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On James and Jordan–von Neumann Constants of Lorentz Sequence Spaces

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The nonsquare or James constant $J(X)$ and the Jordan–von Neumann constant $C_{NJ}(X)$ are computed for two-dimensional Lorentz sequence spaces $d^{(2)}(w, q)$ in the case where $2 \le q < \infty$. The Jordan–von Neumann constant is also calculated in the case where $1 \le q < 2$.

Key Words: uniformly nonsquare spaces; James constant; Jordan–von Neumann constant; n-dimensional Lorentz sequence spaces; Lorentz sequence spaces.

Several results on the nonsquare constant $J(X)$ of James and also the Jordan–von Neumann constant $C_{NJ}(X)$ [which is usually called the von Neumann–Jordan constant, and so we use the notation $C_{NJ}(X)$] of a Banach space X have been recently obtained by Casini [1], Gao-Lau [2, 3], Kato and Takahashi [8], and Kato, Maligranda and Takahashi [6, 7] (see also [5] for the classical result). In particular, they calculated $J(X)$ and $C_{NJ}(X)$ for various spaces X and showed that some properties of X, such as uniform nonsquareness, superreflexivity, type, and cotype, can be described in terms of the constant $C_{NJ}(X)$.

The aim of this paper is to compute the constants $J(X)$ and $C_{NJ}(X)$ for two-dimensional Lorentz sequence spaces $X = d^{(2)}(w, q)$ in the case where $1 \le q < \infty$. The paper is organized as follows. In Section 1 we collect

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properties of constants $J(X)$ and $C_{NJ}(X)$, and also relations between them. In Section 2 we present results on $J(X)$ and $C_{NJ}(X)$ for two-dimensional Lorentz sequence spaces $X = d^{(2)}(w, q)$ in the case where $2 \le q < \infty$. In Section 3 we give the precise value of $C_{NJ}(d^{(2)}(w, q))$ in the case where $1 \le q < 2$.

1. PRELIMINARIES

Let $X = (X, \|\cdot\|)$ be a real Banach space with $\dim X \ge 2$, $B_X = \{x \in X : \|x\| \le 1\}$ its unit ball and $S_X = \{x \in X : \|x\| = 1\}$ its unit sphere. The constant

$$J(X) = \sup\{\min(\|x + y\|, \|x - y\|) : x, y \in S_X\}$$

is called the *James constant*, or the *nonsquare constant*, of a Banach space X.

We collect properties of the James constant $J(X)$ (see Casini [1], Gao and Lau [2, 3], and Kato, Maligranda, and Takahashi [6, 7]):

(i) $J(X) = \sup\{\min(\|x + y\|, \|x - y\|) : x, y \in B_X\}$.

(ii) $\sqrt{2} \le J(X) \le 2$; $J(X) = \sqrt{2}$ if X is a Hilbert space and the converse is not true.

(iii) If $1 \le p \le \infty$ and $\dim L_p(\mu) \ge 2$, then $J(L_p(\mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$.

(iv) $J(X) = \sup\{\epsilon \in (0, 2) : \delta_X(\epsilon) \le 1 - \epsilon/2\}$, where $\delta_X(\epsilon) = \inf\{1 - \|x + y\|/2 : x, y \in S_X, \|x - y\| \ge \epsilon\}$ is the modulus of convexity of X.

(v) $J(X) < 2$ if and only if the space X is uniformly nonsquare; that is, there exists a $\delta \in (0, 1)$ such that for any $x, y \in S_X$ either $\|x + y\|/2 \le 1 - \delta$ or $\|x - y\|/2 \le 1 - \delta$.

(vi) $J(X^{**}) = J(X)$, $2J(X) - 2 \le J(X^*) \le J(X)/2 + 1$, and there exists X such that $J(X^*) \ne J(X)$, where X^* and X^{**} are the dual and bidual spaces of X, respectively.

The *Jordan–von Neumann constant*, of a Banach space X $C_{NJ}(X)$, is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both } 0\right\}.$$

We again collect its properties (see Jordan and von Neumann [5], Kato and Takahashi [8], and Kato, Maligranda, and Takahashi [6, 7]):

(vii) $1 \le C_{NJ}(X) \le 2$; X is a Hilbert space $\iff C_{NJ}(X) = 1$.

(viii) $C_{NJ}(X) = C_{NJ}(X^*)$.

(ix) If $1 \le p \le \infty$ and $\dim L_p(\mu) \ge 2$, then $C_{NJ}(L_p(\mu)) = 2^{2/r-1}$ with $r = \min\{p, p'\}$.

(x) X is uniformly nonsquare $\iff C_{NJ}(X) < 2$.

(xi) $J(X)^2/2 \le C_{NJ}(X) \le J(X)^2/(J(X)-1)^2 + 1$; If X is not uniformly nonsquare, then we have equalities, and there exists a two-dimensional Banach space X for which $J(X)^2/2 < C_{NJ}(X)$.

Let $w = (w_1, w_2, \ldots, w_n)$ with $w_1 \ge w_2 \ge \cdots \ge w_n > 0$ and $n = 2, 3, \ldots$. For $1 \le q < \infty$, the n-dimensional *Lorentz sequence space*, $d^{(n)}(w, q)$, is $\mathbb{R}^n$ with norm

$$\|x\|_{w,q} = (w_1 x_1^{*q} + w_2 x_2^{*q} + \cdots + w_n x_n^{*q})^{1/q},$$

where $(x_1^*, x_2^*, \ldots, x_n^*)$ is the nonincreasing rearrangement of $(|x_1|, |x_2|, \ldots, |x_n|)$; that is, $x_1^* \ge x_2^* \ge \cdots \ge x_n^*$ (cf. [9]). In the case when $w_k = k^{q/p-1}$, $k = 1, 2, \ldots, n$ and $1 \le q < \infty$, we have the classical n-dimensional Lorentz sequence space $\ell_{p,q}^n$.

Next, we compute the constants $J(X)$ and $C_{NJ}(X)$ for two-dimensional Lorentz sequence spaces $X = d^{(2)}(w, q)$.

2. LORENTZ SEQUENCE SPACES $d^{(2)}(w, q)$, FOR THE CASE WHERE $q \ge 2$

Our computation of the James and the Jordan–von Neumann constants for two-dimensional Lorentz sequence spaces $X = d^{(2)}(w, q)$ in the case where $2 \le q < \infty$, begins with the following theorem.

THEOREM 1. *If $q \ge 2$, then*

$$J(d^{(2)}(w, q)) = 2\left(\frac{w_1}{w_1 + w_2}\right)^{1/q} \tag{2.1}$$

and

$$C_{NJ}(d^{(2)}(w, q)) = \frac{1}{2} J(d^{(2)}(w, q))^2 = 2\left(\frac{w_1}{w_1 + w_2}\right)^{2/q}. \tag{2.2}$$

In the proof, we will need the following lemma.

LEMMA 1. *If $1 \le q < \infty$, then*

$$\left(\frac{w_1 + w_2}{2}\right)^{1/q} \|x\|_q \le \|x\|_{w,q} \le w_1^{1/q} \|x\|_q \tag{2.3}$$

for all $x \in \mathbb{R}^2$.

Proof. The first inequality means that

$$(w_1 + w_2)(x_1^{*q} + x_2^{*q}) \le 2(w_1 x_1^{*q} + w_2 x_2^{*q})$$

for any $x = (x_1, x_2) \in \mathbb{R}^2$ or, equivalently,

$$w_1 x_2^{*q} + w_2 x_1^{*q} \le w_1 x_1^{*q} + w_2 x_2^{*q},$$

which is true by the fact of the Hardy–Littlewood type; that is, we have

$$\sum_{k=1}^{2} |u_k v_k| \le \sum_{k=1}^{2} u_k^* v_k^*$$

for any $u_1, u_2, v_1, v_2 \in \mathbb{R}$. The second inequality in (2.3) follows immediately from the assumption $w_1 \ge w_2$. ∎

Proof of Theorem 1. Using Lemma 1, we have

$$\begin{aligned}
\|x+y\|_{w,q}^2 + \|x-y\|_{w,q}^2 &\le w_1^{2/q}[\|x+y\|_q^2 + \|x-y\|_q^2] \\
&\le w_1^{2/q} C_{NJ}(\ell_q^{(2)}) 2(\|x\|_q^2 + \|y\|_q^2) \\
&\le C_{NJ}(\ell_q^{(2)}) 2\left\{\left(\frac{2w_1}{w_1+w_2}\right)^{2/q} \|x\|_{w,q}^2 \right. \\
&\qquad \left. + \left(\frac{2w_1}{w_1+w_2}\right)^{2/q} \|y\|_{w,q}^2\right\} \\
&= 2^{\frac{2}{\min\{q,q'\}}-1}\left(\frac{2w_1}{w_1+w_2}\right)^{2/q} \cdot 2(\|x\|_{w,q}^2 + \|y\|_{w,q}^2),
\end{aligned}$$

which for $q \ge 2$, gives,

$$C_{NJ}(d^{(2)}(w,q)) \le 2^{2/q'-1} 2^{2/q}\left(\frac{w_1}{w_1+w_2}\right)^{2/q} = 2\left(\frac{w_1}{w_1+w_2}\right)^{2/q}.$$

On the other hand, let $\alpha > 0$ be a constant such that $\|(\alpha, \alpha)\|_{w,q} = 1$; that is, $\alpha = 1/(w_1 + w_2)^{1/q}$. For $x_0 = (\alpha, \alpha)$ and $y_0 = (\alpha, -\alpha)$, we have

$$\begin{aligned}
C_{NJ}(d^{(2)}(w,q)) &\ge \frac{\|x_0+y_0\|_{w,q}^2 + \|x_0-y_0\|_{w,q}^2}{2(\|x_0\|_{w,q}^2 + \|y_0\|_{w,q}^2)} \\
&= \frac{(2\alpha w_1^{1/q})^2 + (2\alpha w_1^{1/q})^2}{4} = 2\alpha^2 w_1^{2/q} = 2\left(\frac{w_1}{w_1+w_2}\right)^{2/q}.
\end{aligned}$$

Also,

$$J(d^{(2)}(w,q)) \ge 2\alpha w_1^{1/q} = 2\left(\frac{w_1}{w_1+w_2}\right)^{1/q},$$

and by the first estimate in (xi) we obtain

$$J(X) \le \sqrt{2C_{NJ}(X)} = \sqrt{4\alpha^2 w_1^{2/q}} = 2\alpha w_1^{1/q} = 2\left(\frac{w_1}{w_1+w_2}\right)^{1/q},$$

and the proof is complete. ∎

Remark 1. The foregoing proofs show also that for $1 \le q < \infty$, we have the following estimates for the n-dimensional Lorentz spaces $d^{(n)}(w, q)$. Let $W_n = w_1 + w_2 + \cdots + w_n$. Then

$$\left(\frac{W_n}{n}\right)^{1/q} \|x\|_q \le \|x\|_{w,q} \le w_1^{1/q} \|x\|_q$$

for any $x \in \mathbb{R}^n$, and so $J(d^{(n)}(w, q)) \le \max\{2^{1/q}, 2^{1-1/q}\}(nw_1/W_n)^{1/q}$ and $C_{NJ}(d^{(n)}(w, q)) \le 2^{2/r-1}(nw_1/W_n)^{2/q}$, where $r = \min\{q, q'\}$.

COROLLARY 1. *If* $p \ge q \ge 2$, *then*

$$J(\ell^2_{p,q}) = \frac{2}{(1 + 2^{q/p-1})^{1/q}} \tag{2.4}$$

and

$$C_{NJ}(\ell^2_{p,q}) = \frac{2}{(1 + 2^{q/p-1})^{2/q}}. \tag{2.5}$$

In the case where $q = 2$, the foregoing equalities were found in [7] (cf. also [11], where (2.5) is calculated for the case $q = 2$).

3. LORENTZ SEQUENCE SPACES $d^{(2)}(w, q)$, FOR THE CASE WHERE $1 \le q < 2$

In the case where $1 \le q < 2$, we calculate precisely only the Jordan–von Neumann constant.

THEOREM 2. *If* $1 \le q < 2$, *then*

$$C_{NJ}(d^{(2)}(w,q)) = \left[w_1^{2/(2-q)} + w_2^{2/(2-q)}\right]^{2/q-1} \max\left\{\frac{2}{(w_1+w_2)^{2/q}}, \frac{1}{w_1^{2/q}}\right\}. \tag{3.1}$$

In the proof we will need the following lemma.

LEMMA 2. *(a) If* $2 \le q < \infty$, *then*

$$\frac{(w_1 + w_2)^{1/q}}{\sqrt{2}} \|x\|_2 \le \|x\|_{w,q} \le w_1^{1/q} \|x\|_2. \tag{3.2}$$

(b) If $1 \le q < 2$, *then*

$$a\|x\|_2 \le \|x\|_{w,q} \le b\|x\|_2, \tag{3.3}$$

where

$$a = \min\left\{ \frac{(w_1 + w_2)^{1/q}}{\sqrt{2}},\ w_1^{1/q} \right\} \textit{ and } b = \left[w_1^{2/(2-q)} + w_2^{2/(2-q)} \right]^{1/q-1/2}.$$

Proof. To show all of the foregoing estimates, we must calculate the supremum,

$$A := \sup\{(u^2 + v^2)^{1/2} :\ (w_1 u^q + w_2 v^q)^{1/q} = 1 \text{ and } u \ge v > 0\}.$$

Taking $u = \lambda v$, we obtain

$$\begin{aligned} A &= \sup\{(\lambda^2 + 1)^{1/2} v :\ (w_1 \lambda^q + w_2)^{1/q} v = 1 \text{ and } \lambda \ge 1\} \\ &= \sup\{(\lambda^2 + 1)^{1/2} / (w_1 \lambda^q + w_2)^{1/q} :\ \lambda \ge 1\}. \end{aligned}$$

The function $f(\lambda) = (\lambda^2 + 1)^{1/2} / (w_1 \lambda^q + w_2)^{1/q}$ has the derivative

$$\begin{aligned} f'(\lambda) &= \big[\lambda(\lambda^2+1)^{-1/2}(w_1\lambda^q + w_2)^{1/q} \\ &\qquad - w_1 \lambda^{q-1}(w_1\lambda^q + w_2)^{1/q-1}(\lambda^2+1)^{1/2}\big](w_1\lambda^q + w_2)^{-2/q} \\ &= \big[\lambda w_2 - \lambda^{q-1} w_1\big](w_1\lambda^q + w_2)^{1/q-1}(\lambda^2+1)^{-1/2}(w_1\lambda^q + w_2)^{-2/q}. \end{aligned}$$

Now in the case of $2 \le q < \infty$, we have that $f(\lambda)$ is decreasing on $[1, \infty)$ and that

$$\inf_{x \ne 0} \frac{\|x\|_{w,q}}{\|x\|_2} = \frac{1}{A} = \frac{1}{f(1)} = \frac{(w_1 + w_2)^{1/q}}{\sqrt{2}}$$

and

$$\sup_{x \ne 0} \frac{\|x\|_{w,q}}{\|x\|_2} = \frac{1}{\inf\{f(\lambda) :\ \lambda \ge 1\}} = \frac{1}{f(\infty)} = w_1^{1/q}.$$

Let $1 \le q \le 2$. Assume that $w_1 > w_2$, because otherwise the estimates in (3.3) are clear. Then $f(\lambda)$ is decreasing on $[1, \lambda_0]$ and increasing on $[\lambda_0, \infty)$, where $\lambda_0 = (w_1/w_2)^{1/(2-q)} > 1$. Therefore,

$$\begin{aligned} \inf_{x \ne 0} \frac{\|x\|_{w,q}}{\|x\|_2} &= \frac{1}{A} = \frac{1}{\max\{f(1), f(\infty)\}} \\ &= \min\left\{ \frac{(w_1 + w_2)^{1/q}}{\sqrt{2}}, w_1^{1/q} \right\} = a \end{aligned}$$

and

$$\sup_{x\neq 0}\frac{\|x\|_{w,q}}{\|x\|_2} = \frac{1}{\inf\{f(\lambda):\ \lambda\geq 1\}} = \frac{1}{f(\lambda_0)}$$
$$= [w_1(w_1/w_2)^{q/(2-q)} + w_2]^{1/q}/[(w_1/w_2)^{2/(2-q)}+1]^{1/2} = b. \quad \blacksquare$$

Proof of Theorem 2. Using Lemma 2(b), we have

$$\|x+y\|^2_{w,q} + \|x-y\|^2_{w,q} \leq b^2\big[\|x+y\|^2_2 + \|x-y\|^2_2\big]$$
$$= b^2(\|x\|^2_2 + \|y\|^2_2) \leq b^2 2(\|x\|^2_{w,q} + \|y\|^2_{w,q})/a^2$$

or

$$C_{NJ}(d^{(2)}(w,q)) \leq \frac{b^2}{a^2}.$$

On the other hand, if $x_0 = \big(w_1^{1/(2-q)}, 0\big)$ and $y_0 = \big(0, w_2^{1/(2-q)}\big)$, then

$$C_{NJ}(d^{(2)}(w,q)) \geq \frac{\|x_0+y_0\|^2_{w,q} + \|x_0-y_0\|^2_{w,q}}{2(\|x_0\|^2_{w,q} + \|y_0\|^2_{w,q})}$$
$$= \frac{2\big(w_1 w_1^{q/(2-q)} + w_2 w_2^{q/(2-q)}\big)^{2/q}}{2[\big(w_1 w_1^{q/(2-q)}\big)^{2/q} + \big(w_1 w_2^{q/(2-q)}\big)^{2/q}]}$$
$$= \frac{\big(w_1^{2/(2-q)} + w_2^{2/(2-q)}\big)^{2/q}}{\big(w_1^{2/(2-q)} + w_2^{2/(2-q)}\big)w_1^{2/q}}$$
$$= \big[w_1^{2/(2-q)} + w_2^{2/(2-q)}\big]^{2/q-1}\frac{1}{w_1^{2/q}} = b^2\frac{1}{w_1^{2/q}}.$$

Also, if we take two points $x_1 = \big(w_1^{1/(2-q)} + w_2^{1/(2-q)}\big)(1,1)$ and $y_1 = \big(w_1^{1/(2-q)} - W_2^{1/(2-q)}\big)(1,-1)$, then $x_1 + y_1 = 2\big(w_1^{1/(2-q)}, w_2^{1/(2-q)}\big)$ and $x_1 - y_1 = 2\big(w_2^{1/(2-q)}, w_1^{1/(2-q)}\big)$, and so

$$C_{N,J}(d^{(2)}(w,q))$$
$$\geq \frac{4\big[w_1 w_1^{q/(2-q)} + w_2 w_2^{q/(2-q)}\big]^{2/q}}{(w_1+w_2)^{2/q}\big[\big(w_1^{1/(2-q)} + w_2^{1/(2-q)}\big)^2 + \big(w_1^{1/(2-q)} - w_2^{1/(2-q)}\big)^2\big]}$$
$$= \frac{2\big(w_1^{2/(2-q)} + w_2^{2/(2-q)}\big)^{2/q-1}}{(w_1+w_2)^{2/q}} = b^2\frac{2}{(w_1+w_2)^{2/q}}.$$

The last two estimates from below show that we have equality in (3.1), and the proof is complete. $\blacksquare$

Remark 2. (a) Estimates (3.2) in Lemma 2(a) also give the result in Theorem 1.

(b) Property (xi) and equality (2.2) give the estimate $J(d^{(2)}(w,q)) \leq \sqrt{2C_{NJ}(d^{(2)}(w,q))} = \sqrt{2}b/a$, but we do not know whether the equality holds here. Note also that

$$2\left(\frac{w_1}{w_1+w_2}\right)^{1/q} \leq J(d^{(2)}(w,q)) \leq 2^{2/q}\left(\frac{w_1}{w_1+w_2}\right)^{1/q}$$

and

$$2\left(\frac{w_1}{w_1+w_2}\right)^{2/q} \leq C_{NJ}(d^{(2)}(w,q)) \leq 2^{4/q-1}\left(\frac{w_1}{w_1+w_2}\right)^{2/q}.$$

COROLLARY 2. *If $w_1 = 1$ and $w_2 = 2^{q/p-1}$ with $p \geq q$ and $1 \leq q < 2$, then*

$$C_{NJ}(\ell^2_{p,q}) = [1 + 2^{2(q-p)/p(2-q)}]^{2/q-1}.$$

In particular, $C_{NJ}(\ell^2_{p,1}) = 1 + 4^{1/p-1}$.

COROLLARY 3. *If $q = 1$, then*

$$C_{NJ}(d^{(2)}(w,1)) = \max\left\{\frac{2(w_1^2+w_2^2)}{(w_1+w_2)^2}, \frac{w_1^2+w_2^2}{w_1^2}\right\}.$$

PROBLEM 1. Compute $J(X)$ and $J(X^*)$ for $X = d^{(2)}(w,q)$ when $1 \leq q < 2$.

Note that the dual norm of $d^{(2)}(w,q)$ is not known for $q > 1$ and that for $q = 1$ we have that the dual space to the two-dimensional Lorentz space $d^{(2)}(w,1)$ is a two-dimensional Marcinkiewicz space m_w given by the norm

$$\|x\|_{m_w} = \max\left\{\frac{x_1^*}{w_1}, \frac{x_1^*+x_2^*}{w_1+w_2}\right\}.$$

PROBLEM 2. Compute $J(X)$, $J(X^*)$ and $C_{NJ}(X)$ for the n-dimensional Lorentz sequence spaces $X = d^{(n)}(w,q)$ when $n \geq 3$, and for the infinite-dimensional Lorentz sequence spaces $X = d(w,q)$ (see [9] for the definition).

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