View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Linear Algebra and its Applications 402 (2005) 111–125
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Biorthogonal multivariate filter banks from centrally symmetric matrices

Qiuhui Chen [a,1], Charles A. Micchelli [b,2], Yuesheng Xu [c,d,*,3]

[a] *School of Mathematics and Computer Sciences, Hubei University, Wuhan 430062, PR China*
[b] *Department of Mathematics and Statistics, State University of New York, The University at Albany, Albany, NY 12222, USA*
[c] *Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA*
[d] *Institute of Mathematics, Academy of Mathematics and System Science, Chinese Academy of Science, Beijing 100080, PR China*

Received 9 February 2004; accepted 15 December 2004
Available online 19 February 2005
Submitted by R.A. Brualdi

Abstract

We provide a practical characterization of block centrally symmetric and anti-symmetric matrices which arise in the construction of multivariate filter banks and use these matrices for the construction of biorthogonal filter banks with linear phase.
© 2005 Elsevier Inc. All rights reserved.

AMS classification: 42C40; 65T60

Keywords: Biorthogonal multivariate filter banks; Centrally symmetric matrices

* Corresponding author. Address: Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA.

E-mail addresses: mcscqh@zsu.edu.cn (Q. Chen), cam@math.albany.edu (C.A. Micchelli), yxu06@syr.edu (Y. Xu).

[1] Supported in part by NSFC under grants 10201034 and 10226036.

[2] Supported in part by the US National Science of Foundation under grant CCR-0407476.

[3] Supported in part by the US National Science Foundation under grants CCR-0407476, by the Natural Science Foundation of China under grant 10371122 and by the Chinese Academy of Sciences under the program of "One Hundred Distinguished Chinese Scientists".

0024-3795/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2004.12.028

1. Introduction

Filter banks with linear phase have the desirable property that for any input signal with energy confined to the pass-band of the filter, the corresponding output signal is approximately equal to the input [11]. In the univariate case the only two-channel conjugate mirror filters and finite impulse response filter banks which have linear phase are the Haar filter. In the multivariate multiple channel vector-valued case, examples of linear phase filter are given in [3,6,10]. A general construction of filter banks having linear phase was proposed in [1] by a matrix factorization approach and the use of block centrally symmetric matrices. This approach demands a computationally convenient method to generate block centrally symmetric matrices. Inspired by this practical issue, we provide such a characterization here and also extend the results in [1] to cover the important case of multivariate biorthogonal filter banks.

We begin our discussion in Section 2 with a study of block centrally symmetric and anti-symmetric matrices. The structure of these matrices is completely resolved and some of their most useful properties are presented. In the last section we show how to use these matrices for the design of biorthogonal multivariate filter banks.

2. Centrally symmetric matrices

We start this section with a description of two ancillary results. They will be used later in the section for the study of block centrally symmetric and anti-symmetric matrices. Suppose that m is a positive integer and E, G, H are in M_m, the space of all matrices of order m with real elements. Here we shall assume that these matrices satisfy several conditions. First, we require that they are nonzero and symmetric. Next, we demand that they satisfy the equations

$$G^2 = H^2 = I, \quad E^2 = E. \tag{2.1}$$

We also require that there is a real constant ν such that

$$GH + HG = \nu E. \tag{2.2}$$

Note that Eqs. (2.1) and (2.2) imply that

$$\nu EG = GHG + H = \nu GE$$

and so we see, at least for $\nu \neq 0$, that E and G *commute*. Similarly, in this case we also see that the matrices H and E commute. We assume, even when $\nu = 0$, that this property holds and, to conveniently state our first observation below, we simply say, matrices E, G, H which satisfy *all* of these conditions are *admissible*. Thus, when the matrices E, G and H are admissible they satisfy Eqs. (2.1) and (2.2) and E commutes with G and H.

Lemma 2.1. *If $E, G, H \in M_m$ are admissible matrices and μ is a real number then the matrix*

$$S := \frac{\sqrt{2}}{2}(I + HG + \mu E) \tag{2.3}$$

is orthogonal if and only if $\nu + (\nu + 2)\mu + \mu^2 = 0$.

Proof. Using the symmetry of the matrices E, G and H and Eq. (2.1) we have that

$$S^{\mathrm{T}} S = I + \frac{1}{2}(HG + 2\mu E + GH + \mu GHE + \mu EHG + \mu^2 E).$$

Since these matrices are admissible this equation simplifies to

$$S^{\mathrm{T}} S = I + \frac{1}{2}(\nu + (\nu + 2)\mu + \mu^2)E,$$

which proves the result. □

In the next lemma we need another property of the matrices E, G and H when they are admissible.

Lemma 2.2. *If* $E, G, H \in M_m$ *are admissible matrices such that*

$$HE = GE, \tag{2.4}$$

then

$$S^{\mathrm{T}} HS = G$$

if and only if $\nu + 4\mu + \mu^2 = 0$.

Proof. Using the symmetry of matrices E, G and H and Eq. (2.1) a direct computation leads to the equation

$$S^{\mathrm{T}} HS = G + \frac{1}{2}(H + \mu HE + GHG + \mu GE + \mu EH + \mu EG + \mu^2 EHE).$$

Since the matrices are admissible we derive from this equation the formula

$$S^{\mathrm{T}} HS = G + \frac{1}{2}[(2\mu + \mu^2)HE + (2\mu + \nu)GE].$$

Now, we use property (2.4) and observe that the above equation reduces to

$$S^{\mathrm{T}} HS = G + \frac{1}{2}(\nu + 4\mu + \mu^2)GE,$$

from which the lemma follows. □

We now recall the definition of *block centrally symmetric* and *anti-symmetric matrices*. The definition uses a special matrix defined for any positive integers r, n by the equation

$$H_n := \begin{pmatrix} 0 & 0 & \cdots & I_r \\ 0 & \cdots & I_r & 0 \\ \vdots & \vdots & \vdots & \vdots \\ I_r & 0 & \cdots & 0 \end{pmatrix}.$$

where I_r stands for the identity matrix of order r. A matrix $B \in M_{rn}$ is called *block centrally σ-symmetric* if it satisfies the matrix equation

$$B = \sigma H_n B H_n.$$

When $\sigma = 1$ (respectively $\sigma = -1$), we say B is a centrally symmetric (respectively anti-symmetric) matrix. We call the matrix H_n the *basic* block centrally symmetric matrix.

Since we shall always work with block matrices in M_{rn} whose elements are matrices of order r we will frequently drop our reference to the block size. Also, for the sake of notational simplicity we shall generally *not* subscript the matrix H_n as long as no confusion arises. When doing so we always distinguish this special matrix from the general matrices which appeared in the first two lemmas.

Since the matrix H_n plays a crucial role in the study of centrally σ-symmetric matrices, we identify its eigenvalues. For this purpose, we introduce two positive integers

$$n_1 := \left[\frac{n+1}{2}\right], \quad n_2 := \left[\frac{n}{2}\right],$$

where $[x]$ denotes the greatest integer less than or equal to x. An induction on n confirms that the *characteristic polynomial* p of the matrix H_n is given, at any complex number λ, by the formula

$$p(\lambda) := |\lambda I_{rn} - H_n| = (\lambda - 1)^{rn_1}(\lambda + 1)^{rn_2}. \tag{2.5}$$

In other words, the matrix H_n has only 1 as an eigenvalue with algebraic multiplicity rn_1 and -1 with algebraic multiplicity rn_2. The formula (2.5) also follows from Proposition 2.3 below.

To make use of the previous lemmas we use the diagonal matrix $G \in M_{rn}$ defined by the equation

$$G := \begin{pmatrix} I_{rn_1} & 0 \\ 0 & -I_{rn_2} \end{pmatrix} \tag{2.6}$$

and for the matrix $E \in M_{rn}$ we use the identity matrix for n even while for n odd E is defined by the equation

$$E := \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We also set $\mu = \nu = 0$ when n is even and we set $\mu = -2 + \sqrt{2}$ and $\nu = 2$ when n is odd. Therefore, for this value of ν these matrices are admissible and satisfy Eq. (2.4) with $H = H_n$. With them and the value of μ we prescribed, the matrix $S \in M_{rn}$ defined by (2.3) has the alternative form

$$S = \frac{\sqrt{2}}{2}\begin{pmatrix} I_{rk} & 0 & -H_k \\ 0 & \sqrt{2} I_r & 0 \\ H_k & 0 & I_{rk} \end{pmatrix},$$

when $n = 2k + 1$ for some nonnegative integer k and when $n = 2k$ it is given by the formula

$$S = \frac{\sqrt{2}}{2}\begin{pmatrix} I_{rk} & -H_k \\ H_k & I_{rk} \end{pmatrix}.$$

From Lemmas 2.1 and 2.2, we obtain the next proposition.

Proposition 2.3. *The matrix S is orthogonal and has the property that*

$$S^{-1}HS = G. \tag{2.7}$$

Proof. As we have already mentioned above the matrices E, G and H are admissible and, in addition, their definitions ensure the validity of the formula

$$HE = GE.$$

Therefore, we conclude by Lemma 2.1 that S is an orthogonal matrix and by Lemma 2.2 that they satisfy Eq. (2.7). □

From this fact follows our first theorem on a characterization of block centrally σ-symmetric matrices.

Theorem 2.4. *If $Z \in M_{rn}$ and $B := SZS^{-1}$ then B is block centrally σ-symmetric if and only if*

$$\sigma GZG = Z. \tag{2.8}$$

Proof. Suppose that B is block centrally σ-symmetric. Using this hypothesis and Proposition 2.3, we have that

$$\sigma GZG = \sigma(S^{-1}HS)Z(S^{-1}HS) = \sigma S^{-1}(HBH)S = S^{-1}BS = Z.$$

Conversely, suppose that Eq. (2.8) holds. We then have that

$$\sigma HBH = \sigma HSZS^{-1}H = \sigma SGZGS^{-1} = SZS^{-1} = B,$$

which completes the proof. □

This theorem gives a complete characterization of centrally σ-symmetric matrices as it is an easy matter to solve Eq. (2.8) for Z since the matrix G has the simple form (2.6). The details are postponed until later.

We shall now use this theorem to say more about block centrally symmetric and anti-symmetric matrices. For this purpose, we recall that the Frobenious inner product of any F and G in M_{rn} is defined by the equation

$$\langle F, G\rangle := \mathrm{trace}(FG^{\mathrm{T}}).$$

We denote by M^{c} the subspace of M_{rn} of block centrally symmetric matrices and M^{a} the subspace of block centrally anti-symmetric matrices. We need the linear operator $\mathscr{H} : M_{rn} \longrightarrow M_{rn}$ defined for any $F \in M_{rn}$ by the equation

$$\mathscr{H}(F) := HFH,$$

where H is the basic centrally symmetric matrix defined earlier. In particular, $H^2 = I$ which guarantees that the operator

$$\mathscr{G} := \frac{1}{2}(\mathscr{I} + \mathscr{H})$$

satisfies $\mathscr{G}^2 = \mathscr{G}$. By noting the fact $\mathrm{trace}(A) = \mathrm{trace}(\mathscr{H}A)$ for any matrix A, we conclude that $\mathscr{G}$ is self-adjoined and then an orthogonal projection onto M^{c} with orthogonal complement M^{a}. These facts are directly verifiable and lead to the following result.

Proposition 2.5. *The space M_{rn} is the orthogonal direct sum of M^{c} and M^{a}, that is,*

$$M_{rn} = M^{\mathrm{c}} \oplus^{\perp} M^{\mathrm{a}}.$$

Let us now use this fact to identify the dimension of M^{c} and M^{a}.

Proposition 2.6. *The dimensions of M^{c} and M^{a} are given respectively by*

$$\dim M^{\mathrm{c}} = r^2\left(n_1^2 + n_2^2\right), \quad \text{and} \quad \dim M^{\mathrm{a}} = 2r^2 n_1 n_2.$$

Proof. When $\sigma = 1$ the dimension of the subspace of all matrices $Z \in M_{rn}$ which satisfy Eq. (2.8) is seen to be $r^2(n_1^2 + n_2^2)$. By Proposition 2.5, we know that the dimension of M^{a} is given by

$$\dim M^{\mathrm{a}} = \dim M_{rn} - \dim M^{\mathrm{c}} = 2r^2 n_1 n_2,$$

which proves the result. □

The association of matrices Z which satisfy Eq. (2.8) in Theorem 2.4 with block centrally σ-symmetric matrices B allows us to easily transfer properties of B directly to Z and visa verse. To explain the next corollary it is now appropriate to characterize these matrices. For $\sigma = 1$ we see that $Z \in M_{rn}$ satisfies (2.8) exactly when it has the form

$$Z := \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \tag{2.9}$$

with $Z_1 \in M_{rn_1}$ and $Z_2 \in M_{rn_2}$ while for $\sigma = -1$ it has the form

$$Z := \begin{pmatrix} 0 & Z_2 \\ Z_1 & 0 \end{pmatrix}$$

with $Z_1 \in M_{rn_1 \times rn_2}$ and $Z_2 \in M_{rn_2 \times rn_1}$. Let us make use of these formulas and denote by $M^{\mathrm{c,i}}$ and $M^{\mathrm{c,o}}$ the group of all invertible matrices and orthogonal matrices in $M^{\rm c}$, respectively. The next result characterizes these matrices in terms of the corresponding matrices Z_1 and Z_2. The proof follows directly from Theorem 2.4.

Corollary 2.7. *The following statements hold*:

(i) $B \in M^{\rm c,i}$ *if and only if it has the form*

$$B = SZS^{-1} \tag{2.10}$$

where Z *has the form* (2.9) *with* $Z_1 \in M^i_{rn_1}$ *and* $Z_2 \in M^i_{rn_2}$.

(ii) $B \in M^{\rm c,o}$ *if and only if it has the form* (2.10) *where* Z *has the form* (2.9) *with* $Z_1 \in M^o_{rn_1}$ *and* $Z_2 \in M^o_{rn_2}$.

We now turn to discussion of centrally symmetric and centrally anti-symmetric matrices, which are symmetric and anti-symmetric (as *block* matrices). Specifically, we denote by $M^{\rm c,s}$ and $M^{\rm c,a}$ the group of all symmetric and anti-symmetric matrices in $M^{\rm c}$, respectively. Likewise, we use $M^{\rm a,s}$ and $M^{\rm a,a}$ for the group of all symmetric and anti-symmetric matrices in $M^{\rm a}$, respectively. Moreover, we use $M_n^{\rm s}$ and $M_n^{-\rm s}$ to denote the symmetric and anti-symmetric matrices in M_n, respectively.

Corollary 2.8. *The following statements hold*:

(i) $B \in M^{\rm c,s}$ *if and only if it has the form* (2.10) *with* Z *having the form* (2.9), *where* $Z_1 \in M^{\rm s}_{rn_1}$ *and* $Z_2 \in M^{\rm s}_{rn_2}$; $B \in M^{\rm c,a}$ *if and only if it has the form* (2.10) *with* Z *having the form* (2.9) *where* $Z_1 \in M^{-\rm s}_{rn_1}$ *and* $Z_2 \in M^{-\rm s}_{rn_2}$.

(ii) *The dimensions of these spaces are given by the formulas*

$$\dim M^{\rm c,s} = r^2 n_1(n_2+1), \quad \textit{and} \quad \dim M^{\rm c,a} = r^2 n_2(n_1-1).$$

(iii) *There holds the decomposition*

$$M^{\rm c} = M^{\rm c,s} \oplus^{\perp} M^{\rm c,a}.$$

Proof. Note that both $M^{\rm c,s}$ and $M^{\rm c,a}$ are linear subspaces of $M^{\rm c}$. The first assertion (i) follows from Theorem 2.4. while the second is apparent by just counting the dimensions of the matrices Z_1 and Z_2. Indeed, for the first formula in (ii) Z_1 and Z_2 are symmetric matrices of order rn_1 and rn_2, respectively, and so they have total dimension

$$\frac{r^2}{2} n_1(n_1+1) + \frac{r^2}{2} n_2(n_2+1) = r^2 n_1(n_2+1).$$

Similarity, for the second formula in (ii) Z_1 and Z_2 are anti-symmetric matrices of order rn_1 and rn_2 and so their total dimension

$$\frac{r^2}{2} n_1(n_1-1) + \frac{r^2}{2} n_2(n_2-1) = r^2 n_2(n_1-1).$$

Finally, we show that (iii) is valid. Choose any $B \in M^{\rm c}$, which by Theorem 2.4 can be written in the form such

$$B = SZS^{\mathrm{T}}, \quad \text{for some matrix} \quad Z \in M^{\mathrm{s}}.$$

We decompose B in the form

$$B = \frac{1}{2}S(Z + Z^{\mathrm{T}})S^{\mathrm{T}} + \frac{1}{2}S(Z - Z^{\mathrm{T}})S^{\mathrm{T}}.$$

Since $Z + Z^{\mathrm{T}}$ and $Z - Z^{\mathrm{T}}$ are in M^{s} it follows that $\frac{1}{2}S(Z + Z^{\mathrm{T}})S^{\mathrm{T}} \in M^{\mathrm{c,s}}$ and $\frac{1}{2}S(Z - Z^{\mathrm{T}})S^{\mathrm{T}} \in M^{\mathrm{c,a}}$. Moreover, the sets $M^{\mathrm{c,s}}$ and $M^{\mathrm{c,a}}$ are perpendicular because for any symmetric matrix A and anti-symmetric matrix B we have that $\langle A, B\rangle = 0$ since

$$\langle A, B\rangle = \mathrm{trace}(AB^{\mathrm{T}}) = -\mathrm{trace}(AB) = -\mathrm{trace}(A^{\mathrm{T}}B) = -\langle A, B\rangle.$$

Thus, (iii) is valid and the proof is complete. □

The proof of the next corollary is similar to that of Corollary 2.8 and so we omit the details.

Corollary 2.9. *The following statements hold*:

(i) *Both of* $M^{\mathrm{a,s}}$ *and* $M^{\mathrm{a,a}}$ *are subspaces of* M^{a}. *Moreover*, $B \in M^{\mathrm{a,s}}$ *if and only if it has the form* (2.10) *with* Z *having the form*

$$Z = \begin{pmatrix} 0 & Z_1 \\ Z_1^{\mathrm{T}} & 0 \end{pmatrix} \tag{2.11}$$

where $Z_1 \in M_{rn_1 \times rn_2}$; $B \in M^{\mathrm{a,a}}$ *if and only if it has the form* (2.10) *with* Z *having the form*

$$Z = \begin{pmatrix} 0 & Z_1 \\ -Z_1^{\mathrm{T}} & 0 \end{pmatrix}$$

where $Z_1 \in M_{rn_1 \times rn_2}$.

(ii) *The dimensions of* $M^{\mathrm{a,s}}$ *and* $M^{\mathrm{a,a}}$ *are given by*

$$\dim M^{\mathrm{a,s}} = \dim M^{\mathrm{a,a}} = r^2 n_1 n_2.$$

(iii) *There holds the decomposition*

$$M^{\mathrm{a}} = M^{\mathrm{a,s}} \oplus^{\perp} M^{\mathrm{a,a}}.$$

3. Construction of biorthogonal filter banks

In this section, we describe a general construction of vector-valued multivariate biorthogonal and orthogonal filter banks having a matrix factorization. For these filter banks, the low-pass filter, its corresponding high-pass filters and its dual filters have the same support. Moreover, when we choose the matrices to be block centrally symmetric, the filters enjoy the property of uniform linear phase which leads to the symmetry of the corresponding refinable function and wavelets (provided that they exist). Our presentation and point of view follow closely that were presented in [1] where only the orthonormal case was considered. We begin by recalling the setup in that paper.

Let A be a $d \times d$ matrix with integer entries such that all its eigenvalues are greater than one. Let $s := |\det A|$, $\mathbb{Z}_s := \{0, 1, \ldots, s-1\}$ and $\Omega(A) := \{\gamma_j : j \in \mathbb{Z}_s\}$ be a complete set of representatives of the distinct coset of $\mathbb{Z}^d/A\mathbb{Z}^d$ with $\gamma_0 = 0$. For each $\gamma \in \Omega(A)$, we define the corresponding coset $\overline{\gamma} := A\mathbb{Z}^d + \gamma$. The collection of cosets $\{\overline{\gamma}_j : j \in \mathbb{Z}_s\}$ form a partition of $\mathbb{Z}^d$. Recall that, it was shown in [1] that, for any dilation matrix A, there exists a complete set of representers of the distinct coset of $\mathbb{Z}^d/A\mathbb{Z}^d$ given by $\{\gamma_j : j \in \mathbb{Z}_s\}$ with $\gamma_0 = 0$ satisfying the additional conditions

$$\gamma_{s-1} - \gamma_j = \gamma_{s-1-j}, \quad j \in \mathbb{Z}_s. \tag{3.1}$$

For an $r \times r$ matrix h of trigonometric polynomials defined by the equation

$$h(\xi) := \sum_{\alpha \in \mathbb{Z}^d} a_\alpha \mathrm{e}^{-\mathrm{i}\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^d,$$

where $a_\alpha \in M_r$, its polyphase factors are the $r \times r$ matrices of trigonometric polynomials $\{h_l : l \in \mathbb{Z}_s\}$ defined for $l \in \mathbb{Z}_s$ and $\xi \in \mathbb{R}^d$ as

$$h_l(\xi) = \sum_{\alpha \in \mathbb{Z}^d} a_{A\alpha+\gamma_l} \mathrm{e}^{-\mathrm{i}\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^d.$$

We can reverse this process and construct the matrix of trigonometric polynomials h from its polyphase factors h_l, $l \in \mathbb{Z}_s$, by the formula

$$h(\xi) = \sum_{l \in \mathbb{Z}_s} h_l(A^{\mathrm{T}}\xi) \mathrm{e}^{-\mathrm{i}\gamma_l\cdot\xi}, \quad \xi \in \mathbb{R}^d.$$

The construction of multivariate compactly supported biorthogonal or orthonormal multi-wavelets using multiresolution analysis (MRA) leads to the following two questions (see, for example, [2]).

(i) Find an $r \times r$ matrix of trigonometric polynomials m_0 and it its dual $\tilde{m}_0$ such that their polyphase factors $m_{0,l}, \tilde{m}_{0,l}$, $l \in \mathbb{Z}_s$, satisfy the perfect reconstruction condition

$$W_0(\xi)^* \widetilde{W}_0(\xi) = \frac{1}{s} I_r, \quad \xi \in \mathbb{R}^d, \tag{3.2}$$

where W_0 and $\widetilde{W}_0$ are $rs \times r$ matrices defined by

$$W_0(\xi) := \big(m_{0,l}(\xi) : l \in \mathbb{Z}_s\big), \quad \widetilde{W}_0(\xi) := \big(m_{0,l}(\xi) : l \in \mathbb{Z}_s\big), \quad \xi \in \mathbb{R}^d.$$

We call W_0 and $\widetilde{W}_0$ the polyphase vectors corresponding to m_0 and $\tilde{m}_0$, respectively.

(ii) Find $r \times r$ matrices $m_j, \tilde{m}_j$, $j \in \mathbb{Z}_s \setminus \{0\}$, of trigonometric polynomials such that the $rs \times rs$ block matrices composed of their polyphase factors given by

$$W(\xi) := \left(m_{j,l}(\xi) : j, l \in \mathbb{Z}_s\right), \quad \widetilde{W}(\xi) := \left(\tilde{m}_{j,l}(\xi) : j, l \in \mathbb{Z}_s\right), \quad \xi \in \mathbb{R}^d$$

satisfies

$$W(\xi)^* \widetilde{W}(\xi) = \frac{1}{s} I_{rs}, \quad \xi \in \mathbb{R}^d. \tag{3.3}$$

There is an extensive literature associated with the problems (i) and (ii), especially for dyadic dilation (see, for example, [2,5–9,12,13]). However, even in this special case there is no general method to solve them. By using block central symmetric matrices, we resolve these problems for a large family of filter banks which have a matrix factorization.

To explain our method, for any $\xi \in \mathbb{R}^d$, we let $D(\xi)$ be the $rs \times rs$ diagonal block diagonal matrix of trigonometric entries defined by

$$D(\xi) := \operatorname{diag}\left(\mathrm{e}^{-\mathrm{i}\gamma_0\cdot\xi} I_r, \ldots, \mathrm{e}^{-\mathrm{i}\gamma_{s-1}\cdot\xi} I_r\right)$$

and denote by X the $r \times rs$ matrix function defined by

$$X(\xi) := \left(\mathrm{e}^{-\mathrm{i}\gamma_0\cdot\xi} I_r, \ldots, \mathrm{e}^{-\mathrm{i}\gamma_{s-1}\cdot\xi} I_r\right), \quad \xi \in \mathbb{R}^d.$$

We choose the low-pass filters $m_0, \tilde{m}_0$ to have the form

$$\begin{cases} m_0(\xi) = \frac{1}{s} X(\xi) \left(\prod_{j\in\mathbb{Z}_N} U_j D(A^{\mathrm{T}}\xi) U_j^{-1}\right) V_0, & \xi \in \mathbb{R}^d, \\ \tilde{m}_0(\xi) = \frac{1}{s} X(\xi) \left(\prod_{j\in\mathbb{Z}_N} \widetilde{U}_j D(A^{\mathrm{T}}\xi) \widetilde{U}_j^{-1}\right) \widetilde{V}_0, & \xi \in \mathbb{R}^d, \end{cases} \tag{3.4}$$

where the matrices $U_j, \widetilde{U}_j$, $j \in \mathbb{Z}_N$, are arbitrary $rs \times rs$ real block centrally symmetric matrices with

$$\widetilde{U}_j = (U_j^{\mathrm{T}})^{-1}, \tag{3.5}$$

and $V_0, \widetilde{V}_0$ are arbitrary $rs \times r$ real matrices satisfying

$$V_0^{\mathrm{T}} \widetilde{V}_0 = s I_r. \tag{3.6}$$

We first show below that m_0 and $\tilde{m}_0$ defined in (3.4) form a dual pair of filters satisfying the perfect reconstruction condition (3.2).

Theorem 3.1. *Suppose that $m_0, \tilde{m}_0$ are trigonometric polynomials defined in* (3.4) *for some $rs \times rs$ real matrices $U_j, \widetilde{U}_j$, $j \in \mathbb{Z}_N$ and some $rs \times r$ real constant matrices $V_0, \widetilde{V}_0$. The following statements about m_0 and $\tilde{m}_0$ hold*:

(i) *If U_j and $\widetilde{U}_j$, $j \in \mathbb{Z}_N$, satisfy* (3.5) *and $V_0, \widetilde{V}_0$ satisfy* (3.6), *then m_0 and $\tilde{m}_0$ satisfy the perfect reconstruction condition* (3.2).

(ii) *The conditions $m_0(0) = I_r$, $\tilde{m}_0(0) = I_r$ hold if and only if*

$$X(0)V_0 = sI_r, \quad X(0)\widetilde{V}_0 = sI_r.$$

(iii) *If $V_0 = \widetilde{V}_0$, then both m_0 and $\tilde{m}_0$ have accuracy of order at least* 1, *that is*,

$$m_0(2\pi(A^{\mathrm{T}})^{-1}\omega_l) = \tilde{m}_0(2\pi(A^{\mathrm{T}})^{-1}\omega_l) = \delta_{0l} I_r,$$

for all $\omega_l, l \in \mathbb{Z}_s$ belonging to $\Omega(A^{\mathrm{T}})$, an arbitrary complete set of the representatives of the distinct coset $\mathbb{Z}^d/A^{\mathrm{T}}\mathbb{Z}^d$ with $\omega_0 = 0$.

Proof. We first show (i). To this end, we note that the polyphase vectors W_0 and $\widetilde{W}_0$ corresponding to m_0 and $\tilde{m}_0$ are

$$W_0(\xi) = \frac{1}{s} \prod_{j \in \mathbb{Z}_N} \left(U_j D(A^{\mathrm{T}}\xi) U_j^{-1} \right) V_0, \quad \xi \in \mathbb{R}^d$$

and

$$\widetilde{W}_0(\xi) = \frac{1}{s} \prod_{j \in \mathbb{Z}_N} \left(\widetilde{U}_j D(A^{\mathrm{T}}\xi) \widetilde{U}_j^{-1} \right) \widetilde{V}_0, \quad \xi \in \mathbb{R}^d,$$

respectively. Therefore, we conclude that

$$W_0^* \widetilde{W}_0 = \frac{1}{s^2} V_0^{\mathrm{T}} \prod_{j \in \mathbb{Z}_N} \left((U^{-1})_{N-j-1}^{\mathrm{T}} D(-A^{\mathrm{T}}\cdot) U_{N-j-1}^{\mathrm{T}} \right) \times \prod_{j \in \mathbb{Z}_N} \left(\widetilde{U}_j D(A^{\mathrm{T}}\cdot) \widetilde{U}_j^{-1} \right) \widetilde{V}_0.$$

Since $D(-\cdot)D = I_{rs}$, $U_j, \widetilde{U}_j$, $j \in \mathbb{Z}_N$ satisfy (3.5) and $V_0, \widetilde{V}_0$ satisfy (3.6), we conclude that $m_0, \tilde{m}_0$ satisfies the perfect reconstruction condition (3.2).

To show (ii), we use Eq. (3.4) to get that

$$m_0(0) = \frac{1}{s} X(0) V_0, \quad \tilde{m}_0(0) = \frac{1}{s} X(0) \widetilde{V}_0,$$

from which (ii) follows.

Finally, we turn to the proof of (iii). From Theorem 2.2 of [1], we conclude that the condition $V_0 = \widetilde{V}_0$ implies that $V_0 = \widetilde{V}_0 = (I_r, \ldots, I_r)^{\mathrm{T}}$. We furthermore observe for $n \in \mathbb{Z}_s$ that

$$m_0(\pi_n) = \frac{1}{s} X(\pi_n) \prod_{j \in \mathbb{Z}_N} \left(U_j D(2\pi\omega_n) U_j^{-1} \right) V_0,$$

where $\pi_n := 2\pi(A^{\mathrm{T}})^{-1}\omega_n$. Since $D(\xi)$ is 2π-periodic, we see that $D(2\pi\omega_n)$ equals to the identity matrix I_{rs} for $n \in \mathbb{Z}_s$. Noting the definition of $X(\xi)$, we conclude that

$$m_0(\pi_n) = \frac{1}{s} \sum_{j \in \mathbb{Z}_s} \mathrm{e}^{-\mathrm{i}\gamma_j \cdot \pi_n} I_r, \quad n \in \mathbb{Z}_s.$$

Now, using the identity

$$\frac{1}{s} \sum_{j \in \mathbb{Z}_s} \mathrm{e}^{2\pi \mathrm{i}(A^{-1}\gamma_j)\cdot\omega_n} = \delta_{0n}, \quad n \in \mathbb{Z}_s,$$

(see, for example, [4]), we conclude that $m_0(\pi_l) = \delta_{0l} I_r$. Similarly, by changing the symbol U_j to $\widetilde{U}_j$, we can verify $\tilde{m}_0(\pi_l) = \delta_{0l} I_r$. $\square$

Remark. The conditions $m_0(0) = I_r$ and $m_0(0) = I_r$ mean that m_0 and $\tilde{m}_0$ are low-pass filters. In [1], we have shown that the equations

$$X(0)V_0 = sI_r, \quad V_0^{\mathrm{T}} V_0 = sI_r$$

have the unique solution $V_0^{\mathrm{T}} = X(0)$. But in biorthogonal case, V_0 and $\widetilde{V}_0$ are not uniquely determined by the equations

$$X(0)V_0 = X(0)\widetilde{V}_0 = sI_r, \quad V_0^{\mathrm{T}}\widetilde{V}_0 = sI_r.$$

Recall that the low-pass filter m_0 has *uniform linear phase* if there exists a $\mu \in \mathbb{Z}^d$ such that for all $\xi \in \mathbb{R}^d$

$$\overline{m_0(\xi)} = \mathrm{e}^{\mathrm{i}\mu\cdot\xi} m_0(\xi).$$

We call an $rn \times r$ matrix V an block centrally symmetric vector if $V = H_n V$, where H_n is the basic block centrally symmetric matrix defined in Section 2.

Theorem 3.2. *Suppose that the complete set $\Omega(A) = \{\gamma_j : j \in \mathbb{Z}_s\}$ with $\gamma_0 = 0$ of representatives of the distinct coset of $\mathbb{Z}^d/A\mathbb{Z}^d$ satisfies (3.1). If U_j and $\widetilde{U}_j$, $j \in \mathbb{Z}_N$, are block centrally symmetric matrices and V_0, $\widetilde{V}_0$ are block centrally symmetric vectors, then both of m_0 and $\tilde{m}_0$ defined in (3.4) have uniform linear phase.*

Proof. We only need to verify that m_0 has linear phase since its dual $\tilde{m}_0$ can be dealt with similarly. We must find a vector $\mu \in \mathbb{Z}^d$ such that

$$\overline{m_0(\xi)} = \mathrm{e}^{\mathrm{i}\mu\cdot\xi} m_0(\xi), \quad \xi \in \mathbb{R}^d.$$

Our choice for μ is that

$$\mu := (NA + I)\gamma_{s-1}.$$

Let us confirm that this is a correct choice. By (3.4), we have that

$$\overline{m_0(\xi)} = \frac{1}{s} X(-\xi) \prod_{j\in Z_N} \left(U_j D(-A^{\mathrm{T}}\xi) U_j^{-1} \right) V_0, \quad \xi \in \mathbb{R}^d$$

while our hypothesis on $\Omega(A)$ lead us to conclude that

$$\mathrm{e}^{-\mathrm{i}\gamma_{s-1}\cdot A^{\mathrm{T}}\xi} D(-A^{\mathrm{T}}\xi) = HD(A^{\mathrm{T}}\xi)H, \quad \xi \in \mathbb{R}^d,$$

and

$$\mathrm{e}^{-\mathrm{i}\gamma_{s-1}\cdot\xi} X(-\xi) = X(\xi)H, \quad \xi \in \mathbb{R}^d.$$

Combining these equations, we get that

$$\overline{m_0(\xi)} = \frac{1}{s} \mathrm{e}^{\mathrm{i}\mu\cdot\xi} X(\xi)H \prod_{j\in Z_N} \left(U_j HD(A^{\mathrm{T}}\xi) HU_j^{-1} \right) V_0, \quad \xi \in \mathbb{R}^d,$$

from which it follows that

$$\overline{m_0(\xi)} = \frac{1}{s} \mathrm{e}^{\mathrm{i}\mu\cdot\xi} X(\xi) \prod_{j\in Z_N} \left((HU_jH) D(A^{\mathrm{T}}\xi)(HU_j^{-1}H) \right) HV_0, \quad \xi \in \mathbb{R}^d.$$

Theorem 2.4 implies that the inverse of U_j, $j \in \mathbb{Z}_N$, is a centrally symmetric from which the result follows. □

Our next task is to construct high-pass filters corresponding to the dual pair of low-pass filters given by (3.4) which is accomplished by matrix extensions for m_0 and $\tilde{m}_0$. Indeed, in this case, the matrix extension for the low-pass filters m_0 and $\tilde{m}_0$ is realizable. Specifically, we extend the $rs \times r$ matrix V_0 and $\widetilde{V}_0$ which satisfy (3.6) to $rs \times rs$ real matrices

$$V := (V_0, V_1, \ldots, V_{s-1}), \quad \widetilde{V} := \left(\widetilde{V}_0, \widetilde{V}_1, \ldots, \widetilde{V}_{s-1}\right),$$

respectively, such that

$$V^{\mathrm{T}}\widetilde{V} = sI_{rs}.$$

Now, we define $r \times r$ matrices of trigonometric polynomials m_j and $\tilde{m}_j$, $j \in \mathbb{Z}_s \setminus \{0\}$ by the equation

$$\begin{cases} m_j(\xi) = \frac{1}{s}X(\xi)\left(\prod_{k\in\mathbb{Z}_N} U_k D(A^{\mathrm{T}}\xi)U_k^{-1}\right)V_j, & \xi \in \mathbb{R}^d, \\ \tilde{m}_j(\xi) = \frac{1}{s}X(\xi)\left(\prod_{k\in\mathbb{Z}_N} \widetilde{U}_k D(A^{\mathrm{T}}\xi)\widetilde{U}_k^{-1}\right)\widetilde{V}_j, & \xi \in \mathbb{R}^d. \end{cases} \tag{3.7}$$

The next theorem ensures that the matrices m_j and $\tilde{m}_j$ of trigonometric polynomials constructed in (3.7) are the high-pass filters corresponding to the low-pass filters m_0 and $\tilde{m}_0$.

Theorem 3.3. *The trigonometric polynomials m_j and $\tilde{m}_j$, $j \in Z_s \setminus \{0\}$, defined by (3.7) are high-pass filters corresponding to the low-pass filter m_0 and $\tilde{m}_0$ defined by (3.4) with $V_0 = \widetilde{V}_0$.*

Proof. We have known that the condition $V_0 = \widetilde{V}_0$ implies that

$$V_0 = (I_r, \ldots, I_r)^{\mathrm{T}}.$$

Furthermore, by using Eq. $V^{\mathrm{T}}\widetilde{V} = sI_{rs}$, we know that

$$V_0^{\mathrm{T}}V_j = 0 \text{ and } V_0^{\mathrm{T}}\widetilde{V}_j = 0, \quad \text{for all } j \in \mathbb{Z}_s \setminus \{0\}.$$

It is clear that $m_j, \tilde{m}_j$, $j \in Z_s \setminus \{0\}$, are high-pass filters because

$$m_j(0) = \frac{1}{s}V_0^{\mathrm{T}}V_j = 0, \quad \tilde{m}_j(0) = \frac{1}{s}V_0^{\mathrm{T}}\widetilde{V}_j = 0.$$

It remains to prove that the two polyphase matrices W and $\widetilde{W}$ formed from m_j and $\tilde{m}_j$, $j \in Z_s$, satisfy

$$W^*(\xi)\widetilde{W}(\xi) = \frac{1}{s}I_{rs}, \quad \xi \in \mathbb{R}^d.$$

It follows from (3.4) and (3.7) that the polyphase matrices W and $\widetilde{W}$ corresponding to m_j and $\tilde{m}_j$, $j \in \mathbb{Z}_s$ are as follows

$$W(\xi) = \frac{1}{s} \prod_{j \in \mathbb{Z}_N} \left(U_j D(A^{\mathrm{T}}\xi) U_j^{-1} \right) (V_0, V_1, \ldots, V_{s-1}), \quad \xi \in \mathbb{R}^d,$$

and

$$\widetilde{W}(\xi) = \frac{1}{s} \prod_{j \in \mathbb{Z}_N} \left(\widetilde{U}_j D(A^{\mathrm{T}}\xi) \widetilde{U}_j^{-1} \right) \left(\widetilde{V}_0, \widetilde{V}_1, \ldots, \widetilde{V}_{s-1}\right), \quad \xi \in \mathbb{R}^d.$$

Thus

$$W^* \widetilde{W} = \frac{1}{s^2} V^{\mathrm{T}} \prod_{j \in \mathbb{Z}_N} \left((U_{N-j-1}^{-1})^{\mathrm{T}} D(-A^{\mathrm{T}} \cdot) U_{N-j-1}^{\mathrm{T}} \right) \prod_{j \in \mathbb{Z}_N} \left(\widetilde{U}_j D(A^{\mathrm{T}} \cdot) \widetilde{U}_j^{-1} \right) \widetilde{V}.$$

Since $D(-\cdot)D$ is the identity matrix of order rs, matrices $U_j, \widetilde{U}_j, j \in \mathbb{Z}_N$ satisfy (3.5) and $V^{\mathrm{T}}\widetilde{V} = sI_{rs}$, we conclude that

$$W^* \widetilde{W} = \frac{1}{s} I_{rs},$$

proving the result. □

To close this paper, we remark that the matrices V_j, $\widetilde{V}_j$, U_j, $\widetilde{U}_j$, $j \in \mathbb{Z}_N$ in the construction of the low pass filters and high pass filters may be chosen such that the refinable functions and wavelets have certain desirable accuracy, support and vanishing moment properties. This is an interesting future research project.

References

[1] Q.H. Chen, C.A. Micchelli, S. Peng, Y. Xu, Multivariate filter banks having a matrix factorization, SIAM J. Matrix Anal. Appl. 25 (2003) 517–531.

[2] W. Dahmen, C.A. Micchelli, Biorthogonal wavelet expansions, Constr. Approx. 13 (1997) 293–328.

[3] O. Herrmann, On the approximation problem in nonrecursive digital filter design, IEEE Trans. Circuit Theory CT-18 (1971) 411–413.

[4] R.Q. Jia, Approximation properties of multivariate wavelets, Math. Comp. 67 (1998) 647–665.

[5] R.Q. Jia, C.A. Micchelli, Using the refinement equation for the construction of prewavelets V: extensibility of trigonometric polynomials, Computing 48 (1992) 61–72.

[6] J. Kovacevic, M. Vetterli, Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for R^n, IEEE Trans. Inf. Theory 38 (1992) 533–555.

[7] C.A. Micchelli, Using the refinement equation for the construction of pre-wavelets, Numer. Algorithms 1 (1991) 75–116.

[8] C.A. Micchelli, Using the refinement equation for the construction of pre-wavelets VI: Shift invariant subspaces, in: Approximation Theory, Spline Functions and Applications (Maratea, 1991), 213–222, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 356, Kluwer Acad. Publ., Dordrecht, 1992.

[9] C.A. Micchelli, Y. Xu, Reconstruction and decomposition algorithms for biorthogonal multiwavelets, Multidimen. Syst. Signal Process. 8 (1997) 31–69.

[10] T.Q. Nguyen, P.P. Vaidyanathan, Two-channel perfect reconstruction FIR QMF structure which yield linear phase FIR analysis and synthesis filters, IEEE Trans. Acoust. Speech Signal Proc. 37 (1989) 676–690.

[11] P.P. Vaidyanathan, Multivariate Systems and Filter Banks, Prentice Hall, Englewood Cliffs, NJ, 1993.

[12] P.P. Vaidyanathan, Quadrature mirror filter banks, M-band extensions and perfect reconstruction techniques, IEEE ASSP Mag. 4 (3) (1987) 4–20.
[13] M. Vetterli, Filter banks allowing perfect reconstruction, Signal Process. 10 (3) (1986) 219–244.