollary.

**Corollary 4.1.** Suppose that $A$ is a strictly generalized ultrametric matrix, then there exists a split $A = M + N$ such that the spectrum radius $\rho(M^{-1}N) < 1$ where $M, N$ are nonnegative matrices.

**Proof.** Let $P$ be permutate matrix such that
\[
\overline{A} := P^TAP = \begin{bmatrix}
C & 0 \\
\delta(A)\xi_n\xi_n^T & D
\end{bmatrix} + \tau(A)\xi_n\xi_n^T = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]
\[
\overline{M} = \begin{bmatrix}
A_{11} \\
A_{22}
\end{bmatrix}, \quad \overline{N} = \begin{bmatrix}
0 & A_{12} \\
A_{21} & 0
\end{bmatrix}
\]
then by Theorems 4.5 and 4.4, we get $\rho(\overline{M}^{-1}\overline{N}) < 1$. Let $M = P\overline{M}P^T$, $N = P\overline{N}P^T$, then $A = M + N$ and
\[
\rho(M^{-1}N) = \rho(P\overline{M}^{-1}P^TP\overline{N}^{-1}P^T) = \rho(\overline{M}^{-1}\overline{N}) < 1. \quad \square
Acknowledgements

We are grateful to the referee for many useful suggestions and to Prof. R.S. Varga for his helpful comments, and thank Prof. Guanggui Ding for his support.

References