Inequalities and Comparisons of the Cauchy, Gauss, and Logistic Distributions

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Abstract—In this note, some inequalities and basic results including characterizations and comparisons of the Cauchy, logistic, and normal distributions are given. These results lead to necessary and sufficient conditions for the stochastic and dispersive ordering of the corresponding absolute random variables. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this note, some inequalities and comparisons of the Cauchy, logistic, and normal distributions are presented. These results shed further insight into these well-known distributions. Also, necessary and sufficient conditions for dispersive and stochastic ordering of the corresponding absolute random variables are given. A general Cauchy law \( C(\alpha, \beta) \) is a distribution with the form \( Y = \beta Z + \alpha \), where \( Z \) has the density function of the form \( f(z) = (1/\pi)(1 + z^2) \), \(-\infty < z < \infty\), denoted by \( C(0,1) \). The characteristic function is given by \( \Psi_Y(t) = \exp\{i\alpha t - \beta |t|\} \). A random variable \( X \) has a logistic distribution with parameters \( \alpha \) and \( \beta \) if the distribution function can be written as

\[
F(x) = P(X \leq x) = \left[1 + \exp\left\{-\frac{x - \alpha}{\beta}\right\}\right]^{-1},
\]

\(-\infty < \alpha < \infty, \beta > 0\). Furthermore, the distribution can be written as

\[
F(x) = \frac{1}{2} \left\{1 + \tan h\left(\frac{x - \alpha}{\beta}\right)\right\}, \quad -\infty < \alpha < \infty, \beta > 0.
\]

(1.1)

This distribution has been used to model tolerance levels in bioassay problems. It is also useful for economic and demographic purposes. A standard logistic distribution is (1.1) with \( \alpha = 0 \) and \( \beta = 1 \). A general Gaussian law \( N(\mu, \sigma^2) \) has a distribution of the form \( X = \sigma Z + \mu \), where \( Z \) is \( N(0,1) \). In Section 2, some basic results including characterizations are given. Some
comparisons are given in Section 3. Section 4 contains discussions and concluding remarks. We use the notation ~ to mean distributed as.

2. SOME RESULTS

In this section, we present characterization results on the Cauchy and logistic distributions. We also present results on the stochastic ordering of the absolute value of logistic and Gaussian random variables. Specifically, we show that the behaviour of the probability density functions at zero and restrictions on the first moment (mean) and standard deviation are equivalent in the establishment of the necessary and sufficient conditions for the stochastic ordering of the random variables.

It is well known that if a random variable $T$ can be expressed as a ratio of two independent standard normal random variables, then $T$ has the standard Cauchy distribution; that is, $T \sim C(0,1)$. In the proposition below, a characterization of the Cauchy distribution via a convex function of a symmetric random variable is presented. This result characterizes the Cauchy distribution without recourse to the well-known result involving the standard normal distribution. Also, Theorem 1 deals with a characterization of the general Cauchy law with potential extension to multivariate stable distributions.

**PROPOSITION 1.** If $X$ is symmetric about 0 and $X^2 \sim g(u) = (1/\pi)(1 + u)(\sqrt{u})^{-1}$, $u > 0$, then $X \sim C(0,1)$.

**PROOF.** Let $g(\cdot)$ be the probability density function of $U = X^2$ and $f(\cdot)$ the probability density function of $X$. Then

$$g(u) = \frac{1}{\pi}(1 + u)(\sqrt{u})^{-1} = \left(\frac{2}{\sqrt{\pi}}\right)^{-1} \left[f(\sqrt{u}) + f(-\sqrt{u})\right].$$

Therefore, $f(\sqrt{u}) + f(-\sqrt{u}) = (2/\pi)(1 + u)$. Let $\Psi$ be the characteristic function of $X$. Then

$$\Psi(t) + \Psi(-t) = \int_{-\infty}^{\infty} e^{itx} f(x) \, dx + \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx$$

$$= \int_{-\infty}^{\infty} e^{itx} [f(x) + f(-x)] \, dx$$

$$= \int_{-\infty}^{\infty} e^{itx} \left[\frac{2}{\pi} (1 + x^2)\right] \, dx$$

$$= 2\exp\{-|t|\}, \quad -\infty < t < \infty.$$

Since $X$ is symmetric about zero,

$$\Psi(t) = \Psi(-t) \quad \text{and} \quad \Psi(t) = \exp\{-|t|\}, \quad -\infty < t < \infty.$$

Consequently, $X \sim C(0,1)$. 

**THEOREM 1.** Let $\Psi_{X_i}$ be the characteristic function of the random variables $X_i$, $i = 1, 2$ and $g$ a differentiable function defined on $[0, \infty)$. Suppose $\Psi_{X_i}$ are differentiable and $\lim_{t \to 0^+} \frac{d^2 \Psi_{X_i}}{dt^2} \neq 0$, $i = 1, 2$, and the distribution of $U X_1 + g(U)X_2$ does not depend on $U$ for $U$ in the domain of $g$. Then $X_1$ has the Cauchy distribution and $g(u) = \mu - \gamma u \geq 0$, for $\mu, \gamma > 0$.

**PROOF.** Let $A = UX_1 + g(U)X_2$. Then

$$\Psi_A(t) = \Psi_{UX_1 + g(U)X_1}(t) = \Psi_{X_1}(ut) \Psi_{X_2}(g(u)t).$$

(2.1)

Now,

$$\frac{d\Psi_A}{du} = \Psi_{X_2}(g(u)t) \frac{d}{du} \Psi_{X_1}(ut) + \Psi_{X_1}(ut) \frac{d\Psi_{X_2}(g(u)t)}{du} \cdot \frac{dg(u)}{du}.$$
Since the distribution of $A$ does not depend on $U$, we have

$$
\Psi_X(g(u)t) \frac{d}{du} \Psi_X(ut) = -\Psi_X(ut) \frac{d}{du} \Psi_X(g(u)t) \frac{dg(u)}{du}
$$

so that

$$
\Psi_X((\mu - \gamma u)t)\Psi_X'(ut) = \Psi_X(ut)\Psi_X'((\mu - \gamma u))(-\gamma).
$$

Letting $u \to 0^+$, we obtain a differential equation in $\Psi_X(t)$ given by

$$
(i\alpha_1 - \beta_1)\Psi_X(ut) = \gamma\Psi_X'(ut),
$$

where we have set $\lim_{t \to 0^+} \Psi_X(t) = i\alpha_1 - \beta_1$.

Consequently,

$$
\Psi_X(t) = \exp\left\{ \frac{i\alpha_1}{\gamma} t - \frac{\beta_1}{\gamma} t \right\}, \quad \text{for } t \geq 0,
$$

$$
= \exp\{i\alpha_2 t - \beta_2 t\}, \quad \text{for } t \geq 0.
$$

With $\Psi_X(t)$ in (2.2), we obtain a differential equation in $\Psi_X(t)$ from which

$$
\Psi_X(t) = \exp\{i\alpha_1 t - \beta_1 t\}, \quad \text{for } t \geq 0.
$$

Letting $t \to 0^+$ in (2.2), with $\lim_{t \to 0^+} \Psi_X'(t) = i\alpha_j - \beta_j$, $j = 1, 2$, we have

$$
g'(u) = -\frac{i\alpha_1 - \beta_1}{i\alpha_2 - \beta_2} = \gamma.
$$

It follows, therefore, that $g(u) = \mu - \gamma u$, for $\mu \geq \gamma u \geq 0$.

The results presented in Theorems 2 and 3 are concerned with the characterization of the logistic distribution. These results shed further light into the logistic distribution which has tremendous practical applications in a wide variety of areas including the analysis of quantal response and bioassay data as well economic and demographic data.

**Theorem 2.** A random variable $X$ is distributed as the standard logistic distribution if and only if for real numbers $u > 1$ and $v > 1$,

$$
P(X \leq -\ln(uv - 1)) = P(X \leq -\ln(u - 1))P(X \leq -\ln(v - 1)).
$$

**Proof.** Let $\delta(u) = P(X \leq -\ln(u - 1))$. Then

$$
\delta(uv) = \delta(u)\delta(v).
$$

This is a functional form, the solution of which is given by [1] as $\delta(s) = Cs^d$. In this case, $C = 1$.

Let $x = -\ln(u - 1)$. Then $u = 1 + \exp(-x)$ and $P(X \leq x) = [1 + \exp(-x)]^d$.

Now, $\lim_{x \to -\infty} F(x) = 1$ and for $\lim_{x \to -\infty} F(x) = 0$, we must have $d < 0$. Consequently, choose $d = -1$.

For the necessity, let $X$ be distributed as the standard logistic. Then,

$$
P(X \leq -\ln(uv - 1)) = [1 + \exp\{\ln(uv - 1)\}]^{-1}
$$

$$
= (uv)^{-1}
$$

$$
= P(X \leq -\ln(u - 1))P(X \leq -\ln(v - 1)).$$

"
COROLLARY 1. The random variable $X$ is distributed as the logistic $L(\alpha, \beta)$ if for real numbers $u > 1$ and $v > 1$, $\beta > 0$, $-\infty < \alpha < \infty$,

$$P(X \leq -\beta \ln(uv - 1) + \alpha) = P(X \leq -\beta \ln(u - 1) + \alpha)P(X \leq -\beta \ln(v - 1) + \alpha).$$

PROOF. Let $\gamma(u) = P(X \leq -\beta \ln(u - 1) + \alpha)$. Then $\gamma(uv) = \gamma(u)\gamma(v)$. This is a functional form, the solution of which is known, and is given by [1], as $\gamma(s) = Cs^d$. In this case, $C = 1$. Let $x = -\beta \ln(u - 1) + \alpha$. Then $u = 1 + \exp\left\{-\frac{x - \alpha}{\beta}\right\}$ and

$$P(X \leq x) = \left[1 + \exp\left\{-\frac{x - \alpha}{\beta}\right\}\right]^d.$$ 

Now $\lim_{x \to \infty} F(x) = 1$ and for $\lim_{x \to -\infty} F(x) = 0$, we must have $d < 0$. Choose $d = -1$.

THEOREM 3. Let $X$ be a random variable with distribution function $F(x)$ which is symmetric about $\alpha \in \mathbb{R}$. Then $X$ has the logistic distribution with parameters $\alpha$ and $\beta > 0$ if and only if

$$P\{|X - \alpha| \leq y\} = \left[1 - \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]\left[1 + \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]^{-1}, \quad (2.6)$$

for all $y \geq 0$.

PROOF. Suppose $X$ has the logistic distribution with parameters $\alpha$ and $\beta > 0$. Then for $y \geq 0$

$$P\{|X - \alpha| \leq y\} = P\{\alpha - y \leq X \leq \alpha + y\} = F(y + \alpha) - F(\alpha - y) = 2F(y + \alpha) - 1 = \left[1 - \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]\left[1 + \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]^{-1}.$$

Conversely, supposing (2.6) holds, we show that the distribution of $X$ is the logistic distribution with parameters $\alpha$ and $\beta > 0$.

Since

$$P\{|X - \alpha| \leq y\} = 2G(y + \alpha) - 1,$$

for some distribution function $G$, it follows that

$$2G(y + \alpha) - 1 = \left[1 - \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]\left[1 + \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]^{-1}$$

and

$$G(y + \alpha) = \frac{1}{2} \left\{\left[1 - \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]\left[1 + \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]^{-1} + 1\right\} = \left[1 + \exp\left\{-\left(\frac{y}{\beta}\right)\right\}\right]^{-1}.$$ 

Consequently,

$$F(x) = \left[1 + \exp\left\{-\frac{x - \alpha}{\beta}\right\}\right]^{-1},$$

which is the logistic distribution.
3. COMPARISONS

Let $F$ and $G$ be the distribution functions of the random variable $X$ and $Y$, respectively. We say $F$ is stochastically larger than $G$ if $F(x) \geq G(x)$ for all $x$. Also, we say the distribution of $F$ is more spread out than $G$ if

$$G^{-1}(v) - G^{-1}(u) \leq F^{-1}(v) - F^{-1}(u),$$

for all $0 < u < v < 1$, where $G^{-1}$ is the quantile function. If $F^{-1}$ and $G^{-1}$ are differentiable, then (3.1) is equivalent to

$$\frac{f(F^{-1}(y))}{g(G^{-1}(y))} \leq 1, \quad \text{for all } 0 < y < 1.$$  

We now present necessary and sufficient conditions for the stochastic ordering of the absolute value of an arbitrary logistic random variable with respect to the standard logistic random variable. This result also applies to the normal distribution and is stated in Theorem 5, where the value of the density function at zero is necessary and sufficient for the stochastic ordering of the absolute value of an arbitrary normal random variable with respect to the standard normal random variable.

THEOREM 4. Let $X$ be distributed as the logistic distribution with parameters $\alpha$ and $\beta$, and $Y$ as the standard logistic distribution. Then $|X| \prec_Y |Y|$ if and only if $\beta \leq 1$ and $\alpha \leq 0$.

PROOF. Suppose $|X| \prec_Y |Y|$, that is

$$P(-x \leq X \leq x) \geq P(-x \leq Y \leq x), \quad \text{for all } x > 0,$$

$$\iff P\left(\frac{-x - \alpha}{\beta} \leq Y^* \leq \frac{x - \alpha}{\beta}\right) \geq P(-x \leq Y \leq x), \quad \text{for all } x > 0,$$

where $Y^*$ has a standard logistic distribution. We show that $\beta < 1$ and $\alpha < 0$ is necessary for (3.3) to hold. We note that (3.3) can be written as

$$G(x) = \tanh\left[\frac{1}{2} \frac{x - \alpha}{\beta}\right] + \tanh\left[\frac{1}{2} \frac{x + \alpha}{\beta}\right] - \tanh\left(\frac{1}{2} x\right)$$

Clearly $G(0) = 0$, so $G'(0) \geq 0$ is necessary. Equivalently, $\sech^2(\alpha/2\beta) \geq \beta$. But $1 \geq \sech^2(\alpha/2\beta) \geq \beta > 0$, so the last inequality reduces to $\alpha/\beta \geq 0$. Consequently, $\beta \leq 1$ and $\alpha \leq 0$. Conversely, suppose $\beta \leq 1$ and $\alpha \geq 0$. If $\beta = 1$ and $\alpha = 0$, then $G(x) \equiv 0$. Suppose $\beta < 1$ and $\alpha < 0$. If $x \geq |\alpha|/(1 - \beta)$, then

$$\frac{1 - \exp(x)}{1 + \exp(x)} \geq \frac{1 - \exp\{(x - \alpha)/\beta\}}{1 + \exp\{(x - \alpha)/\beta\}}$$

and

$$G(x) = \tanh\left[\frac{1}{2} \frac{x - \alpha}{\beta}\right] + \tanh\left[\frac{1}{2} \frac{x + \alpha}{\beta}\right] - \tanh\left(\frac{1}{2} x\right)$$

$$= \frac{2[\exp(2x/\beta) - 1]}{[\exp\{(x - \alpha)/\beta\} + 1][\exp\{(x + \alpha)/\beta\} + 1]} + \frac{1 - \exp(x)}{1 + \exp(x)}$$

$$\geq \frac{\exp(2x/\beta) - 1 + \exp\{(x + \alpha)/\beta\} - \exp\{(x - \alpha)/\beta\}}{[\exp\{(x - \alpha)/\beta\} + 1][\exp\{(x + \alpha)/\beta\} + 1]}$$

$$\geq \frac{\exp(2x/\beta) - 1 + (2\alpha/\beta)\exp\{(x - \alpha)/\beta\}}{[\exp\{(x - \alpha)/\beta\} + 1][\exp\{(x + \alpha)/\beta\} + 1]}$$

$$\geq \frac{1 + (2x/\beta) - 1 + (2\alpha/\beta)\exp\{(x - \alpha)/\beta\}}{[\exp\{(x - \alpha)/\beta\} + 1][\exp\{(x + \alpha)/\beta\} + 1]}$$

$$> 0.$$
The last inequality follows from the fact that
\[
\exp(x) \geq 1 + x \quad \text{and} \quad \left( \frac{x - \alpha}{\beta} \right) \exp \left( \frac{x - \alpha}{\beta} \right) > 0.
\]

Note that
\[
G'(x) = \frac{1}{2} \beta \left\{ \text{sech}^2 \left( \frac{1}{2} \frac{x - \alpha}{\beta} \right) + \text{sech}^2 \left( \frac{1}{2} \frac{x + \alpha}{\beta} \right) \right\} - \frac{1}{2} \text{sech}^2 \left( \frac{1}{2} \frac{x}{\beta} \right)
\]
and
\[
G''(x) < 0, \quad \text{on } (0, \infty).
\]
Since \(G(0) = 0\) and \(G(x) \to 0\) as \(x \to \infty\), it follows therefore that \(G(x) \geq 0\) for all \(x > 0\).

**Theorem 5.** Let \(X\) be distributed as the normal distribution with mean \(\mu\) and variance \(\sigma^2\), and \(Y\) as the standard normal distribution. Then \(|X| <_{st} Y\) if and only if the coefficient of variation of the random variable \(X\) is less than or equal to zero.

The proof of Theorem 5 can be obtained along the lines of the proof of Theorem 4 and is omitted.

**4. REMARKS**

We note that for all \(f\) convex (concave) nondecreasing (nonincreasing) functions, \(f(X) <_{st} f(Y)\) if and only if \(P(f(X) \leq t) \geq P(f(Y) \leq t)\) for all \(t\). It turns out that if \(X <_{st} Y\), then \(X <_{\text{disp}} Y\) if and only if \(f(X) <_{\text{disp}} f(Y)\). Consequently, for even convex transformations, dispersive ordering \(<_{\text{disp}}\), and hence, stochastic ordering \(<_{st}\) is preserved, where \(<_{\text{disp}}\) is known to mean the distribution function of \(X\) is less dispersive than the distribution function of \(Y\) [2]. It follows that if \(X \sim L(\alpha, \beta)\) or \(C(\alpha, \beta)\) and \(Y \sim L(0, 1)\) or \(C(0, 1)\), then \(|X - \alpha| <_{\text{disp}} |Y|\) if and only if \(\beta \leq 1\), a consequence of which is \(|X - \alpha| <_{\text{disp}} |Y|\) if and only if \(X <_{\text{disp}} Y\). Similar results hold for the normal distributions. Note that Theorem 1 can be extended to multivariate stable distributions with Cauchy marginals.

**REFERENCES**