Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces

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Abstract
In this paper, first a new fixed point theorem is established, and then, by the use of it, the existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces are investigated. The results obtained in this paper generalize and improve the results corresponding to those obtained by others.

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1. Introduction

Initial value problems for nonlinear integro-differential equations arise from many nonlinear problems in science (see [6]). Over the last couple of decades, many attempts have been made to study the existence of solutions of first-order or second-order initial value problems with or without impulses in abstract spaces [2,3,6–10]. Extensive studies have also been carried out to study the global or iterative solutions of initial value problems.
In particular, in [2], Guo studied the global solutions of the following initial value problem (IVP) for first-order nonlinear integro-differential equations of mixed type in a real Banach space $E$:

\[
\begin{align*}
    u' &= f(t, u, Tu, Su), \quad t \in J, \\
    u(0) &= u_0,
\end{align*}
\]  

(1.1)

where

\[
    J = [0, a], \quad u_0 \in E, \quad f \in C(J \times E \times E \times E, E),
\]

in which

\[
    \begin{align*}
        Tu(t) &= \int_0^t k(t, s)u(s)\,ds, \\
        Su(t) &= \int_0^t h(t, s)u(s)\,ds, \quad t \in J,
    \end{align*}
\]

(1.2)

where $k \in C(D, \mathbb{R})$, $h \in C(D_0, \mathbb{R})$, $\mathbb{R}$ denotes the set of real numbers, $D = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq a\}$, $D_0 = \{(t, s) \in \mathbb{R}^2 : 0 \leq t, s \leq a\}$.

Guo obtained the following existence theorem of global solutions of IVP (1.1) by Darbo’s fixed point theorem.

**Theorem** (Guo). Let $E$ be a real Banach space. For any $R > 0$, $f$ is uniformly continuous on $J \times T_R \times T_R \times T_R$. Suppose that there exist constants $L_i \geq 0$ ($i = 1, 2, 3$) such that

\[
    2a(L_1 + ak_0L_2 + ah_0cL_3) < 1,
\]

(1.3)

and for any bounded sets $U, V, W \subset E$, $t \in J$ such that

\[
    \alpha(f(t, U, V, W)) \leq L_1\alpha(U) + L_2\alpha(V) + L_3\alpha(W).
\]

Suppose, further, that

\[
    \limsup_{R \to \infty} \frac{M(R)}{R} < \frac{1}{aa_0},
\]

where

\[
    M(R) = \sup \{\|f(t,x,y,z)\| : (t,x,y,z) \in J \times T_R \times T_R \times T_R, t \in J\},
\]

$T_R = \{x \in E : \|x\| \leq R\}$, $a_0 = \max\{1, ah_0, ak_0\}$, $k_0 = \max\{|k(t,s)| : (t,s) \in D\}$, $h_0 = \max\{|h(t,s)| : (t,s) \in D_0\}$.

Then IVP (1.1) has at least one global solution in $C^1(J, E)$.

Recently, Liu [8] established the existence theorem of global solutions of IVP (1.1) by Mönch’s fixed point theorem (see [1,4]) and a new established comparison result, which improves the results in [2]. More recently, Liu, Wu and Guo [10] generalized and improved the results of [2,8] ulteriorly. In all the above mentioned studies, the conditions of the Kuratowski measure of noncompactness play an important role in the proof of the main results. However, on the other hand, it is difficult to establish a comparison result for higher order differential equations. Thus, the problem of searching global solutions for (1.1) is an interesting and important question. In order to study IVP (1.1), we consider the following Volterra type integral equation:

\[
    u(t) = h(t) + \int_0^t G(t,s)f\left(s, u(s), Tu(s), Su(s)\right)\,ds,
\]

(1.4)
where \( h \in C[J, E] \), \( G \in C[D, R] \), operators \( T \) and \( S \) are defined by (1.2).

In this paper, we will use utterly different method from [2,4,8,10] to investigate the global solutions of Volterra type integral equation (1.4). We first establish a new fixed point theorem, which is a generalization of the well-known Darbo's fixed point theorem. And then by making use of it, the existence of global solutions of Eq. (1.4) is investigated under much weaker conditions compared with those in [2,4,8,10]. The results obtained in this paper generalize and improve the related results in [2,4,8,10].

As applications of our results, we get the global solutions of IVP (1.1) which improves the results in [2,4,8,10], and under the proper conditions we obtain the global solutions in \( C^3[J, E] \) of the following two classes of third-order mixed boundary value problem (MBVP):

\[
\begin{aligned}
&x'''(t) = f(t, x'', x), \quad 0 \leq t \leq 1, \\
&\alpha_1 x(0) + \alpha_2 x'(0) = \theta, \\
&\beta_1 x(1) + \beta_2 x'(1) = \theta, \\
&x''(0) = x_0, \\
&x'''(t) = f(t, x'', x', x), \quad 0 \leq t \leq 1, \\
&x'(0) = \theta, \\
&\beta_1 x(1) + \beta_2 x'(1) = \theta, \\
&x''(0) = x_0. 
\end{aligned}
\]

This paper is organized as follows. In Section 2, we give some preliminaries and establish several lemmas, and the main theorems are formulated and proved in Section 3. Finally, in Section 4, the global solutions of IVP (1.1) are obtained, and the existence theorem of global solutions of two classes of third-order mixed boundary value problem are proved.

2. Preliminaries and lemmas

Set

\[
C[J, E] = \{u: J \to E \text{ is continuous}\},
\]

\[
C^1[J, E] = \{u: J \to E \text{ is continuously differentiable}\}.\]

It is easy to see \( C[J, E] \) and \( C^1[J, E] \) are Banach spaces with norms

\[
\|u\|_C = \max \{\|u(t)\|: t \in J\}, \quad \|u\|_{C^1} = \max \{\|u\|_C, \|u'\|_C\},
\]

respectively. Without confusion, let \( \alpha \) denotes the Kuratowski measure of noncompactness in \( E \) and \( C[J, E] \), the properties of which may be found in [1,3–5]. For any \( B \subset C[J, E] \), \( t \in J \), let \( B(t) = \{u(t): u \in B\} \subset E \). For any \( R > 0 \), let \( B_R = \{u \in C[J, E]: \|u\|_C \leq R\} \).

**Lemma 2.1.** Let \( B \subset C[J, E] \) be bounded and equicontinuous, then \( \overline{\bigcup B} \subset C[J, E] \) is also bounded and equicontinuous.
Lemma 2.1 is quite well-known and its proof is standard and trivial. So its proof is omitted here.

**Lemma 2.2** [4]. Let $B \subset C[J, E]$ be bounded and equicontinuous. Then $m(t) = \alpha(B(t))$ is continuous on $J$ and

$$\alpha\left( \int_J B(s) \, ds \right) \leq \int_J \alpha(B(s)) \, ds.$$ 

**Lemma 2.3** [9]. Assume for all $R > 0$, $f$ is bounded and uniformly continuous on $J \times T_R \times T_R \times T_R \times T_R$, $B \subset C[J, E]$ is bounded and equicontinuous, then $\{f(t, u(t), (Tu)(t), (Su)(t)) : u \in B\}$ is bounded and equicontinuous in $C[J, E]$.

**Lemma 2.4** (Fixed Point Theorem). Let $F$ be a closed and convex subset of a real Banach space $E$, operator $A : F \to F$ be continuous and $A(F)$ be bounded. For any bounded subset $B \subset F$, set

$$\hat{A}^1(B) = A(B), \quad \hat{A}^n(B) = A(\hat{A}^{n-1}(B)), \quad n = 2, 3, \ldots.$$ (2.1)

If there exist a constant $0 \leq k < 1$ and a positive integer $n_0$ such that for any bounded subset $B \subset F$

$$\alpha(\hat{A}^{n_0}(B)) \leq k \alpha(B),$$ (2.2)

then $A$ has a fixed point in $F$.

**Proof.** Let

$$D_0 = F, \quad D_1 = \hat{A}^{n_0}(D_0), \quad \ldots, \quad D_n = \hat{A}^{n_0}(D_{n-1}), \quad n = 2, 3, \ldots.$$ 

We shall prove

(i) $D_0 \supset D_1 \supset D_2 \supset \cdots \supset D_{n-1} \supset D_n \supset \cdots$;

(ii) $\lim_{n \to \infty} \alpha(D_n) = 0$.

By (2.1), we know $D_1 = \hat{A}^{n_0}(D_0) \subset F = D_0$, hence $\hat{A}^{n_0}(D_1) \subset \hat{A}^{n_0}(D_0)$, therefore $D_2 = \hat{A}^{n_0}(D_1) \subset \hat{A}^{n_0}(D_0) = D_1$. By the method of mathematical induction, we can prove $D_n \subset D_{n-1} (n = 1, 2, \ldots)$. Thus, we have proved (i). By (2.2), we have

$$\alpha(D_n) = \alpha(\hat{A}^{n_0}(D_{n-1})) \leq k \alpha(D_{n-1}) = k \alpha(\hat{A}^{n_0}(D_{n-2}))$$

$$\leq k^2 \alpha(D_{n-2}) \leq \cdots \leq k^{n-1} \alpha(D_1) \to 0, \quad \text{as } n \to \infty.$$ 

Hence, (ii) is true. From (i) and (ii), it follows by using [1, Exercise 7.4] that $\hat{D} = \bigcap_{n=0}^{\infty} D_n$ is a nonempty, compact and convex subset in $F$.

Now, we prove $AD \subset \hat{D}$. First, we prove

$$AD_n \subset D_n, \quad n = 0, 1, 2, \ldots.$$ (2.3)

From $\hat{A}^1(D_0) = A(D_0) \subset D_0$, we know $\hat{A}^1(D_0) \subset D_0$. Therefore
\[ \tilde{A}^2(D_0) = A(\tilde{\mathcal{O}}(\tilde{A}^1(D_0))) \subset A(D_0) = \tilde{A}^1(D_0), \]
\[ \tilde{A}^3(D_0) = A(\tilde{\mathcal{O}}(\tilde{A}^2(D_0))) \subset A(\tilde{\mathcal{O}}(\tilde{A}^1(D_0))) = \tilde{A}^2(D_0), \]
\[ \vdots \]
\[ \tilde{A}^{n_0}(D_0) = A(\tilde{\mathcal{O}}(\tilde{A}^{n_0-1}(D_0))) \subset A(\tilde{\mathcal{O}}(\tilde{A}^{n_0-2}(D_0))) = \tilde{A}^{n_0-1}(D_0). \]

Hence, \( D_1 = \mathcal{O}(\tilde{A}^{n_0}(D_0)) \subset \mathcal{O}(\tilde{A}^{n_0-1}(D_0)), \) so \( A(D_1) \subset A(\mathcal{O}(\tilde{A}^{n_0-1}(D_0))) = \tilde{A}^{n_0}(D_0) \subset \mathcal{O}(\tilde{A}^{n_0-2}(D_0)) = D_1. \) Employing the same method, we can prove \( A(D_n) \subset D_n \) \((n = 1, 2, \ldots)\). By (2.3), we get
\[ A(D_1) \subset A(\mathcal{O}(\tilde{A}^{n_0-2}(D_0))) = \tilde{A}^{n_0-1}(D_0). \]

It follows by using Schauder’s fixed point theorem that \( A \) has a fixed point in \( \check{D} \subset F \). Thus, we complete the proof. \( \square \)

Remark 2.1. Our new fixed point theorem is the generalization of the well-known Darbo’s fixed point theorem, since when \( n_0 = 1 \), Lemma 2.4 is the latter. In Section 3, we will give its application.

3. Main results

In this section, we give the existence theorem of global solutions for Eq. (1.4).

Theorem 3.1. Let \( E \) be a real Banach space. Assume

\((H_1)\) For any \( R > 0 \), \( f \) is bounded and uniformly continuous on \( J \times T_R \times T_R \times T_R \), and
\[ \limsup_{R \to \infty} \frac{M(R)}{R} < \frac{1}{a_0 b^2}, \]

where \( a_0 = \max\{1, a_0k, ah_0\} \), \( b = \max\{G(t, s) \mid (t, s) \in D\} \), \( M(R) \) is defined as in Theorem (Guo).

\((H_2)\) There exist nonnegative Lebesgue integrable functions \( L_i \in L(J, R^+ \} (i = 1, 2, 3) \) such that for any bounded sets \( D_i \subset E (i = 1, 2, 3) \) and \( t \in J \),
\[ \alpha\left(f(t, D_1, D_2, D_3)\right) \leq \sum_{i=1}^{3} L_i(t) \alpha(D_i). \]

Then Eq. (1.4) has at least one global solution in \( C[J, E] \).

Proof. Define operator \( A : C[J, E] \to C[J, E] \) by
\[ Au(t) = h(t) + \int_{0}^{t} G(t, s)f(s, u(s), (Tu)(s), (Su)(s))ds, \quad u \in C[J, E]. \]

Since \( f \) is uniformly continuous on \( J \times T_R \times T_R \times T_R \), we can see easily \( A : C[J, E] \to C[J, E] \) is continuous and bounded. On account of (3.1), there exist \( 0 < r < (a_0 b)^{-1} \) and \( R_0 > 0 \) such that for any \( R \geq a_0 R_0, \)
\[ M(R) < r R. \]
Let $R^* = \max\{R_0, \|h\|C(1 - a_0 b)^{-1}\}$. For any $u \in B_{R^*}$, we have
\[
\|u\|_C \leq R^* \leq a_0 R^*,
\]
\[
\|Tu\|_C \leq a_k \|u\|_C \leq a_k R^* \leq a_0 R^*,
\]
\[
\|Su\|_C \leq a_h \|u\|_C \leq a_h R^* \leq a_0 R^*.
\]
Thus, from (3.4) we have
\[
\|Au\|_C \leq \|h\|C + abM(a_0 R^*) \leq \|h\|C + a_0 a_0 b R^* \leq R^*,
\]
which implies $A : B_{R^*} \to B_{R^*}$ is continuous and bounded operator.

We will prove $A(B_{R^*}) \subset C[J, E]$ is equicontinuous. For any $u \in B_{R^*}$, $t_1, t_2 \in J (t_1 < t_2)$, by (3.3) and (3.4) we have
\[
\|(Au)(t_1) - (Au)(t_2)\| \\
\leq \|h(t_1) - h(t_2)\| + \int_0^{t_1} G(t_1, s) f(s, u(s), (Tu)(s), (Su)(s)) \, ds \\
- \int_0^{t_2} G(t_2, s) f(s, u(s), (Tu)(s), (Su)(s)) \, ds \\
\leq \|h(t_1) - h(t_2)\| + \int_0^{t_1} (G(t_1, s) - g(t_2, s)) f(s, u(s), (Tu)(s), (Su)(s)) \, ds \\
+ \int_0^{t_2} G(t_2, s) f(s, u(s), (Tu)(s), (Su)(s)) \, ds \\
\leq \|h(t_1) - h(t_2)\| + \int_0^{t_1} |G(t_1, s) - g(t_2, s)| M(a_0 R^*) \, ds + b M(a_0 R^*) |t_1 - t_2|.
\]
From the continuity of $h(t)$ and $G(t, s)$, we implies $A(B_{R^*})$ is equiuniform.

Let $F = \overline{A}(B_{R^*})$. Then by Lemma 2.1, $F \subset B_{R^*}$ is bounded and equicontinuous, and $A : F \to F$ is continuous and bounded operator. For any $B \subset F$, by the definition of operator $A$ and $f$ is bounded and uniformly continuous, we get $A(B)$ is bounded and equicontinuous on $J$. Hence, we know from Lemmas 2.1, 2.3 and (2.1), $\tilde{A}^n(B)$ ($n = 1, 2, \ldots$) is bounded and equiuniform, and so
\[
\alpha(\tilde{A}^n(B)) = \max_{t \in J} \alpha((\tilde{A}^n(B))(t)), \quad n = 1, 2, \ldots.
\]
We shall prove there exist a constant $0 < k < 1$ and a positive integer $n_0$ such that for any $B \subset F$,
\[
\alpha(\tilde{A}^{n_0}(B)) \leq k \alpha(B).
\]
It follows from (3.2) that
\[
\alpha((\tilde{A}^1(B))(t)) = \alpha((A(B))(t)) \\
= \alpha\left(\int_0^t G(t, s) f(s, B(s), (TB)(s), (SB)(s)) \, ds\right) \\
\leq b \int_0^t \alpha\left(f(s, B(s), (TB)(s), (SB)(s))\right) \, ds \\
\leq b \int_0^t \left[L_1(s)\alpha(B(s)) + L_2(s)\alpha(TB(s)) + L_3(s)\alpha(SB(s))\right] \, ds \\
\leq b \int_0^t \left[L_1(s) + ak_0L_2(s) + ah_0L_3(s)\right]\alpha(B) \, ds \\
= \int_0^t L(s) \, ds \, \alpha(B), \quad t \in J, \quad (3.7)
\]

where \(L(t) = b[L_1(t) + ak_0L_2(t) + ah_0L_3(t)]\) is Lebesgue integrable functions in \(J\). We know there is a continuous function \(\phi: J \to \mathbb{R}^1\) such that for any \(\varepsilon > 0\) (\(\varepsilon < 1\)),

\[
\int_0^a |L(s) - \phi(s)| \, ds < \varepsilon. \quad (3.8)
\]

It follows from (3.7) and (3.8) that

\[
\alpha((\tilde{A}^k(B))(t)) \leq \int_0^t \left[L(s) - \phi(s)\right] \, ds + \int_0^t \phi(s) \, ds \right] \alpha(B) \leq (\varepsilon + Mt)\alpha(B),
\]

where \(\varepsilon = \max(|\phi(t)|: t \in J)\). Suppose

\[
\alpha(\tilde{A}^k(B)(t)) \leq \left[\varepsilon^k + C_k \varepsilon^{k-1} (Mt)^2 + \frac{C_k^2 \varepsilon^{k-2} (Mt)^2}{2!} + \cdots + \frac{(Mt)^k}{k!}\right] \alpha(B), \quad t \in J,
\]

then for any \(t \in J\),

\[
\alpha(\tilde{A}^{k+1}(B)(t)) \\
= \alpha\left(\int_0^t G(t, s) f(s, (\tilde{A}^k(B))(s), (TB)(s), (SB)(s)) \, ds\right) \\
\leq b \int_0^t \alpha\left(f(s, (\tilde{A}^k(B))(s), (TB)(s), (SB)(s))\right) \, ds
\]
\[ \leq b \int_0^t \left[ L_1(s) + ak_0L_2(s) + ah_0L_3(s) \right] \alpha((\tilde{A}^k(B))(s)) \, ds \]
\[ = \int_0^t L(s) \alpha((\tilde{A}^k(B))(s)) \, ds \]
\[ \leq \int_0^t \left[ |L(s) - \phi(s)| + |\phi(s)| \right] \alpha((\tilde{A}^k(B))(s)) \, ds \]
\[ \leq \varepsilon \left[ \varepsilon^k + C_1^k \varepsilon^{k-1}(Mt) + \frac{C_2^k \varepsilon^{k-2}(Mt)^2}{2!} + \cdots + \frac{(Mt)^k}{k!} \right] \alpha(B) \]
\[ + M \int_0^t \left[ \varepsilon^k + C_1^k \varepsilon^{k-1}(Ms) + \frac{C_2^k \varepsilon^{k-2}(Ms)^2}{2!} + \cdots + \frac{(Ms)^k}{k!} \right] \alpha(B) \, ds \]
\[ = \left[ \varepsilon^{k+1} + C_1^k \varepsilon^k (Mt) + \frac{C_2^k \varepsilon^{k-1}(Mt)^2}{2!} + \cdots + \frac{\varepsilon(Mt)^k}{k!} \right] \alpha(B) \]
\[ + \varepsilon^k (Mt) + \frac{C_1^k \varepsilon^{k-1}(Mt)^2}{2!} + \frac{C_2^k \varepsilon^{k-2}(Mt)^3}{3!} + \cdots + \frac{(Mt)^{k+1}}{(k+1)!} \alpha(B) \]
\[ = \left[ \varepsilon^{k+1} + C_1^k \varepsilon^k (Mt) + \frac{C_2^k \varepsilon^{n-1}(Mt)^2}{2!} + \cdots + \frac{(Mt)^n}{n!} \right] \alpha(B). \]

Hence, by the method of mathematical induction, for any positive integer \( n \) and \( t \in J \), we obtain
\[ \alpha(\tilde{A}^n(B)(t)) \leq \left[ \varepsilon^n + C_1^k \varepsilon^{n-1}(Mt) + \frac{C_2^k \varepsilon^{n-2}(Mt)^2}{2!} + \cdots + \frac{(Mt)^n}{n!} \right] \alpha(B). \]

Therefore, by (3.5), for any positive integer \( n \) we have
\[ \alpha(\tilde{A}^n(B)) \leq \left[ \varepsilon^n + C_1^k \varepsilon^{n-1}h + \frac{C_2^k \varepsilon^{n-2}h^2}{2!} + \cdots + \frac{h^n}{n!} \right] \alpha(B), \quad (3.9) \]
where \( h = Ma \). We know easily
\[ \lim_{n \to \infty} \left[ \varepsilon^n - n \left( \frac{n}{n - 1} \right)^{n-1} \right]^{1/n} = \varepsilon < 1. \]

Hence there exists a positive integer \( n_1 > 2 \) such that
\[ \left[ \varepsilon^{n_1-1}n_1 \left( \frac{n_1}{n_1 - 1} \right)^{n_1-1} \right]^{1/n_1} = \varepsilon < 1. \quad (3.10) \]

For any positive integer \( n \), let \( n = mn_1 + j \) (\( 0 \leq j < n_1 \)), where \( n_1 \) is given by (3.10). Then for any sufficiently large positive integer \( n > n_1 \), it follows from the Stirling formula and (3.10) that
\[ S_1 = \varepsilon^n + C_n^1 \varepsilon^{n-1}h + \frac{1}{2!} C_n^2 \varepsilon^{n-2}h^2 + \cdots + \frac{1}{m!} C_n^m \varepsilon^{n-m}h^m \]
\[ \leq \varepsilon^n - mC_m \varepsilon^n \left( 1 + h + \frac{1}{2!} h^2 + \cdots + \frac{1}{m!} h^m \right) \]
\[ = O(1) \varepsilon^n - mC_m \varepsilon^n \sqrt{2\pi n \left( 1 + O(1) \right)} \]
\[ = O \left( \frac{n^m}{\sqrt{m}} \left( \frac{\varepsilon^n}{n-n} \right)^{m-m} \right) \]
\[ = O \left( \frac{\varepsilon^n}{\sqrt{n}} \right), \quad n \to \infty. \quad (3.11) \]

Similarly, we have
\[ S_2 = C_n^{m+1} \varepsilon^{n-m-1}h^{m+1} + \cdots + h^n \]
\[ \leq \frac{C_n^{m+1}}{(m+1)!} \left( \varepsilon^{n-m-1}h^{m+1} + \cdots + h^n \right) \]
\[ = O \left( \frac{\varepsilon^n}{\sqrt{n}} \right) \varepsilon^{m+1} \left( \varepsilon^{n-m-1}h^{m+1} + \cdots + h^n \right) \]
\[ = \sqrt{2\pi(m+1)(m+1)^{m+1}} \left( 1 + O(1) \right) \]
\[ = o \left( \frac{1}{n^s} \right), \quad n \to \infty, \quad \forall s > 1. \quad (3.12) \]

Hence for any \( s > 1 \), by (3.9), (3.11) and (3.12) we have
\[ \alpha(\tilde{A}^n(B)) \leq (S_1 + S_2)\alpha(B) = \left[ O \left( \frac{\varepsilon^n}{\sqrt{n}} \right) + o \left( \frac{1}{n^s} \right) \right] \alpha(B) \]
\[ = o \left( \frac{1}{n^s} \right) \alpha(B), \quad n \to \infty. \]

Thus there exist \( 0 \leq k < 1 \) and positive integer \( n_0 \) such that (3.6) holds. It follows from Lemma 2.4 that \( A \) has at least one fixed point in \( F \), i.e., Eq. (1.4) has at least one global solution \( u^* \) in \( F \subset C[J, E] \). Thus, the proof is completed. \( \square \)

**Remark 3.1.** Theorem 3.1 generalizes and improves the related results in [2,4,8,10].
4. Applications

It is well known that IVP (1.1) is equivalent to the following Volterra integral equation:

\[ u(t) = u_0 + \int_0^t f(s, u(s), Tu(s), Su(s)) \, ds. \]  (4.1)

In this case, \( G(t, s) \equiv 1 \), \( b = 1 \). So, from Theorem 3.1 we obtain the following theorem.

**Theorem 4.1.** Let \( E \) be a real Banach space. Assume

- \((H_1)\) For any \( R > 0 \), \( f \) is bounded and uniformly continuous on \( J \times T_R \times T_R \times T_R \), and

\[ \lim \sup_{R \to \infty} \frac{M(R)}{R} \leq \frac{1}{a a_0}, \]

where \( a_0 = \max\{1, a k_0, a h_0\} \), \( M(R) \) is defined as in Theorem (Guo).

- \((H_2)\) There exist nonnegative Lebesgue integrable functions \( L_i \in L(J, R^+) \) (\( i = 1, 2, 3 \)) such that for any bounded sets \( D_i \subset E \) (\( i = 1, 2, 3 \)) and \( t \in J \),

\[ \alpha \left( f(t, D_1, D_2, D_3) \right) \leq \sum_{i=1}^{3} L_i(t) \alpha(D_i). \]

Then Eq. (1.1) has at least one global solution in \( C^1[J, E] \).

**Remark 4.1.** Compared with Theorem (Guo), we did not require (1.3) at all and replace a constant \( L_i \) (\( i = 1, 2, 3 \)) with Lebesgue integrable functions \( L_i(t) \) (\( i = 1, 2, 3 \)) Generally speaking, the operator \( A \) defined by (3.3) is not a strict set contraction, therefore, the results of this paper cannot be obtained by the method used in [2], and our results also cannot be obtained by Darbo’s fixed point theorem. So, our results do improve main result in [2]. In the special case where the nonnegative integrable functions \( L_i \in L(J, R^+) \) (\( i = 1, 2, 3 \)) in Theorem 4.1 are nonnegative constants, we obtain [10, Theorem 3.1], hence our results also generalize and improve the main results in [10].

**Remark 4.2.** When studying IVP (1.1), Liu [8] requires the following strict compactness type condition:

\((C)\) For any countable equicontinuous bounded set \( B \subset C[J, E] \), and \( t \in J \),

\[ \alpha \left( f \left( t, B(t), (TB)(t), (SB)(t) \right) \right) \leq L_1 \alpha(B(t)) + L_2 \alpha((TB)(t)) + L_3 \alpha((SB)(t)), \]

where \( L_i \geq 0 \) (\( i = 1, 2, 3 \)) are constants satisfying one of the following two conditions:

- (a) \( a h_0 L_3 (e^{2a(L_1 + a k_0 L_2)} - 1) < L_1 + a k_0 L_2 \);
- (b) \( a (2L_1 + a k_0 L_2 + a h_0 L_3) < 1. \)

Obviously, we do not require (a) and (b) in our paper at all.
By using Theorem 4.1, we can obtain the global solutions of two classes of third-order mixed boundary value problems (I) and (II).

Firstly, consider the third-order MBVP (I) in a Banach space. Suppose
$$\alpha_1^2 + \alpha_2^2 > 0, \quad \beta_1^2 + \beta_2^2 > 0, \quad \Delta = \alpha_1 \beta_1 + \alpha_2 \beta_2 - \alpha_2 \beta_1 \neq 0, \quad f \in C[J \times E \times E].$$ Let \(x'' = u\), then we have
$$x(t) = \frac{1}{k(t)} \int_0^t h(t,s)u(s)ds \equiv (Su)(t), \quad t \in J = [0,1],$$
where \(h(t,s)\) is Green function of the following boundary value problem:
$$\begin{cases}
x'' = \theta, \\
\alpha_1 x(0) + \alpha_2 x'(0) = \theta, \\
\beta_1 x(1) + \beta_2 x'(1) = \theta.
\end{cases}$$

Then MBVP (I) can be regarded as an IVP of the following first order integro-differential equation:
$$\begin{cases}
u' = f(t,u,Su), \\
u(0) = x_0.
\end{cases}$$

For any \(R > 0\), let \(M_1(R) = \sup\{\|f(t,x,y)\| : (t,x,y) \in J \times TR \times TR\}\). The conclusion of the following Theorem 4.2 follows from Theorem 4.1.

**Theorem 4.2.** Let \(E\) be a real Banach space. Suppose that \(f \in C[J \times E \times E, E]\) satisfy the following conditions:

\((C_1)\) \(\limsup_{R \to \infty} \frac{M_1(R)}{R} < \frac{1}{\max\{1,\delta_0\}}\);

\((C_2)\) for any \(R > 0\), \(f\) is bounded and uniformly continuous on \(J \times TR \times TR\), and there exist nonnegative Lebesgue integrable functions \(L_i \in L(J, R^+)\) (\(i = 1,3\)) such that for any bounded subsets \(D_i \subset E\) (\(i = 1,3\)) and \(t \in J\),
$$\alpha (f(t,D_1,D_2)) \leq L_1(t)\alpha(D_1) + L_3(t)\alpha(D_3).$$

Then MBVP (I) has at least a global solution in \(C^3[J,E]\).

Secondly, consider the third-order MBVP (II) in a Banach space. Suppose \(\beta_1 \neq 0, \alpha_1 = 0, \alpha_2 = 1, \Delta = -\beta_1 \neq 0\). Let \(x'' = u\), then we have
$$\begin{cases}
x'(t) = \int_0^t u(s)ds \equiv (Tu)(t), \\
k(t,s) \equiv 1, \quad 0 \leq s \leq t \leq 1,
\end{cases}$$
$$x(t) = \frac{1}{k(t)} \int_0^1 h(t,s)u(s)ds \equiv (Su)(t), \quad t \in J,$$
where \( h(t,s) \) is the same as above. Then MBVP (II) can be regarded as an IVP of the following first-order integro-differential equation:

\[
\begin{cases}
  u'(t) = f(t, u, Tu, Su), & t \in J = [0, 1], \\
  u(0) = x_0.
\end{cases}
\]

The conclusion of the following Theorem 4.3 follows from Theorem 4.1.

**Theorem 4.3.** Let \( E \) be a real Banach space. If \( f \in C(J \times E \times E \times E, E) \) satisfy the conditions \((H_1)\) and \((H_2)\) of Theorem 4.1 for \( a_0 = \max\{1, h_0\}, a = 1 \), then MBVP (II) has at least a global solution in \( C^3[J, E] \).

**Remark 4.3.** In the special case where the nonnegative integrable functions \( L_i \in L(J, R^+) \) \((i = 1, 2, 3)\) in Theorems 4.2 and 4.3 are nonnegative constants, we obtain Theorems 4.1 and 4.2 in [10], respectively.

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**References**