JOURNAL OF COMPUTER AND SYSTEM SCIENCES 34, 179–209 (1987)

Parameter-Preserving Data Type Specifications

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Received August 1985; revised September 1986

Term rewriting methods are used for solving the persistency problem of parametrized data type specifications. Such a specification is called persistent if the parameter part of its algebraic semantics agrees with the semantics of the parameter specification. Since persistency mostly cannot be guaranteed for the whole equational variety of the parameter specification, the persistency criteria developed here mainly concern classes of parameter algebras with "built-in" logic. © 1987 Academic Press, Inc.

1. PERSISTENCY, EXTENSIONS, AND INDUCTIVE THEORIES

Starting from a many-sorted signature $\langle S, OP\rangle$ with sorts S and operation symbols OP an algebraic specification in the sense of ADJ [1] is given by a triple SPEC $= \langle S, OP, E\rangle$, where E is a set of equations between OP-terms. Algebras with signature $\langle S, OP\rangle$ which satisfy E are called SPEC-algebras. For reasons discussed extensively in the literature (e.g., in [1]) the isomorphism class of initial SPEC-algebras plays a dominant role.

A parameterized specification PAR is a pair of two specifications PSPEC and SPEC, where the parameter PSPEC is part of the target SPEC. The role of initial algebras is taken over by a class of target algebras each of which is "freely generated" over some algebra in a given class K of parameter algebras (cf. [2]).

Such a class of target algebras is called a parameterized data type. [10] deals with the proof-theoretical characterization of the equational variety of parameterized data types. This variety turned out to be a certain "inductive" theory of the target specification.

In many cases this characterization works only if PAR is persistent, i.e., if each algebra in the corresponding data type "preserves" the parameter algebra where it is "freely generated" upon. Persistency is also a sufficient criterion for the "passing compatibility" of PAR with actual parameter specifications (cf. ADJ [3]). So this paper is devoted to decidable and powerful criteria for persistency. The first step towards such conditions is the decomposition of PAR into a "base" specification BPAR and the remaining operations and equations of PAR. BPAR is supposed to contain those operations and equations of PAR that are necessary for the "contruction" of data. Following this strategy it is mostly simple to show that BPAR is per-

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sistent. Then PAR is persistent, too, if PAR is complete and consistent with respect to BPAR which means that the "base" part of the data type specified by PAR agrees with the data type specified by BPAR.

So the tools for solving the persistency problem are the criteria for persistency of BPAR given by Theorem 2.12 and the completeness and consistency conditions of Theorems 3.4 and 4.7 and 3.5 and 5.14, respectively. They involve normalization and confluence properties of term reductions and are tailor-made for parameter algebras with "built-in" logic upon which the proof-theoretical characterization of parameterized data types given in [10, Sect. 3] is also based.

Besides the well-known notions in term rewriting theory like "confluence" and "critical pair" we use some recently introduced ones like "coherence" (cf. [8]), "contextual reductions" (cf. [12]) and "recursive critical pairs" (cf. [11]). They should support the reader's intuition, although their definitions sometimes deviate from their meaning in the cited papers. Moreover, the corresponding results presented here are different from those given there.

The paper is organized as follows: Section 2 contains basic definitions and proof-theoretical characterizations of completeness, consistency, and persistency (2.12). In Section 3 general completeness and consistency theorems (3.4 and 3.5) are given, that refer to term reductions. Sections 4 and 5 focus on parameter algebras with "built-in" logic and adapt the notions of Section 3 to this case. Decidable criteria for the crucial confluence criteria of Theorem 3.5 are developed in Section 5, which culminates in the critical pair theorem 5.12. The main results of Sections 4 and 5 are summarized by the completeness theorem 4.7, the consistency theorem 5.14, and the persistency theorem 5.16.

Former versions of these results are part of the author's Ph.D. thesis [9]; id, inc, and nat denote identity, inclusion, and natural mappings, respectively. The first occurrences of notions used throughout the paper are printed in boldface.

2. The Syntax and Semantics of Parameterized Specifications

Let $\mathrm{SIG} = \langle S, OP \rangle$ be a many-sorted signature with a set S of sorts and an $(S^* \times S)$-sorted set OP of operation symbols. If $\sigma \in OP_{w,s}$, then $\mathrm{arity}(\sigma) = w$, $\mathrm{sort}(\sigma) = s$, and we often write σ: $w \to s$. If $w = \varepsilon$ (empty word), σ is called a constant. **T(SIG)** denotes the free S-sorted algebra of OP-terms over a fixed infinite S-sorted set X of variables. If $t, t' \in T(\mathrm{SIG})$ and $x \in X$, then $t[t'/x]$ is t with x replaced by t'.

For every S-sorted set A and all $s1,..., sn \in S$, $A_{s1 \cdots sn} := A_{s1} \times \cdots \times A_{sn}$. Let $w \in S^*$, $s \in S$, $\sigma \in OP_{w,s}$ and $t \in T(\mathrm{SIG})_w$. Then $\mathrm{root}(\sigma t) = \sigma$, $\mathrm{arg}(\sigma t) = t$, $\mathrm{sort}(\sigma t) = s$, and $\mathrm{op}(\sigma t)$ (resp. $\mathrm{var}(\sigma t)$) denotes the set of operation symbols (resp. variables) of σt. $\mathrm{Size}(t)$ is the number of operation symbol occurrences in t. A SIG-equation $l = r$ is a pair of SIG-terms l and r with $\mathrm{sort}(l) = \mathrm{sort}(r)$. Let A be a SIG-algebra. $Z(A)$ denotes the S-sorted set of functions from X to A. The unique homomorphic extension of $f \in Z(A)$ to $T(\mathrm{SIG})$ is also written f. If $f \in Z(T(\mathrm{SIG}))$, $t \in T(\mathrm{SIG})$, and $x \in X$, then $f[t/x] \in Z(T(\mathrm{SIG}))$ is defined by $f[t/x](x) = t$ and $f[t/x](y) = fy$ for all $y \in X - \{x\}$.

A satisfies a SIG-equation $l = r$ if for all $f \in Z(A)$ $fl = fr$. (This definition extends to classes of algebras and sets of equations as usual.)

2.1. DEFINITIONS (specification and semantics). An (equational) **specification** $\mathrm{SPEC} = \langle S, OP, E \rangle$ consists of a many-sorted signature $\mathrm{SIG} = \langle S, OP \rangle$ and a set E of SIG-equations. **Alg(SPEC)** denotes the class of SIG-algebras that satisfy E. The **free SPEC-congruence** $=_{\mathrm{SPEC}}$ is the smallest SIG-congruence on $T(\mathrm{SIG})$ that contains all pairs $\langle fl, fr \rangle$ with $l = r$ in E and $f \in Z(T(\mathrm{SIG}))$. $=_{\mathrm{SPEC}}$ is also called the **free theory of SPEC.**

G(SIG) denotes the free S-sorted algebra of OP-terms over the empty set. **Gen(SPEC)** is the class of "finitely generated" SIG-algebras that satisfy E, i.e., every $a \in A$ is the interpretation of some $t \in G(\mathrm{SIG})$. The **inductive SPEC-congruence** $\equiv_{\mathrm{SPEC}}$ is given by all pairs $\langle t, t' \rangle \in T(\mathrm{SIG})^2$ such that for all $f \in Z(G(\mathrm{SIG}))$ $ft =_{\mathrm{SPEC}} ft'$. $\equiv_{\mathrm{SPEC}}$ is also called the **inductive theory of SPEC**. Note that the restriction of $\equiv_{\mathrm{SPEC}}$ to $G(\mathrm{SIG})^2$ coincides with $=_{\mathrm{SPEC}}$.

Two facts are well known (cf. [4] (resp. [1])).

2.2. THEOREM. 1. Alg(SPEC) *satisfies* $t = t'$ *iff* $t =_{\mathrm{SPEC}} t'$.

2. Gen(SPEC) *satisfies* $t = t'$ *iff* $t \equiv_{\mathrm{SPEC}} t'$.

2.3. DEFINITIONS (parameterized data types). A **parameterized specification PAR** is a pair of two specifications PSPEC and SPEC. The forgetful functor from Alg(SPEC) to Alg(PSPEC) is denoted by $\mathbf{U}_{\mathbf{PAR}}$, while $\mathbf{F}_{\mathbf{PAR}}$ stands for its left adjoint. For every class K of PSPEC-algebras the **parameterized data type specified by** $\langle$**PAR,** $K\rangle$ is given by

$$\mathbf{PDT\ PAR},\ K = \{F_{\mathrm{PAR}}(A) \mid A \in K\}.$$

Let $\mathbf{PSIG} = \langle PS, POP \rangle$, $\mathbf{PSPEC} = \langle PS, POP, PE \rangle$, $\mathbf{SPEC} = \langle S, OP, E \rangle$ and $PX = \{x \in X \mid \mathrm{sort}(x) \in PS\}$. Regarding PX as constants we obtain the signature $\mathbf{SIGX} = \langle S, OP \cup PX \rangle$ and the specification $\mathbf{SPECX} = \langle S, OP \cup PX, E \rangle$. $\equiv_{\mathrm{SPECX}}$ is called the **inductive theory of PAR**.

Analogously to Theorem 2.2, there is the following proof-theoretical characterization of the data type specified by $\langle$PAR, Alg(PSPEC)$\rangle$:

2.4. THEOREM [10, 1.7]. PDT(PAR, Alg(PSPEC)) *satisfies* $t = t'$ *iff* $t \equiv_{\mathrm{SPECX}} t'$.

2.5. DEFINITIONS (persistency, completeness, and consistency). Let ID be the identity functor on Alg(PSPEC). We recall from category theory that there is a functor transformation $\eta_{\mathrm{PAR}} : ID \to U_{\mathrm{PAR}} F_{\mathrm{PAR}}$ such that for all $B \in$ Alg(SPEC) each homomorphism h: $A \to U_{\mathrm{PAR}}(B)$ uniquely extends to a homomorphism h^*: $F_{\mathrm{PAR}}(A) \to B$ such that $U_{\mathrm{PAR}}(h^*) \circ \eta_{\mathrm{PAR}}(A) = h$.

Let K be a class of PSPEC-algebras. $\langle$PAR, $K\rangle$ is **persistent** if for all $A \in K$, $\eta_{\mathrm{PAR}}(A)$ is bijective.

Let **BPAR** = $\langle$PSPEC, BSPEC$\rangle$ be a parameterized subspecification of PAR, i.e., **BSPEC** = $\langle$**BS, BOP, BE**$\rangle$ is componentwise included in SPEC. Let **BSIG** = $\langle$BS, BOP$\rangle$ and **EXT** = $\langle$BSPEC, SPEC$\rangle$. Since $U_{\mathrm{PAR}} = U_{\mathrm{BPAR}} \circ U_{\mathrm{EXT}}$, $\eta_{\mathrm{PAR}}(A)$: $A \to U_{\mathrm{PAR}} F_{\mathrm{PAR}}(A)$ uniquely extends to $\boldsymbol{\eta}_{\mathbf{PAR}}(\mathbf{A})^*$: $F_{\mathrm{BPAR}}(A) \to U_{\mathrm{EXT}} F_{\mathrm{PAR}}(A)$ such that $U_{\mathrm{BPAR}}(\eta_{\mathrm{PAR}}(A)^*) \circ \eta_{\mathrm{BPAR}}(A) = \eta_{\mathrm{PAR}}(A)$:

$$\begin{array}{ccc} A & \xrightarrow{\eta_{\mathrm{BPAR}}(A)} & U_{\mathrm{BPAR}} F_{\mathrm{BPAR}}(A) \\ {\scriptstyle \eta_{\mathrm{PAR}}(A)}\downarrow & (1) & \swarrow {\scriptstyle U_{\mathrm{BPAR}}(\eta_{\mathrm{PAR}}(A)^*)} \\ \multicolumn{3}{c}{U_{\mathrm{PAR}} F_{\mathrm{PAR}}(A) = U_{\mathrm{BPAR}} U_{\mathrm{EXT}} F_{\mathrm{PAR}}(A)} \end{array}$$

PAR is **complete (consistent)** w.r.t. $\langle$BPAR, $K\rangle$ if for all $A \in K$ $\eta_{\mathrm{PAR}}(A)^*$ is surjective (injective).

An immediate consequence of these definitions is

2.6. Decomposition Lemma for Persistency. *Let* $\langle$BPAR, $K\rangle$ *be persistent. If* PAR *is complete and consistent with respect to* $\langle$BPAR, $K\rangle$, *then* $\langle$PAR, $K\rangle$ *is persistent.*

2.7. Example. Let BOOL be a specification of Boolean algebras, i.e., BOOL consists of a sort bool, constants true and false, operation symbols $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and the Boolean algebras axioms. Moreover,

DATA = BOOL +

sorts: entry

opns: eq: entry entry $\to$ bool

eqns: $\mathrm{eq}(x, x) = \mathrm{true}$ (e1)

$\mathrm{eq}(x, y) = \mathrm{eq}(y, x)$ (e2)

$(\mathrm{eq}(x, y) \wedge \mathrm{eq}(y, z)) \Rightarrow \mathrm{eq}(x, z) = \mathrm{true}$ (e3)

BSET = DATA +

sorts: set

opns: $\varnothing$: $\to$ set

ins: set entry $\to$ set

eqns: $\mathrm{ins}(\mathrm{ins}(s, x), x) = \mathrm{ins}(s, x)$ (e4)

$\mathrm{ins}(\mathrm{ins}(s, x), y) = \mathrm{ins}(\mathrm{ins}(s, y), x)$ (e5)

SET = BSET +

opns: has: set entry → bool
del: set entry → set
if-bool: bool bool bool → bool
if-set: bool set set → set

eqns:
$$\text{has}(\varnothing, x) = \text{false} \tag{e6}$$
$$\text{has}(\text{ins}(s, x), y) = \text{if-bool}(\text{eq}(x, y), \text{true}, \text{has}(s, y)) \tag{e7}$$
$$\text{del}(\varnothing, x) = \varnothing \tag{e8}$$
$$\text{del}(\text{ins}(s, x), y) = \text{if-set}(\text{eq}(x, y), \text{del}(s, y), \text{ins}(\text{del}(s, y), x)) \tag{e9}$$
$$\text{if-bool}(\text{true}, b, b') = b \tag{e10}$$
$$\text{if-bool}(\text{false}, b, b') = b' \tag{e11}$$
$$\text{if-set}(\text{true}, s, s') = s \tag{e12}$$
$$\text{if-set}(\text{false}, s, s') = s' \tag{e13}$$

Following the strategy developed in this paper we will show that for a certain class Log of DATA-algebras, which will be given in Section 4, $\langle$SET, Log(DATA)$\rangle$ is persistent.

We proceed with the representation of parameterized data types by classes of initial algebras which is essential for the proof-theoretical characterization of completeness and consistency (2.9).

2.8. Definition and Theorem [10, 1.5]. Let $A \in$ Alg(PSPEC). The specification

$$\mathbf{SPEC(A)} = \langle S, OP \cup A, E \cup \Delta(A) \rangle$$

has all operation symbols of SPEC together with all elements of A as constants, while the set of equations of SPEC is extended by the **equational diagram** of A, $\Delta(A)$, that consists of all equations $\sigma(a) = \sigma_A(a)$ with $\sigma \in$ POP and $a \in A_{\text{arity}(\sigma)}$.

$F_{\text{PAR}}(A)$ gets an $(OP \cup A)$-algebra by interpreting each constant $a \in A$ by $\eta_{\text{PAR}}(A)(a)$. Moreover, $F_{\text{PAR}}(A)$ is an initial object in Alg(SPEC(A)).

Using the well-known quotient term algebra representation of initial algebras (cf. ADJ [1]) we can formulate completeness and consistency as free theory properties:

2.9. Theorem. *Let* **BSIG(A)** $= \langle BS, BOP \cup A \rangle$ *and* **SIG(A)** $= \langle S, OP \cup A \rangle$.

1. PAR *is complete w.r.t.* $\langle$BPAR, $K\rangle$ *iff for all* $A \in K$, $s \in BS$ *and* $t \in G(\text{SIG}(A))_s$ *some* $t' \in G(\text{BSIG}(A))$ *satisfies* $t =_{\text{SPEC}(A)} t'$.
2. PAR *is consistent w.r.t.* $\langle$BPAR, $K\rangle$ *iff for all* $A \in K$ *and* $t, t' \in G(\text{BSIG}(A))$,

$$t =_{\text{SPEC}(A)} t' \text{ implies } t =_{\text{BSPEC}(A)} t'.$$

Proof. Let $A \in K$ and **EXT(A)** $= \langle \text{BSPEC}(A), \text{SPEC}(A)\rangle$. Since BSPEC($A$) is a subspecification of SPEC(A), there is a unique homomorphism h such that the following diagram commutes:

$$\begin{array}{ccc} G(\text{BSIG}(A)) & \xrightarrow{\text{inc}} & U_{\text{EXT}(A)}G(\text{SIG}(A)) \\ \downarrow{\scriptstyle \text{nat}} & & \downarrow{\scriptstyle \text{nat}} \\ G(\text{BSIG}(A))/{=_{\text{BSPEC}(A)}} & \xrightarrow[h]{} & U_{\text{EXT}(A)}G(\text{SIG}(A))/{=_{\text{SPEC}(A)}} \end{array} \qquad (1)$$

Let $B := F_{\text{BPAR}}(A)$ and $C := F_{\text{PAR}}(A)$. By Theorem 2.8, B and C are initial objects in Alg(BSPEC(A)) and Alg(SPEC(A)), respectively. Hence

$$B \simeq G(\text{BSIG}(A))/{=_{\text{BSPEC}(A)}} \text{ and } C \simeq G(\text{SIG}(A))/{=_{\text{SPEC}(A)}}. \qquad (2)$$

Furthermore, $\eta_{\text{PAR}}(A)^*\colon B \to U_{\text{EXT}}(C)$ (cf. 2.5) is a BSIG(A)-homomorphism because for all $a \in A$,

$$\eta_{\text{PAR}}(A)^*(a_B) = \eta_{\text{PAR}}(A)^*(\eta_{\text{BPAR}}(A)(a)) = \eta_{\text{PAR}}(A)(a) = a_C.$$

Vice versa, let $\bar{h}\colon B \to U_{\text{EXT}}(C)$ be the composition of h and the isomorphisms given by (2). For all $a \in A$,

$$U_{\text{BPAR}}(\bar{h}) \circ \eta_{\text{BPAR}}(A)(a) = U_{\text{BPAR}}(\bar{h})(a_B) = \bar{h}(a_B) = a_C = \eta_{\text{PAR}}(A)(a).$$

Hence $\bar{h} = \eta_{\text{PAR}}(A)^*$. Therefore, PAR is complete w.r.t. $\langle$BPAR, $K\rangle$

iff for all $A \in K$, $\eta_{\text{PAR}}(A)^*$ is surjective

iff for all $A \in K$, h is surjective

iff for all $A \in K$, $s \in BS$, and $t \in G(\text{SIG}(A))_s$, some

$$t' \in G(\text{BSIG})(A)) \text{ satisfies } t =_{\text{SPEC}(A)} t',$$

and PAR is consistent w.r.t. $\langle$BPAR, $K\rangle$

iff for all $A \in K$, $\eta_{\text{PAR}}(A)^*$ is injective

iff for all $A \in K$, h is injective

iff for all $A \in K$ and $t, t' \in G(\text{BSIG}(A))$,

$$t =_{\text{SPEC}(A)} t' \text{ implies } t =_{\text{BSPEC}(A)} t'. \quad \blacksquare$$

Besides, the persistency theorems 2.8 and 2.9 provide a useful criterion for the validity of equations in parameterized data types: PDT(PAR, K) satisfies a set E' of SIG-equations if PAR is complete w.r.t. $\langle$BPAR, $K\rangle$ and $\langle$PSPEC, $\langle S, OP, E \cup E'\rangle\rangle$ is consistent w.r.t. $\langle$BPAR, $K\rangle$.

The characterization of completeness and consistency given by Theorem 2.9 will be further investigated in the next section. The rest of this section deals with criteria for the other condition for decomposing a persistency proof, namely persistency of the base specification BPAR.

The proof-theoretical conditions "maximal completeness" and "maximal consistency" defined below deal with variables instead of elements of a particular parameter algebra. Hence they characterize persistency of PAR with respect to all parameter algebras (Theorem 2.12).

2.10. DEFINITIONS. PAR is **maximally complete** if for all $s \in PS$ and $t \in G(\text{SIGX})_s$ $t =_{\text{SPECX}} t'$ for some $t' \in T(\text{PSIG})$ (cf. 2.3). PAR is **maximally consistent** if for all $t, t' \in T(\text{PSIG})$ $t =_{\text{SPECX}} t'$ implies $t =_{\text{PSPEC}} t'$.

2.11. DEFINITION. The **simple reduction relation generated by E**, $\rightarrow_E$, is the smallest relation on $T(\text{SIG})$ (resp. $Z(T(\text{SIG}))$) such that

(i) for all $l = r$ in E and $f \in Z(T(\text{SIG}))$, $fl \rightarrow_E fr$,

(ii) for all $\sigma \in OP$, $\sigma(t_1,\ldots, t_i,\ldots, t_n) \rightarrow_E \sigma(t_1,\ldots, t_i',\ldots, t_n)$ if $t_i \rightarrow_E t_i'$,

(iii) for all $f, g \in Z(T(\text{SIG}))$ $f \rightarrow_E g$ if for all $x \in X$, $fx \rightarrow_E gx$. $\overset{=}{\rightarrow}_E$, $\leftrightarrow_E$, and $\overset{*}{\rightarrow}_E$ denote the reflexive, symmetric, and reflexive-transitive closures of E, respectively.

2.12. PERSISTENCY THEOREM I. *$\langle$PAR, Alg(PSPEC)$\rangle$ is persistent iff PAR is maximally complete and maximally consistent.*

Proof. (only if) By assumption, $\eta_{\text{PAR}}(T(\text{PSIG})/=_{\text{PSPEC}})$ is an isomorphism. Since

$$F_{\text{PAR}}(T(\text{PSIG})/=_{\text{PSPEC}}) \simeq G(\text{SIGX})/=_{\text{SPECX}}$$

[10, 1.6], we conclude

$$T(\text{PSIG})/=_{\text{PSPEC}} \simeq U_{\text{PAR}}(G(\text{SIGX})/=_{\text{SPECX}}).$$

The surjective (resp. injective) part of this isomorphism is maximal completeness (resp. consistency) of PAR.

(if) First we observe that for all $K \in \text{Alg(PSPEC)}$, $\langle$PAR, $K\rangle$ is persistent iff PAR is complete and consistent w.r.t. $\langle$PPAR, $K\rangle$, where PPAR = $\langle$PSPEC, PSPEC$\rangle$. So it suffices to use Theorem 2.9 to show that $\langle$PAR, Alg(PSPEC)$\rangle$ is persistent.

Let $A \in \text{Alg(PSPEC)}$, $s \in PS$, and $t \in G(\text{SIG}(A))_s$. Then there are $u \in G(\text{SIGX})$ and $f \in Z(A)$ such that $t = fu$. Since PAR is maximally complete, there is $u' \in T(\text{PSIG})$ with $u =_{\text{SPECX}} u'$.

Hence $t =_{\text{SPEC}(A)} fu' \in G(\text{PSIG}(A))$. Therefore, PAR is complete w.r.t. $\langle$PPAR, Alg(PSPEC)$\rangle$ (cf. 2.9.1). It remains to show that PAR is consistent w.r.t. $\langle$PPAR, Alg(PSPEC)$\rangle$ (cf. 2.9.2). So let $t, t' \in G(\text{PSIG}(A))$ such that $t =_{\text{SPEC}(A)} t'$. Then there are a least number n and $t_1,\ldots, t_n \in G(\text{SIG}(A))$ such that $t_1 = t$, $t_n = t'$, and for all $1 \leqslant i < n$, $t_i \leftrightarrow_{E \cup \Delta(A)} t_{i+1}$. Moreover, there are $f \in Z(A)$ and $u_i \in G(\text{SIGX})$,

$1 \leqslant i \leqslant n$, such that $fu_i = t_i$ and f is injective on $\bigcup_{i=1}^n \text{var}(u_i)$. Maximal completeness of PAR implies $u_i =_{\text{SPECX}} u_i'$ for some $u_i' \in T(\text{PSIG})$.

Next we show that for all $1 \leqslant i < n$,

$$fu_i' =_{\text{PSPEC}(A)} fu_{i+1}'. \qquad (*)$$

If $t_i \leftrightarrow_E t_{i+1}$, then

$$u_i' =_{\text{SPECX}} u_i \xleftrightarrow[E]{} u_{i+1} =_{\text{SPECX}} u_{i+1}',$$

and we conclude $u_i' =_{\text{PSPEC}} u_{i+1}'$ from maximal consistency of PAR. Hence (*) holds true.

If $t_i \leftrightarrow_{\Delta(A)} t_{i+1}$, then there are $v, v' \in T(\text{PSIG})$, $u \in T(\text{SIG})$ and $z \in \text{var}(u) \cap PX$ such that $fv = fv'$ is in $\Delta(A)$, $u_i = u[v/z]$, and $u_{i+1} = u[v'/z]$. Maximal completeness of PAR implies $u =_{\text{SPECX}} u'$ for some $u' \in T(\text{PSIG})$. Hence

$$u_i' =_{\text{SPECX}} u_i = u[v/z] =_{\text{SPECX}} u'[v/z]$$

so that by maximal consistency of PAR, $u_i' =_{\text{PSPEC}} u'[v/z]$. Analogously, $u_{i+1}' =_{\text{PSPEC}} u'[v'/z]$. Therefore,

$$fu_i' =_{\text{PSPEC}(A)} f(u'[v/z]) = u'[fv/z][fx/x \mid x \in X]$$

$$\xleftrightarrow[\Delta(A)]{} u'[fv'/z][fx/x \mid x \in X] = f(u'[v'/z]) =_{\text{PSPEC}(A)} fu_{i+1}'.$$

Hence (*) holds true.

$t, t' \in G(\text{PSIG}(A))$ implies $u_1, u_n \in T(\text{PSIG})$ and thus by maximal consistency of PAR, $u_1 =_{\text{PSPEC}} u_1'$ and $u_n =_{\text{PSPEC}} u_n'$. Finally, (*) yields

$$t = fu_1 =_{\text{PSPEC}(A)} fu_1' =_{\text{PSPEC}(A)} fu_n' =_{\text{PSPEC}(A)} fu_n = t'.$$

Therefore, PAR is consistent w.r.t. ⟨PPAR, Alg(PSPEC)⟩. (The main idea of this proof is due to Ganzinger [6, Theorem 5].) ▮

2.13. COROLLARY. ⟨PAR, Alg(PSPEC)⟩ *is persistent if for all* $\sigma \in OP$, $\text{sort}(\sigma) \in PS$ *implies* $\sigma \in POP$ *and if for all* $l = r$ *in* E $\text{sort}(l) \in PS$ *implies that* $l = r$ *is in* PE.

2.14. EXAMPLE (cf. 2.7). Using Corollary 2.13 we immediately observe that ⟨⟨DATA, BSET⟩, Alg(DATA)⟩ is persistent.

3. COMPLETENESS AND CONSISTENCY PROOFS BY TERM REWRITING

Assuming that the "base" ⟨BPAR, K⟩ is persistent we turn to refinements of the proof-theoretical characterization of completeness and consistency given in the last section (2.9). From now on we suppose that $S = BS$ and for all $A \in \text{Alg(PSPEC)}$ and $s \in S$, $G(\text{SIG}(A))_s$ is nonempty.

A first step is the decomposition of the free SPEC(A)-congruence into simple reductions and the free BSPEC(A)-congruence:

3.1. Definition. Let $A \in \mathrm{Alg}(\mathrm{PSPEC})$. A set R of SIG(A)-equations is **Church–Rosser w.r.t. A** if for all $s \in BS$ and $t, t' \in G(\mathrm{SIG}(A))_s$, $t =_{\mathrm{PSPEC}(A)} t'$ implies

$$\begin{array}{ccc} t & & t' \\ {\scriptstyle R}\downarrow * & & * \downarrow {\scriptstyle R} \\ u & =_{\mathrm{BSPEC}(A)} & u' \end{array}$$

for some $u, u' \in G(\mathrm{BSIG}(A))$.

3.2. Lemma. *Suppose that for each $l = r$ in $E - BE$, $\mathrm{op}(l)$ contains at least one operation symbol of $OP - BOP$. For all $A \in K$ let $E(A)$ be a subset of $=_{\mathrm{BSPEC}(A)}$. If $(E - BE) \cup E(A)$ is Church–Rosser w.r.t. A, then* PAR *is consistent w.r.t* $\langle \mathrm{BPAR}, K \rangle$.

Proof. Let $t, t' \in G(\mathrm{BSIG}(A))$ such that $t =_{\mathrm{SPEC}(A)} t'$. By assumption,

$$\begin{array}{ccc} t & & t' \\ {\scriptstyle (E-BE)\cup E(A)}\downarrow * & & * \downarrow {\scriptstyle (E-BE)\cup E(A)} \\ u & =_{\mathrm{BSPEC}(A)} & u' \end{array}$$

for some $u, u' \in G(\mathrm{BSIG}(A))$. Since for each $l = r$ in $E - BE$, $\mathrm{op}(l) \cap (OP - BOP) \neq \emptyset$, we have $t =_{\mathrm{BSPEC}(A)} u$ and $t' =_{\mathrm{BSPEC}(A)} u'$. Thus by Theorem 2.9.2, PAR is consistent w.r.t. $\langle \mathrm{BPAR}, K \rangle$. ∎

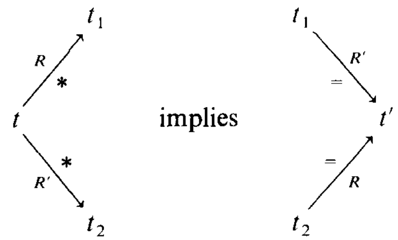
Localizing the Church–Rosser property by "confluence" and "coherence" conditions goes along with restricting equations to "normalizing" ones.

3.3. Definitions. Let $A \in \mathrm{Alg}(\mathrm{PSPEC})$ and R be a set of SIG(A)-equations. $t' \in G(\mathrm{BSIG}(A))$ is an **R-normal form of t** $\in G(\mathrm{SIG}(A))$ if $t \xrightarrow{*}_R t'$. R is **normalizing w.r.t. A** if for all $t \in G(\mathrm{SIG}(A))$ t has an R-normal form. R is **confluent w.r.t. A** if for all $t \in G(\mathrm{SIG}(A))$ all R-normal forms t_1, t_2 of t satisfy $t_1 =_{\mathrm{BSPEC}(A)} t_2$. $\langle t_1, t_2 \rangle \in G(\mathrm{SIG}(A))$ is **uniformly R-convergent w.r.t. A** if some R-normal forms t_1', t_2' of t_1 (resp. t_2) satisfy $t_1' =_{\mathrm{BSPEC}(A)} t_2'$, written $t_1 \downarrow_{R,A} t_2$.

R is **coherent w.r.t. A** if for all $t, t_1, t_2 \in G(\mathrm{SIG}(A))$,

$$\begin{array}{ccc} & t_1 & \\ {\scriptstyle R}\nearrow & & \\ t & & \text{implies} \quad t_1 \downarrow_{R,A} t_2. \\ {\scriptstyle BE}\searrow & & \\ & t_2 & \end{array}$$

R **commutes with** another set R' of SIG(A)-equations if for all $t, t_1, t_2 \in G(\mathrm{SIG}(A))$,



for some $t' \in G(\mathrm{SIG}(A))$.

3.4. Completeness Theorem I. *For all $A \in K$ let $E(A)$ be a subrelation of $=_{\mathrm{BSPEC}(A)}$. If for all $A \in K$, $E \cup E(A)$ is normalizing w.r.t.* $\langle$BPAR, $K\rangle$, *then* PAR *is complete w.r.t.* $\langle$BPAR, $K\rangle$.

Proof. The statement immediately follows from Theorem 2.9.1. ▮

3.5. Consistency Theorem I. *Suppose that for each $l = r$ in BE* $\mathrm{var}(r) \subseteq \mathrm{var}(l)$ *and for each $l = r$ in $E - BE$, l contains at least one operation symbol of $OP - BOP$. For all $A \in K$ let $E(A)$ be a subrelation of $=_{\mathrm{BSPEC}(A)}$. If $(E - BE) \cup E(A)$ is normalizing, confluent, and coherent w.r.t. A and commutes with* $(\Delta(A) \cup \Delta(A)^{-1}) - (E(A) \cup E(A)^{-1})$ (cf. 2.8), *then* PAR *is consistent w.r.t.* $\langle$BPAR, $K\rangle$.

Proof. By Lemma 3.2, it is sufficient to show that $R = (E - BE) \cup E(A)$ is Church-Rosser w.r.t. A. So let $s \in BS$ and $t, t' \in G(\mathrm{SIG}(A))_s$ such that $t =_{\mathrm{SPEC}(A)} t'$. There are a least number n and $t_1, \ldots, t_n, u_1, \ldots, u_n \in G(\mathrm{SIG}(A))$ with $t_1 = t$, $u_n = t'$, and for all $1 \leqslant i < n$ $u_i \xrightarrow{*}_{E-BE} t_i$ and

(i) $u_i \rightarrow_R t_{i+1}$, or

(ii) $u_i \rightarrow_{BE} t_{i+1}$, or

(iii) $t_{i+1} \rightarrow_{BE} u_i$, or

(iv) $t_{i+1} \rightarrow_{E(A)} u_i$, or

(v) $u_i \leftrightarrow_{\Delta(A) - E(A)} t_{i+1}$.

We prove $t \downarrow_R t'$ by induction on n; $n = 1$ implies $t' \xrightarrow{*}_R t$, and $t \downarrow_R t'$ follows from the normalization and confluence of R w.r.t. A. Since $E(A) \subseteq =_{\mathrm{BSPEC}(A)}$ and for each $l = r$ in $E - BE$, $\mathrm{op}(l) \cap (OP - BOP) \neq \varnothing$,

$$\text{for all } u \in G(\mathrm{BSIG}(A)), \qquad u \rightarrow_R u' \qquad \text{implies} \quad u =_{\mathrm{BSPEC}(A)} u'. \qquad (*)$$

Let $n > 1$. By the induction hypothesis, $t_2 \downarrow_{R,A} t'$. Hence by the confluence of R, it remains to show $t_1 \downarrow_{R,A} t_2$.

The proof proceeds by deriving $t_1 \downarrow_{R,A} t_2$ in each of the cases (i)–(iv) for $i=1$. If $i=1$ satisfies (i), we have

$$\begin{array}{ccc} & u_1 & \\ {}^{E-BE}\swarrow^{*} & & \searrow^{R} \\ t_1 & & t_2 \end{array}$$

and infer $t_1 \downarrow_R t_2$ from normalization and confluence of R.

If $i=1$ satisfies (ii), then

$$\begin{array}{ccc} t_1 & \xleftarrow[*]{E-BE} & u_1 \\ {}^{R}\swarrow^{*} & & \searrow^{BE} \\ t_1' & & t_2 \end{array}$$

for some $t_1' \in G(\mathrm{BSIG}(A))$. If u_1 is R-normal, then $t_1 = u_1$ and $u_1 =_{\mathrm{BSPEC}(A)} t_2$. There is an R-normal form t_2' of t_2, and we conclude $t_2 =_{\mathrm{BSPEC}(A)} t_2'$ by (*). Hence $t_1 =_{\mathrm{BSPEC}(A)} t_2'$ so that $t_1 \downarrow_{R,A} t_2$. If u_1 is not R-normal, there is $v \in G(\mathrm{SIG}(A))$ such that

$$\begin{array}{ccccc} & t_1 & \xleftarrow[*]{E-BE} & u_1 & \\ {}^{R}\swarrow^{*} & & {}^{R}\swarrow & & \searrow^{BE} \\ t_1' & \xleftarrow[*]{} & v & & t_2. \end{array}$$

Since R is coherent w.r.t. A, we have $v \downarrow_{R,A} t_2$. Therefore $t_1 \downarrow_{R,A} t_2$ by confluence of R.

If $i=1$ satisfies (iii), we have

$$\begin{array}{ccc} u_1 & \xleftarrow{BE} & t_2 \\ {}^{E-BE}\swarrow^{*} & & {}^{*}\searrow^{R} \\ t_1 & & t_2' \end{array}$$

for some $t_2' \in G(\mathrm{BSIG}(A))$. If t_2 is R-normal, then $t_2 =_{\mathrm{BSPEC}(A)} u_1$ and thus $u_1 = t_1$. There is an R-normal form t_1' of t_1, and we conclude $t_1 =_{\mathrm{BSPEC}(A)} t_1'$ by (*). Hence $t_2 =_{\mathrm{BSPEC}(A)} t_1'$ so that $t_1 \downarrow_{R,A} t_2$. If t_2 is not R-normal, there is $v \in G(\mathrm{SIG}(A))$ such that

$$\begin{array}{ccccc} u_1 & \xleftarrow{BE} & t_2 & \xrightarrow{R} & v \\ {}^{E-BE}\swarrow^{*} & & & & {}^{*}\searrow^{R} \\ t_1 & & & & t_2'. \end{array}$$

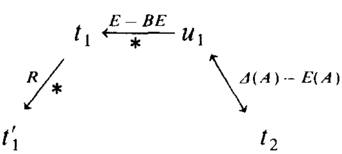
Since R is coherent w.r.t. A, we have $u_1 \downarrow_{R,A} v$. Therefore $t_1 \downarrow_{R,A} t_2$ by the normalization and confluence of R.

If $i = 1$ satisfies (iv), we have

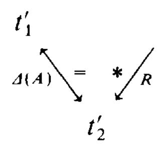
$$t_1 \xleftarrow[*]{E - BE} u_1 \xleftarrow{E(A)} t_2$$

and infer $t_1 \downarrow_R t_2$ from the normalization and confluence of R.

If $i = 1$ satisfies (v), then



for some $t_1' \in G(\mathrm{BSIG}(A))$. Since R commutes with $(\Delta(A) \cup \Delta(A)^{-1}) - (E(A) \cup E(A)^{-1})$, there is $t_2' \in G(\mathrm{SIG}(A))$ such that



Hence $t_2' \in G(\mathrm{BSIG}(A))$ and thus $t_1 \downarrow_{R,A} t_2$. ∎

4. Parameters with "Built-in" Logic

From now on we deal with parameters including Boolean operators and restrict parameter algebras to those where the Boolean operators are interpreted as in propositional logic. In addition, we use if-then-else operators to simulate conditional axioms by equations.

General Assumption. Suppose that BOOL (cf. 2.7) is a subspecification of PSPEC. Moreover, let **if***S* be a subset of S such that for all $s \in \mathrm{if}S$ SIG contains an operation symbol if-s: bool $s\ s \to s$ and E includes the equations

$$\text{if-}s(\text{true}, x, y) = x \quad \text{and} \quad \text{if-}s(\text{false}, x, y) = y.$$

Vice versa, for each $l = r$ in E,

(i) $\mathrm{sort}(l) = \mathrm{bool}$ implies $l \notin \{\text{true}, \text{false}\}$,

(ii) $\mathrm{sort}(l) \neq \mathrm{bool}$ implies $t \in \{\text{true}, \text{false}\}$ for all bool-sorted subterms t of l.

4.1. DEFINITIONS. Let **PEXT** $= \langle$BOOL, PSPEC$\rangle$. The class **Log(PSPEC)** is given by all PSPEC-algebras A such that $U_{\mathrm{PEXT}}(A)$ is the Boolean algebra {true, false}. Hence we drop the equations true = true and false = false from the equational diagram of A (cf. 2.8). For all $A \in$ Log(PSPEC), **LE(A)** denotes the set of BSIG(A)-equations $l = r$ with $l \in G(\mathrm{BSIG}(A))_{\mathrm{bool}} - \{\text{true, false}\}$, $r \in \{\text{true, false}\}$ and $l =_{\mathrm{BSPEC}(A)} r$.

4.2. LEMMA. *Let* $\langle$BPAR, Log(PSPEC)$\rangle$ *be persistent and* $A \in$ Log(PSPEC).

(1) *For all* $t \in G(\mathrm{BSIG}(A))_{\mathrm{bool}}$ *either* $t = true$ *or* $t = false$ *is in* $LE(A)$.

(2) *Suppose that for all* $l' = r'$ *in* BE $\mathrm{var}(r') \subseteq \mathrm{var}(l')$. *Let* $l = r$ *be in* $LE(A)$, $f \in Z(T(\mathrm{SIG}(A)))$ *and* $R \subseteq BE \cup LE(A) \cup \Delta(A) \cup \Delta(A)^{-1}$ (cf. 2.8). *Then* $fl \rightarrow_R t$ *implies* $t \xrightarrow{*}_{LE(A)} fr$.

Proof. Let $A \in$ Log(PSPEC). By assumption, $A \simeq U_{\mathrm{BPAR}} F_{\mathrm{BPAR}}(A)$. By Theorem 2.8, $F_{\mathrm{BPAR}}(A) \simeq G(\mathrm{BSIG}(A))/{=_{\mathrm{BSPEC}(A)}}$. Hence (1) follows from $U_{\mathrm{PEXT}}(A) \simeq \{\text{true, false}\}$.

(2) *Case* 1. ($r =$ true) Since l does not contain variables, $fl = l \in G(\mathrm{BSIG}(A))$. Thus $fl \rightarrow_R t$ implies $fl =_{\mathrm{BSPEC}(A)} t$ because for all $l' = r'$ in BE, $\mathrm{var}(r') \subseteq \mathrm{var}(l')$. Hence

$$t =_{\mathrm{BSPEC}(A)} fl =_{\mathrm{BSPEC}(A)} fr = r = \text{true}. \qquad (*)$$

By (1), $t \neq$ false. So we have either $t =$ true and thus $t = r = fr$ or $t \in G(\mathrm{BSIG}(A))_{\mathrm{bool}} - \{\text{true, false}\}$ and thus by (*), $t \rightarrow_{LE(A)}$ true $= r = fr$.

Case 2. ($r =$ false) The proof proceeds analogously to Case 1. ∎

Next we define a reduction relation with conditions (contexts) to simulate reductions via $LE(A)$.

4.3. DEFINITION. Let **BOOT** $= T(\mathrm{BSIG})_{\mathrm{bool}}$. The **contextual reduction relation generated by E**, $\{\xrightarrow{*}_{E;p}\}_{p \in \mathrm{BOOT}}$, is the family of smallest relations on T(SIG) such that

(i) for all $t \in T(\mathrm{SIG})$ and $p \in$ BOOT $t \xrightarrow{*}_{E;p} t$,

(ii) for all $l = r$ in E, $f \in Z(T(\mathrm{SIG}))$ and $p \in$ BOOT $fl \xrightarrow{*}_{E;p} fr$,

(iii) for all $\sigma \in OP$, $\sigma(t_1, \ldots, t_i, \ldots, t_n) \xrightarrow[E;p]{*} \sigma(t_1, \ldots, t_i', \ldots, t_n)$ if $t_i \xrightarrow{*}_{E;p} t_i'$,

(iv) for all $s \in \mathrm{if}S$ and $t_1, t_2 \in T(\mathrm{SIG})_s$,

$$\text{if-}s(p, t_1, t_2) \xrightarrow[E;p]{*} t_1 \qquad \text{and} \qquad \text{if-}s(p, t_1, t_2) \xrightarrow[E;\neg p]{*} t_2,$$

(v) $t \xrightarrow{*}_{E;p \wedge q} t''$ if $t \xrightarrow{*}_{E;p} t'$ and $t' \xrightarrow{*}_{E;q} t''$,

(vi) $t \xrightarrow{*}_{E;p \vee q} t'$ if $t \xrightarrow{*}_{E;p} t'$ and $t \xrightarrow{*}_{E;q} t'$,

(vii) for all $t, t' \in T(\mathrm{SIG})$, $t \xrightarrow{*}_{E;\mathrm{false}} t'$.

The following lemma draws the connection between contextual and $LE(A)$-reductions. Contexts are now restricted to "base" terms so that contextual reductions can be regarded as "hierarchical" ones.

4.4. LEMMA. *Let* $\langle \mathrm{BPAR}, \mathrm{Log(PSPEC)}\rangle$ *be persistent. Then all* $A \in \mathrm{Log(PSPEC)}$, $f \in Z(G(\mathrm{SIG}(A)))$ *and* $f' \in Z(G(\mathrm{BSIG}(A)))$ *with* $f \xrightarrow{*}_{E \cup LE(A)} f'$ (cf. 2.11) *satisfy*

$$ft \xrightarrow[E \cup LE(A)]{*} ft' \qquad \text{if } t \xrightarrow[E;p]{*} t' \text{ and } f'p = \text{true is in } LE(A). \tag{*}$$

Proof. We show (*) by induction on the least number of derivation steps 4.3(i)–(vii) that lead to $t \xrightarrow{*}_{E;p} t'$.

If the last derivation step that generates $t \xrightarrow{*}_{E;p} t'$ is given by 4.3(i), (ii), or (iii), we are done by the definition of $\xrightarrow{*}_E$ and the induction hypothesis.

If the last step is 4.3(iv), then we have two subcases:

(a) $t = \text{if-}s(p, t', t_2)$ for some $s \in \mathrm{if}S$ and $t_2 \in T(\mathrm{SIG})$. Then $ft = \text{if-}s(fp, ft', ft_2) \rightarrow_{LE(A)} \text{if-}s(\mathrm{true}, ft', ft_2) \rightarrow_E ft'$.

(b) $t = \text{if-}s(q, t_1, t')$ for some $s \in \mathrm{if}S$, $t_1 \in T(\mathrm{SIG})$, and $\neg q = p$. Assume that $f'q = \mathrm{true}$ is in $LE(A)$. Since $f'p = \mathrm{true}$ is in $LE(A)$ by assumption, we would obtain

$$\mathrm{true} =_{\mathrm{BSPEC}(A)} f'q \wedge \neg f'q =_{\mathrm{BSPEC}(A)} \mathrm{false}.$$

This contradicts Lemma 4.2(1). Thus again by Lemma 4.2(1), $f'q = \mathrm{false}$ is in $LE(A)$. Therefore $ft = \text{if-}s(fq, ft_1, ft') \xrightarrow{*}_{E \cup LE(A)} \text{if-}s(\mathrm{false}, ft_1, ft') \rightarrow_E ft'$.

If the last derivation step of $t \xrightarrow{*}_{E;p} t'$ is given by 4.3(v), then there are $q, q' \in \mathrm{BOOT}$ and $t'' \in T(\mathrm{SIG})$ such that $t \xrightarrow{*}_{E;q} t''$, $t'' \xrightarrow{*}_{E;q'} t'$, and $p = q \wedge q'$. Hence $f'q = \mathrm{true}$ and $f'q' = \mathrm{true}$ are in $LE(A)$. Otherwise we would obtain a contradiction to Lemma 4.2 analogously to case (b) above. Thus by the induction hypothesis, $ft \xrightarrow{*}_{E \cup LE(A)} ft''$ and $ft'' \xrightarrow{*}_{E \cup LE(A)} ft'$ so that $ft \xrightarrow{*}_{E \cup LE(A)} ft'$.

If the last derivation step of $t \xrightarrow{*}_{E;p} t'$ is given by 4.3(vi), there are $q, q' \in \mathrm{BOOT}$ such that $t \xrightarrow{*}_{E;q} t'$, $t \xrightarrow{*}_{E;q'} t'$, and $p = q \vee q'$. Hence $f'q = \mathrm{true}$ or $f'q' = \mathrm{true}$ is in $LE(A)$. Otherwise we would obtain a contradiction to Lemma 4.2(1) analogously to case (b) above. Thus w.l.o.g. $f'q = \mathrm{true}$ is in $LE(A)$, and by the induction hypothesis, $ft \xrightarrow{*}_{E \cup LE(A)} ft'$.

If the last derivation step of $t \xrightarrow{*}_{E;p} t'$ is given by 4.3(vii), we have $p = \mathrm{false}$. This contradicts Lemma 4.2(1) because by assumption, $f'p = \mathrm{true}$ is in $LE(A)$. Hence (*) is trivial in this case. ∎

Contextual reduction properties that correspond to 3.3 are defined by 4.5 and 4.10 below.

4.5. DEFINITION. Let $t \in G(\text{SIGX})$ (cf. 2.3). t **has contextual E-normal forms** $t_1, \dots, t_n \in G(\text{BSIGX})$ if there are $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \text{BOOT}$ such that $p_1 \vee \cdots \vee p_n =_{\text{BSPEC}}$ true and for all $1 \leqslant i \leqslant n$ $t \xrightarrow{*}_{E;p_i} t_i$. E is **contextually normalizing** if all $t \in G(\text{SIGX})$ have contextual E-normal forms.

To reduce normalization of $E \cup LE(A)$ to contextual normalization of E we have to guarantee that SPEC(A) does not identify true and false.

4.6. LEMMA. *Suppose that* ⟨BPAR, Log(PSPEC)⟩ *is persistent. Let* $A \in \text{Log(PSPEC)}$. $E \cup LE(A)$ *is normalizing w.r.t.* A *if* E *is contextually normalizing.*

Proof. Let $t \in G(\text{SIG}(A))$. There are $u \in G(\text{SIGX})$ and $f \in Z(A)$ such that $t = fu$. By assumption, there are $n \in \mathbb{N}$, $p_1, \dots, p_n \in \text{BOOT}$, and $u_1, \dots, u_n \in G(\text{BSIGX})$ such that $p_1 \vee \cdots \vee p_n =_{\text{BSPEC}}$ true and for all $1 \leqslant i \leqslant n$ $u \xrightarrow{*}_{E;p_i} u_i$. Assume that for all $1 \leqslant i \leqslant n$, $fp_i =$ false is in $LE(A)$. Then

$$\text{true} =_{\text{BSPEC}(A)} fp_1 \vee \cdots \vee fp_n =_{\text{BSPEC}(A)} \text{false},$$

which contradicts Lemma 4.2(1). Thus for some $1 \leqslant i \leqslant n$, $fp_i =$ true is in $LE(A)$. Hence by Lemma 4.4,

$$t = fu \xrightarrow[E \cup LE(A)]{*} fu_i \in G(\text{BSIG}(A)). \quad \blacksquare$$

4.7. COMPLETENESS THEOREM II. *Suppose that* ⟨BPAR, Log(PSPEC)⟩ *is persistent. If* E *is contextually normalizing, then* PAR *is complete w.r.t.* ⟨BPAR, Log(PSPEC)⟩.

Proof. The statement immediately follows from Lemma 4.6 and completeness theorem 3.4. ∎

4.8. EXAMPLE. (cf. 2.7) Let $E = \{e6, \dots, e13\}$. One easily observes that E is contextually normalizing if

for all $t, t' \in G(\text{BSIGX})_{\text{set}}$ has(t, x), del(t, x), and if-set(x, t, t') have contextual E-normal forms. (*)

Condition (*) follows by induction on size(t) + size(t') because we obtain

$$\text{has}(\varnothing, x) \xrightarrow[e8]{} \text{false},$$

$$\text{has}(\text{ins}(t, x), y) \xrightarrow[e7;\, eq(x,y)]{*} \text{true},$$

$$\text{has}(\text{ins}(t, x), y) \xrightarrow[e7;\, \neg eq(x,y)]{*} \text{has}(t, y),$$

$$\text{del}(\varnothing, x) \xrightarrow[e8]{} \varnothing,$$

$$\text{del}(\text{ins}(t, x), y) \xrightarrow[e9;\, eq(x,y)]{*} \text{del}(t, y),$$

$$\text{del}(\text{ins}(t, x), y) \xrightarrow[e9;\, \neg eq(x,y)]{*} \text{ins}(\text{del}(t, y), x),$$

$$\text{if-set}(x, t, t') \xrightarrow[E;x]{} t,$$

$$\text{if-set}(x, t, t') \xrightarrow[E;\neg x]{} t'$$

for all $t, t' \in G(\text{BSIGX})_{\text{set}}$. Since $\langle\langle \text{DATA}, \text{BSET}\rangle, \text{Log}(\text{DATA})\rangle$ is persistent (cf. Example 2.14), we conclude from Theorem 4.7 that $\langle \text{DATA}, \text{SET}\rangle$ is complete w.r.t. $\langle\langle \text{DATA}, \text{BSET}\rangle, \text{Log}(\text{DATA})\rangle$.

Local criteria for confluence and coherence require the "new" equations $E - BE$ to be normalizing (cf. Theorem 3.5). "Base" equations (BE) are often not normalizing. Hence we can use Noetherian induction—to lift local criteria—only with respect to $E - BE$. But BE must be considered, too. The lack of normalization of BE is circumvented by working with parallel BE-reductions which combine independent simple reductions in one step.

4.9. Definition. The **parallel reduction relation generated by E,** $\Rightarrow_E$, and its reflexive closure $\stackrel{=}{\Rightarrow}_E$ are the smallest relations on $T(\text{SIG})$ (resp. $Z(T(\text{SIG}))$) such that

(i) for all $t \in T(\text{SIG})$, $t \stackrel{=}{\Rightarrow}_E t$,

(ii) for all $f, g \in Z(T(\text{SIG}))$, $f \stackrel{=}{\Rightarrow}_E g$ if for all $x \in X$ $fx \stackrel{=}{\Rightarrow}_E gx$,

(iii) for all $l = r$ in E, $fl \Rightarrow_E gr$ if $f \stackrel{=}{\Rightarrow}_E g$,

(iv) $t \stackrel{=}{\Rightarrow}_E t'$ if $t \Rightarrow_E t'$,

(v) for all $\sigma \in OP$, $\sigma(t_1, \ldots, t_n) \Rightarrow_E \sigma(t'_1, \ldots, t'_n)$ if $\exists\ 1 \leqslant i \leqslant n$: $t_i \Rightarrow_E t'_i$ and $\forall\ 1 \leqslant i \leqslant n$: $t_i \stackrel{=}{\Rightarrow}_E t'_i$.

Hence a parallel reduction step may replace "horizontally" as well as "vertically" independent redices (see Fig. 1).

4.10. Definition. $\langle t_1, t_2\rangle \in T(\text{SIG})^2$ is **contextually E-convergent** if there are $n \in \mathbb{N}$, $p_1, \ldots, p_n$, $q_1, \ldots, q_n \in \text{BOOT}$, $t_1^1, \ldots, t_n^1$, $t_1^2, \ldots, t_n^2 \in T(\text{SIG})$ such that

(i) $(p_1 \wedge q_1) \vee \cdots \vee (p_n \wedge q_n) =_{\text{BSPEC}} \text{true}$,

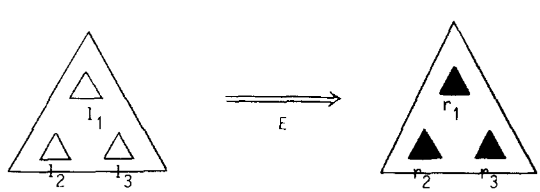


Figure 1

(ii) for all $1 \leqslant i \leqslant n$,

$$\begin{array}{ccc} t_1 & \xrightarrow[*]{E;\,p_i} & t_i^1 \\ & & \big\Downarrow {=}\ BE \\ t_2 & \xrightarrow[E;\,q_i]{*} & t_i^2. \end{array}$$

4.11. LEMMA. *Let* $\langle \mathrm{BPAR}, \mathrm{Log(PSPEC)} \rangle$ *be persistent and* E *be contextually normalizing. Let* $A \in \mathrm{Log(PSPEC)}$. *If* $\langle t_1, t_2 \rangle \in T(\mathrm{SIG})$ *is contextually E-convergent, then for all* $f, g \in Z(G(\mathrm{SIG}(A)))$ $f \Rrightarrow_{BE} g$ *implies*

$$\begin{array}{ccc} ft_1 & \xrightarrow[*]{E \cup LE(A)} & u_1 \\ & & \big\Downarrow {=}\ BE \\ gt_2 & \xrightarrow[E \cup LE(A)]{*} & u_2 \end{array}$$

for some $u_1, u_2 \in G(\mathrm{SIG}(A))$.

Proof. By assumption, there are $n \in \mathbb{N}$, $p_1,..., p_n$, $q_1,..., q_n \in \mathrm{BOOT}$, $t_1^1,..., t_n^1$, $t_1^2,..., t_n^2 \in T(\mathrm{SIG})$ such that 4.10(i), (ii) hold true. Let $f, g \in Z(G(\mathrm{SIG}(A)))$ with $f \Rrightarrow_{BE} g$.

Since E is contextually normalizing, Lemma 4.6 implies $f \xrightarrow{*}_{E \cup LE(A)} f'$ and $g \xrightarrow{*}_{E \cup LE(A)} g'$ for some $f', g' \in Z(G(\mathrm{BSIG}(A)))$.

Assume that for all $1 \leqslant i \leqslant n$ $(f'p_i \wedge f'q_i) = \mathrm{false}$ is in $LE(A)$. Then

$$\mathrm{true} =_{\mathrm{BSPEC}(A)} (f'p_1 \wedge f'q_1) \vee \cdots \vee (f'p_n \wedge f'q_n) =_{\mathrm{BSPEC}(A)} \mathrm{false}$$

which contradicts Lemma 4.2(1). Hence again by Lemma 4.2(1), there is $1 \leqslant i \leqslant n$ such that $(f'p_i \wedge f'q_i) = \mathrm{true}$ is in $LE(A)$. Thus $f'p_i = \mathrm{true}$ is in $LE(A)$. Analogously, one obtains that $g'q_i = \mathrm{true}$ is in $LE(A)$. Therefore Lemma 4.4 implies

$$\begin{array}{ccc} ft_1 & \xrightarrow[*]{E \cup LE(A)} & ft_i^1 \\ & & \big\Downarrow {=}\ BE \\ gt_2 & \xrightarrow[E \cup LE(A)]{*} & gt_i^2. \end{array} \quad \blacksquare$$

5. CRITICAL PAIR CONDITIONS FOR CONSISTENCY

This section is the most technical one. We show that contextual convergence of certain critical pairs is sufficient for confluence, coherence, and commutativity of $(E - BE) \cup LE(A)$ (cf. 3.3, 3.5). The assumptions of Section 4 are still valid.

To prepare the critical pair conditions we introduce superposition relations (5.1 and 5.8) as those reductions where the left-hand side of the applied equation $l = r$ overlaps a given prefix t of the term to be reduced (see Fig. 2).

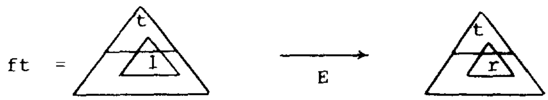


FIGURE 2

5.1. DEFINITION. The **simple superposition relation generated by E**, $\{\rightarrow_{E;f;t}\}_{f\in Z(T(\mathrm{SIG})), t\in T(\mathrm{SIG})-X}$, is the family of smallest relations on $T(\mathrm{SIG})$ such that

(i) for all $l=r$ in E $ft \rightarrow_{E;f;t} fr$ if $ft=fl$,

(ii) for all $\sigma\in OP$, $f\sigma(t_1,..., t_i,..., t_n) \rightarrow_{E;f;\sigma(t_1,...,t_i,...,t_n)} f\sigma(t_1,..., t_i',..., t_n)$, if $ft_i \rightarrow_{E;f;t_i} ft_i'$.

Let $n(ft \rightarrow_{E;f;t} t')$ (resp. $n(t\rightarrow_E t')$) denote the least number of derivation steps 5.1(i), (ii) (resp. 2.11(i), (ii)) that lead to $ft\rightarrow_{E;f;t} t'$ (resp. $t\rightarrow_E t'$).

5.2. PROPOSITION. *If $ft \rightarrow_{E;f;t} t'$, then there are $l=r$ in E, $t_0\in T(\mathrm{SIG})$, $t_1\in T(\mathrm{SIG})-X$ and $x\in X$ such that $t=t_0[t_1/x]$, $ft_1=fl$,* and $t'=f[fr/x](t_0)$, *i.e., l "overlaps" t in ft.*

Proof. Straightforward induction on $n(ft\rightarrow_{E;f;t} t')$. ∎

5.3. PROPOSITION. *Let $t, t'\in T(\mathrm{SIG})$, and $f\in Z(T(\mathrm{SIG}))$ such that $ft\rightarrow_E t'$, but not $ft\rightarrow_{E;f;t} t'$. Then there are $x\in \mathrm{var}(t)$ and $t_x\in T(\mathrm{SIG})$ such that $fx\rightarrow_E t_x$, $n(fx\rightarrow_E t_x)\leqslant n(ft\rightarrow_E t')$, and*

(i) *$t'=f[t_x/x](t)$ if t has unique variable occurrences,*

(ii) *$t' \overset{*}{\rightarrow}_E f[t_x/x](t)$, otherwise.*

Proof. Straightforward induction on $n(ft\rightarrow_E t')$. ∎

5.4. DEFINITION. E is **linear** if for each $l=r$ in E each variable occurs at most once in l.

The next lemma provides a syntactical criterion for the commutativity property in the consistency theorem 3.5.

5.5. LEMMA. *Suppose that $\langle$BPAR, Log(PSPEC)$\rangle$ is persistent, $E-BE$ is linear, for each $l=r$ in $E-BE$ $\mathrm{var}(r)\subseteq\mathrm{var}(l)$ and l does not contain operation symbols of POP $-$ $\{true, false\}$. Then for all $A\in$ Log(PSPEC) $(E-BE)\cup LE(A)$ commutes with $(\Delta(A)\cup\Delta(A)^{-1})-(LE(A)\cup LE(A)^{-1})$.*

Proof. Let $R=(E-BE)\cup LE(A)$, $R'=(\Delta(A)\cup\Delta(A)^{-1})-(LE(A)\cup LE(A)^{-1})$ and

$$\begin{array}{ccc} & & t_1 \\ & \overset{R}{\nearrow} & \\ t & & \\ & \underset{R'}{\searrow} & \\ & & t_2. \end{array}$$

We show

$$\begin{array}{ccc} t_1 & & \\ & \overset{=}{\underset{R'}{\searrow}} & \\ & & u \\ & \overset{=}{\underset{R}{\nearrow}} & \\ t_2 & & \end{array} \qquad \text{for some } u, \qquad (*)$$

by induction on $n(t\to_R t_1)+n(t\to_{R'} t_2)$.

Case 1. $t=fl$ and $t_1=fr$ for some $l=r$ in R and $f\in Z(T(\mathrm{SIG}(A)))$.

Case 1.1. $t\to_{R';f;l} t_2$. By Proposition 5.2, there are $l'=r'$ in R', $t_0\in T(\mathrm{SIG}(A))$, $t_3\in T(\mathrm{SIG}(A))-X$ and $x\in X$ such that $l=t_0[t_3/x]$, $ft_3=fl'$, and $t_2=f[fr'/x](t_0)$.

Case 1.1.1. $l=r$ is in $E-BE$. Hence $l'\in\{\text{true, false}\}$ of l contains some $\sigma\in(\mathrm{POP}-\{\text{true, false}\})\cup A$. The second case contradicts an assumption of the lemma. Assume that $l'=\text{true}$. Since $l'=r'$ is in $\Delta(A)\cup\Delta(A)^{-1}$, we have $r'\in G(\mathrm{BSIG}(A))$. By Lemma 4.2(1), $r'\neq\text{false}$. By the assumption in 4.1, $r'\neq\text{true}$. Hence $r'\in G(\mathrm{BSIG}(A))-\{\text{true, false}\}$ and thus $l'=r'$ is in $LE(A)^{-1}$, in contradiction to the fact that $l'=r'$ is in R'. Therefore $l'\neq\text{true}$. Analogously, $l'\neq\text{false}$.

Case 1.1.2. $l=r$ is in $LE(A)$. Since $R'\subseteq\Delta(A)\cup\Delta(A)^{-1}$, Lemma 4.2(2) implies $t_2 \overset{=}{\to}_{LE(A)} t_1$.

Case 1.2. Not $t\to_{R';f;l} t_2$. By Proposition 5.3, there are $x\in\mathrm{var}(l)$ and $t_x\in T(\mathrm{SIG}(A))$ such that $fx\to_{R'} t_x$ and $t_2=f[t_x/x](l)$. Hence

$$\begin{array}{ccc} t_1=fr & & \\ & \overset{=}{\underset{R'}{\searrow}} & \\ & & f[t_x/x](r). \\ & \overset{}{\underset{R}{\nearrow}} & \\ t_2 & & \end{array}$$

Case 2. $t = fl'$ and $t_2 = fr'$ for some $l' = r'$ in R' and $f \in Z(T(\mathrm{SIG}(A)))$.

Case 2.1. $t \to_{R;f;l'} t_1$. By Proposition 5.2, there are $l = r$ in R, $t_0 \in T(\mathrm{SIG}(A))$, $t_3 \in T(\mathrm{SIG}(A)) - X$ and $x \in X$ such that $l' = t_0[t_3/x]$, $ft_3 = fl$, and $t_1 = f[fr/x](t_0)$.

Case 2.1.1. $l = r$ is in $E - BE$. Hence $l \in X \cup \{\text{true, false}\}$ or l contains some $\sigma \in (\mathrm{POP} - \{\text{true, false}\}) \cup A$. Both cases contradict an assumption.

Case 2.1.2. $l = r$ is in $LE(A)$. Hence $t_3 = l \notin \{\text{true, false}\}$ implies $t_0 = x$ and thus $l = l'$ and $t_1 = fr = r = r' = fr' = t_2$ because $r, r' \in \{\text{true, false}\}$, but by Lemma 4.2(1), true $\neq_{\mathrm{BSPEC}(A)}$ false.

Case 2.2. Not $t \to_{R;f;l'} t_1$. By Proposition 5.3, there are $x \in \mathrm{var}(l')$ and $t_x \in T(\mathrm{SIG}(A))$ such that $fx \to_R t_x$ and $t_1 = f[t_x/x](l')$. This contradicts the fact that l' does not contain variables.

Case 3. $t = \sigma(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_n)$, $t_1 = \sigma(u_1, \ldots, r_i, \ldots, u_j, \ldots, u_n)$, $t_2 = \sigma(u_1, \ldots, u_i, \ldots, s_j, \ldots, u_n)$, $u_i \to_R r_i$, and $u_j \to_{R'} s_j$.

Case 3.1. $i = j$. By the induction hypothesis,

$$r_i \overset{=}{\underset{R'}{\searrow}} v_i \overset{=}{\underset{R}{\nwarrow}} s_i \qquad \text{for some } v_i.$$

Hence,

$$t_1 \overset{=}{\underset{R'}{\searrow}} \sigma(u_1, \ldots, v_i, \ldots, u_n) \overset{=}{\underset{R}{\nwarrow}} t_2.$$

Case 3.2. $i \neq j$. Then

$$t_1 \underset{R'}{\searrow} \sigma(u_1, \ldots, r_i, \ldots, s_j, \ldots, u_n) \underset{R}{\nwarrow} t_2.$$

Hence (*) holds true and thus R commutes with R'. ∎

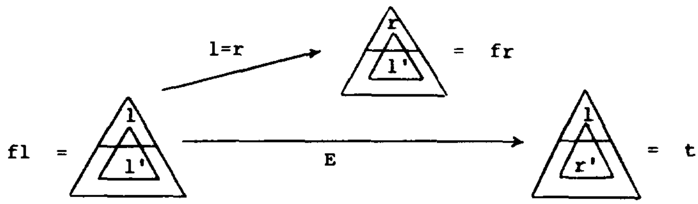


FIGURE 3

Using the superposition relation we can easily define a **critical pair** of E into $l=r$ as a pair of

(i) the substituted right-hand side fr and

(ii) the result t of reducing fl by some equation $l'=r'$, where l' overlaps l, as shown in Fig. 3.

5.6. DEFINITION. Let $fl \to_{E;f;l} t$ and $l=r$ be a SIG-equation. $\langle fr, t\rangle$ is called a **critical pair of E into l = r**.

5.7. LEMMA. *Let* $A \in \text{Log(PSPEC)}$. *Suppose that for each* $l=r$ *in* $E-BE$ $\text{op}(l)$ *contains at least one operation symbol of* $OP-BOP$. *Then there are no critical pairs of* $E-BE$ *into* $LE(A)$ *or of* $LE(A)$ *into* $E-BE$.

Proof. Assume that $\langle t_1, t_2\rangle$ is a critical pair of $E-BE$ into $LE(A)$. Then there are $l=r$ in $LE(A)$ and $f \in Z(T(\text{SIG}(A)))$ such that $fl \to_{E-BE;f;l} t_2$ and $t_1 = fr$. By Proposition 5.2, there are $l'=r'$ in $E-BE$, $t_0 \in T(\text{SIG}(A))$, $t_3 \in T(\text{SIG}(A)) - X$ and $x \in X$ such that $l = t_0[t_3/x]$, $ft_3 = fl'$, and $t_2 = f[fr'/x](t_0)$. Since $\text{var}(l) = \varnothing$, we have

$$l = t_0[ft_3/x] = t_0[fl'/x],$$

which contradicts the fact that $\text{op}(l) \subseteq BOP \cup A$, but $\text{op}(l') \cap (OP-BOP) \neq \varnothing$.

Assume that $\langle t_1, t_2\rangle$ is a critical pair of $LE(A)$ into $E-BE$. Then there are $l=r$ in $E-BE$ and $f \in Z(T(\text{SIG}(A)))$ such that $fl \to_{LE(A);f;l} t_1$ and $t_2 = fr$. By Proposition 5.2, there are $l'=r'$ in $LE(A)$, $t_0 \in T(\text{SIG}(A))$, $t_3 \in T(\text{SIG}(A)) - X$ and $x \in X$ such that $l = t_0[t_3/x]$, $ft_3 = fl'$, $t_1 = f[fr'/x](t_0)$. Hence $\text{sort}(t_3) = \text{bool}$, but $t_3 \notin \{\text{true, false}\}$. Thus $t_0 = x$ and we obtain

$$l = t_3 \qquad \text{and} \qquad l' = fl' = ft_3$$

so that $\text{op}(l) \subseteq BOP$, which contradicts the assumption that $\text{op}(l) \cap (OP-BOP) \neq \varnothing$. ∎

In parallel reductions we may have several equations $l_i = r_i$ applied to the same term u. If all outermost l_i overlap a given prefix t of u, we get a "superposing" parallel reduction (see Fig. 4).

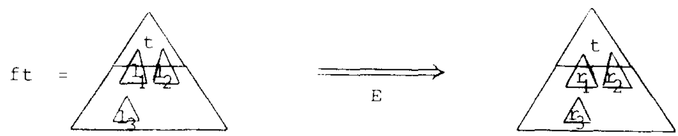


FIGURE 4

5.8. DEFINITION. The **parallel superposition relation generated by E**, $\{\Rightarrow_{E;f;t}\}_{f\in Z(T(\mathrm{SIG})),\, t\in T(\mathrm{SIG})-X}$, is the family of smallest relations on $T(\mathrm{SIG})$ (resp. $Z(T(\mathrm{SIG}))$) such that

(i) for all $l=r$ in E, $ft \Rightarrow_{E;f;t} gr$ if $ft=fl$ and $f \Rrightarrow_E g$,

(ii) for all $\sigma\in OP$, $f\sigma(t_1,\dots,t_n) \Rightarrow_{E;f;\sigma(t_1,\dots,t_n)} \sigma(t_1',\dots,t_n')$ if $\exists\ 1\leqslant i\leqslant n$: $ft_i \Rightarrow_{E;f;t_i} t_i'$ and $\forall\ 1\leqslant i\leqslant n$: $ft_i \Rrightarrow_E t_i'$.

Let $n(ft \Rightarrow_{E;f;t} t')$ (resp. $n(t \Rightarrow_E t')$) denote the least number of derivation steps 5.8(i), (ii) (resp. 4.9.(i)–(v)) that lead to $ft \Rightarrow_{E;f;t} t'$ (resp. $t \Rightarrow_E t'$).

5.9. PROPOSITION. *If $ft \Rightarrow_{E;f;t} t'$, then there are $t_0\in T(\mathrm{SIG})$, $g\in Z(T(\mathrm{SIG}))$, $n>0$, and for all $1\leqslant i\leqslant n$, $l_i=r_i$ in E, $t_i\in T(\mathrm{SIG})-X$, and $x_i\in X$ such that $t=t_0[t_i/x_i \mid 1\leqslant i\leqslant n]$, $ft_i=fl_i$, $f\Rrightarrow_E g$ and $t'=f[gr_i/x_i \mid 1\leqslant i\leqslant n](t_0)$, i.e., $l_1,\dots,l_n$ "overlap" t in ft.*

Proof. Straightforward induction on $n(ft \Rightarrow_{E;f;t} t')$. ▌

5.10. PROPOSITION. *Let $t, t'\in T(\mathrm{SIG})$ and $f\in Z(T(\mathrm{SIG}))$ such that t has unique variable occurrences, $ft \Rightarrow_E t'$, but not $ft \Rightarrow_{E;f;t} t'$. Then there are $n>0$, $x_1,\dots,x_n\in \mathrm{var}(t)$ and $t_1,\dots,t_n\in T(\mathrm{SIG})$ such that $fx_i \Rightarrow_E t_i$ and $t'=f[t_i/x_i \mid 1\leqslant i\leqslant n](t)$.*

Proof. Straightforward induction on $n(ft \Rightarrow_E t')$. ▌

Parallel critical pairs of E into $l=r$ arise in situations like the one shown in Fig. 5, where $l_1=r_1$, $l_2=r_2$, and $l_3=r_3$ are in E.

A more complicated case of a parallel overlapping can occur if l_1 shares a subterm of l **and** a prefix of l_2, e.g., see Fig. 6. Applying $(l_1=r_1)\in E$ on one hand and $(l=r)$, $(l_2=r_2)\in E'$ on the other hand leads to a **recursive critical pair** of E into E'.

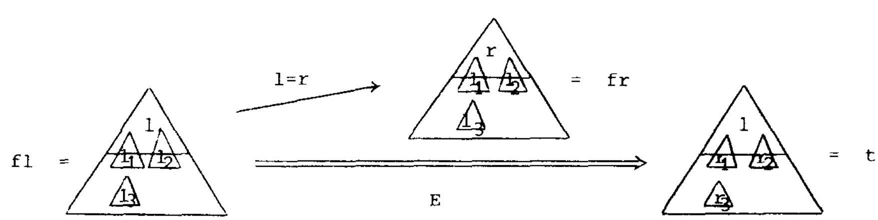


FIGURE 5

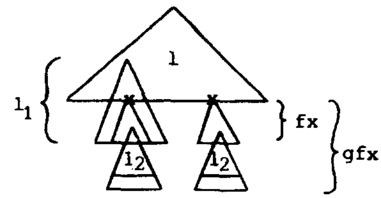


FIGURE 6

5.11. DEFINITIONS. Let $fl \Rightarrow_{E;f;l} t$ and $l = r$ be a SIG-equation. $\langle fr, t \rangle$ is called a **parallel critical pair of E into l = r.**

Let E' be a set of SIG-equations, $l = r$ in E', and $g, h \in Z(T(\mathrm{SIG}))$. Suppose that $fl \rightarrow_{E;f;l} t$ and $f \Rrightarrow_{E'} g$. Then $\langle t, gr \rangle$ is a **recursive critical pair of E into E′.**

E is **terminating** if there are no infinite sequences $t_1 \rightarrow_E t_2 \rightarrow_E t_3 \rightarrow_E \cdots$ t is **E-reducible** if $t \rightarrow_E t'$ for some t'.

5.12. CRITICAL PAIR THEOREM. *Suppose that* $\langle \mathrm{BPAR}, \mathrm{Log}(\mathrm{PSPEC}) \rangle$ *is persistent.*

Let $E - BE$ be linear, terminating, and contextually normalizing (cf. 4.5), *for each $l = r$ in BE* $\mathrm{var}(r) \subseteq \mathrm{var}(l)$ *and for each $l = r$ in $E - BE$, l contains at least one operation symbol of $OP - BOP$.*

Let $A \in \mathrm{Log}(\mathrm{PSPEC})$. *$(E - BE) \cup LE(A)$ is confluent and coherent w.r.t. A (cf.* 3.3) *if*

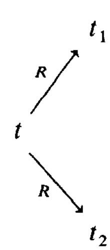
(i) *all critical pairs of $E - BE$ into $E - BE$,*

(ii) *all parallel critical pairs of BE into $E - BE$,*

(iii) *all recursive critical pairs of $E - BE$ into BE*

are contextually $(E - BE)$-convergent (cf. 4.10).

Proof. Let $R = (E - BE) \cup LE(A)$. A simple proof by Noetherian induction w.r.t. $\rightarrow_R$ shows that R is confluent w.r.t. A if for all $t, t_1, t_2 \in G(\mathrm{SIG}(A))$



implies $t_1 \downarrow_{R,A} t_2$ (cf. 3.3). (1)

Suppose that for all $t, t_1, t_2 \in G(\mathrm{SIG}(A))$

$$\begin{array}{ccc} & & t_1 \\ & \nearrow_R & \\ t & & \\ & \searrow_R & \\ & & t_2 \end{array} \quad \text{or} \quad \begin{array}{ccc} & & t_1 \\ & \nearrow_R & \\ t & & \\ & \Searrow_{BE} & \\ & & t_2 \end{array} \quad \text{implies} \quad \begin{array}{ccc} t_1 & \xrightarrow[*]{R} & t_3 \\ & & \Big\Downarrow {=}\ BE \\ t_2 & \xrightarrow[R]{*} & t_4 \end{array} \tag{2}$$

for some $t_3, t_4 \in G(\mathrm{SIG}(A))$. We prove by Noetherian induction w.r.t. $\rightarrow_R$ that (2) implies (1) and coherence of R w.r.t. A.

By Lemma 4.6, R is normalizing w.r.t. A. Hence if t_3 is not R-reducible, we have $t_3 \in G(\mathrm{BSIG}(A))$ and thus $t_4 \in G(\mathrm{BSIG}(A))$ so that $t_1 \downarrow_{R,A} t_2$. If t_3 is R-reducible, then $t_3 \rightarrow_R t_5$ for some $t_5 \in \mathrm{G}(\mathrm{SIG}(A))$. We obtain $t_5 \downarrow_{R,A} t_4$ by induction hypothesis and thus $t_1 \downarrow_{R,A} t_2$. Hence it remains to show (2):

(a) Let

$$\begin{array}{ccc} & & t_1 \\ & \nearrow_R & \\ t & & \\ & \searrow_R & \\ & & t_2 \end{array}$$

Induction on $n(t \rightarrow_R t_1) + n(t \rightarrow_R t_2)$ leads to (2):

Case 1. $t = fl$ and $t_1 = fr$ for some $l = r$ in R and $f \in Z(T(\mathrm{SIG}(A)))$.

Case 1.1. $t \rightarrow_{R;f;l} t_2$. Then $\langle t_1, t_2 \rangle$ is a critical pair of R into $l = r$.

Case 1.1.1. $\langle t_1, t_2 \rangle$ is a critical pair of $E - BE$ into $l = r$ and $l = r$ is in $E - BE$. By assumption, $\langle t_1, t_2 \rangle$ is contextually $(E - BE)$-convergent. Since $t_1, t_2 \in G(\mathrm{SIG}(A))$, we conclude

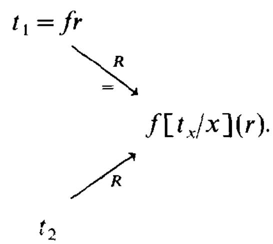
$$\begin{array}{ccc} t_1 & \xrightarrow[*]{R} & u_1 \\ & & \Big\Downarrow {=}\ BE \\ t_2 & \xrightarrow[R]{*} & u_2 \end{array}$$

for some $u_1, u_2 \in G(\mathrm{SIG}(A))$ from Lemma 4.11.

Case 1.1.2. $\langle t_1, t_2 \rangle$ is a critical pair of $E - BE$ into $LE(A)$ or of $LE(A)$ into $E - BE$. This contradicts Lemma 5.7.

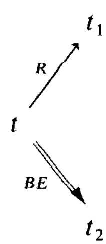
Case 1.1.3. $\langle t_1, t_2 \rangle$ is a critical pair of $LE(A)$ into $l=r$ and $l=r$ is in $LE(A)$. Hence $t \to_{LE(A)} t_2$, and we conclude from Lemma 4.2(2) that $t_2 \overset{=}{\Rightarrow}_{LE(A)} t_1$.

Case 1.2. Not $t \to_{R;f;l} t_2$. By Proposition 5.3, there are $x \in \text{var}(l)$ and $t_x \in T(\text{SIG}(A))$ such that $fx \to_R t_x$ and $t_2 = f[t_x/x](l)$. Hence



Case 2. Analogously to Case 3 of Lemma 5.5 (with $R' = R$).

(b) Let

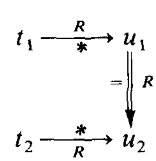


Induction on $n(t \to_R t_1) + n(t \Rightarrow_{BE} t_2)$ leads to (2):

Case 1. $t = fl$ and $t_1 = fr$ for some $l = r$ in R and $f \in Z(T(\text{SIG}))$.

Case 1.1. $t \Rightarrow_{BE;f;l} t_2$. Then $\langle t_1, t_2 \rangle$ is a parallel critical pair of BE into $l = r$.

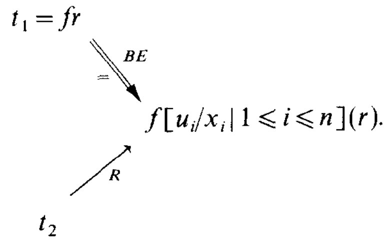
Case 1.1.1. $l = r$ is in $E - BE$. By assumption, $\langle t_1, t_2 \rangle$ is contextually $(E - BE)$-convergent. Since $t_1, t_2 \in G(\text{SIG}(A))$, we conclude



for some $u_1, u_2 \in G(\text{SIG}(A))$ from Lemma 4.11.

Case 1.1.2. $l = r$ is in $LE(A)$. Since $t \overset{*}{\to}_{BE} t_2$, Lemma 4.2(2) implies $t_2 \overset{=}{\Rightarrow}_{LE(A)} t_1$.

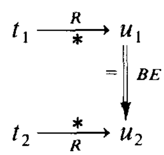
Case 1.2. Not $t \Rightarrow_{BE;f;l} t_2$. By Proposition 5.10, there are $n > 0$, $x_1, \dots, x_n \in \text{var}(l)$ and $u_1, \dots, u_n \in T(\text{SIG}(A))$ such that $fx_i \Rightarrow_{BE} u_i$ and $t_2 = f[u_i/x_i \mid 1 \leqslant i \leqslant n](l)$. Hence



Case 2. $t = fl$, $t_2 = gr$, and $f \overset{=}{\Rightarrow}_{BE}\, g$ for some $l = r$ in BE and $f, g \in Z(T(\mathrm{SIG}(A)))$.

Case 2.1. $t \rightarrow_{R;f;l} t_1$. Then $\langle t_1, gr\rangle$ is a recursive critical pair of R into BE.

Case 2.1.1. $t \rightarrow_{E-BE;f;l} t_1$. By assumption, $\langle t_1, t_2\rangle$ is contextually $(E-BE)$-convergent. Hence by Lemma 4.11,

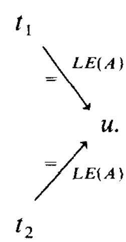


for some $u_1, u_2 \in G(\mathrm{SIG}(A))$.

Case 2.1.2. $t \rightarrow_{LE(A);f;l} t_1$. By Proposition 5.2, there are $l' = r'$ in $LE(A)$, $t_0 \in T(\mathrm{SIG}(A))$, $t_4 \in T(\mathrm{SIG}(A)) - X$, and $x \in X$ such that $l = t_0[t_4/x]$ and $ft_4 = fl'$. By general assumptions on E and $LE(A)$ (cf. Sect. 4), $t_0 = x$. Hence $t = fl = ft_4 = l' \in \mathrm{G(BSIG}(A))$ so that

$$t_1 =_{\mathrm{BSPEC}(A)} t =_{\mathrm{BSPEC}(A)} t_2.$$

By Lemma 4.2, persistency of $\langle \mathrm{BPAR}, \mathrm{Log(PSPEC)}\rangle$ implies $t_i =_{\mathrm{BSPEC}(A)} r'$ for $i = 1, 2$. Therefore



Case 2.2. Not $t \rightarrow_{R;f;l} t_1$. By Proposition 5.3, there are $x \in \mathrm{var}(l)$ and $t_x \in T(\mathrm{SIG}(A))$ such that $fx \rightarrow_R t_x$, $n(fx \rightarrow_R t_x) \leqslant n(t \rightarrow_R t_1)$, and $t_1 \overset{*}{\rightarrow}_R f[t_x/x](l)$.

Case 2.2.1. $fx = gx$. Then

$$\begin{array}{ccc} t_1 & \xrightarrow[*]{R} & f[t_x/x](l) \\ & & \Big\Downarrow {\scriptstyle BE} \\ t_2 = gr = g[fx/x](r) & \xrightarrow[R]{=} & g[t_x/x](r). \end{array}$$

Case 2.2.2. $fx \Rightarrow_{BE} gx$. Since $n(fx \Rightarrow_{BE} gx) < n(t \Rightarrow_{BE} t_2)$, we conclude by induction hypothesis that

$$\begin{array}{ccc} t_x & \xrightarrow[*]{R} & u_x \\ & & {\scriptstyle =}\Big\Downarrow {\scriptstyle BE} \\ gx & \xrightarrow[R]{*} & v_x \end{array}$$

for some $u_x, v_x \in G(\mathrm{SIG}(A))$. Therefore

$$\begin{array}{ccccc} t_1 & \xrightarrow[*]{R} & f[t_x/x](l) & \xrightarrow[*]{R} & f[u_x/x](l) \\ & & & & \Big\Downarrow {\scriptstyle BE} \\ & & t_2 = gr & \xrightarrow[R]{*} & g[v_x/x](r). \end{array}$$

Case 3. $t = \sigma(u_1,..., u_i,..., u_n)$, $t_1 = \sigma(u_1,..., u_i',..., u_n)$, $t_2 = \sigma(v_1,..., v_n)$, $u_i \rightarrow_R u_i'$, and there is $I \subseteq \{1,..., n\}$ such that for all $k \in I$ $u_k \Rightarrow_{BE} v_k$ and for all $k \in \{1,..., n\} - I$ $u_k = v_k$.

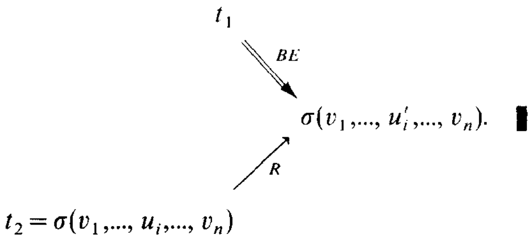
Case 3.1. $i \in I$. By the induction hypothesis,

$$\begin{array}{ccc} u_i' & \xrightarrow[*]{R} & r_i \\ & & {\scriptstyle =}\Big\Downarrow {\scriptstyle BE} \\ v_i & \xrightarrow[R]{*} & s_i \end{array}$$

for some $r_i, s_i \in G(\mathrm{SIG}(A))$. Hence

$$\begin{array}{ccc} t_1 & \xrightarrow[*]{R} & \sigma(u_1,..., r_i,..., u_n) \\ & & {\scriptstyle =}\Big\Downarrow {\scriptstyle BE} \\ t_2 & \xrightarrow[R]{*} & \sigma(v_1,..., s_i,..., v_n). \end{array}$$

Case 3.2. $i \notin I$. Then



5.13. Example (cf. 2.7). Let PAR = ⟨DATA, SET⟩ and BPAR = ⟨DATA, BSET⟩. One immediately verifies all assumptions of Theorem 5.12 (cf. Examples 2.14 and 4.8) except for the termination of $E-BE$ and the critical pair conditions. For the termination we refer to the recursive path ordering method (cf. [5, 7]), which applied to $E-BE = \{(e6), \ldots, (e13)\}$ provides a straightforward termination proof.

Assume that there is a critical pair of $E-BE$ into $E-BE$ or a recursive critical pair of $E-BE$ into BE. In both cases we would have $l=r$ in E, $f \in Z(T(\mathrm{SIG}))$ and $t \in T(\mathrm{SIG})$ such that $fl \rightarrow_{E-BE;f;l} t$. By Proposition 5.2, there would be $l'=r'$ in $E-BE$, $t_0 \in T(\mathrm{SIG})$, $t_1 \in T(\mathrm{SIG})-X$ and $x \in X$ such that $l = t_0[t_1/x]$, $ft_1 = fl'$, and $t = f[fr'/x](t_0)$. Since for all $l=r$ in $E-BE$ $\mathrm{op}(l) \cap (OP-BOP) = \{root(l)\}$, we conclude $t_0 = x$, $l = l'$, and $r = r'$. Thus we have no recursive critical pair of $E-BE$ into BE, and if $\langle fr, t\rangle$ is a critical pair of $E-BE$ into $E-BE$, then $fr = fr' = t$.

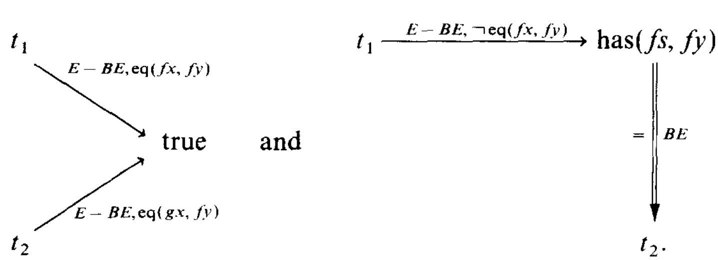
Let $\langle t_1, t_2\rangle$ be a parallel critical pair of BE into $E-BE$. Then $t_1 = fr$ and $fl \Rightarrow_{BE;f;l} t_2$ for some $l=r$ in $E-BE$ and $f \in Z(T(\mathrm{SIG}))$. By Proposition 5.9, there are $t_0 \in T(\mathrm{SIG})$, $g \in Z(T(\mathrm{SIG}))$, $n>0$, and for all $1 \leqslant i \leqslant n$, $l_i = r_i$ in BE, $u_i \in T(\mathrm{SIG})-X$, and $x_i \in X$ such that $l = t_0[u_i/x_i \mid 1 \leqslant i \leqslant n]$, $fu_i = fl_i$, $f \overset{=}{\Rightarrow}_{BE} g$, and $t_2 = f[gr_i/x_i \mid 1 \leqslant i \leqslant n](t_0)$.

Case 1. $t_0 = \mathrm{has}(x_1, y)$, $u_1 = \mathrm{ins}(s, x)$, and $l=r$ is $e7$.

Case 1.1. $fs = \mathrm{ins}(fs', fx)$ and $l_1 = r_1$ is $e4$. Then

$$t_1 = \mathrm{ifb}(\mathrm{eq}(fx, fy), \mathrm{true}, \mathrm{has}(fs, fy)),$$
$$t_2 = \mathrm{has}(\mathrm{ins}(gs', gx), fy)$$

so that



Since

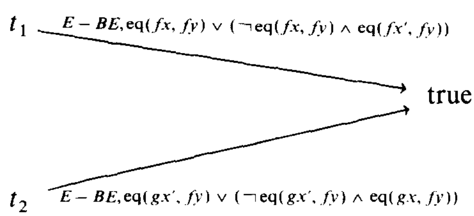
$$(\mathrm{eq}(fx, fy) \wedge \mathrm{eq}(gx, fy)) \vee \neg\mathrm{eq}(fx, fy)$$
$$=_{\mathrm{BSPEC}} \mathrm{eq}(fx, fy) \vee \neg\mathrm{eq}(fx, fy) =_{\mathrm{BSPEC}} \mathrm{true},$$

$\langle t_1, t_2 \rangle$ is contextually $(E - BE)$-convergent.

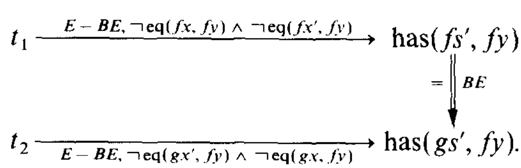
Case 1.2. $fs = \mathrm{ins}(fs', fx')$ and $l_1 = r_1$ is $e5$. Then

$$t_1 = \mathrm{ifb}(\mathrm{eq}(fx, fy), \mathrm{true}, \mathrm{has}(fs, fy)),$$
$$t_2 = \mathrm{has}(\mathrm{ins}(\mathrm{ins}(gs', gx), gx'), fy)$$

so that



and



Since

$$((\mathrm{eq}(fx, fy) \vee (\neg\mathrm{eq}(fx, fy) \wedge \mathrm{eq}(fx', fy)))$$
$$\wedge (\mathrm{eq}(gx', fy) \vee (\neg\mathrm{eq}(gx', fy) \wedge \mathrm{eq}(gx, fy))))$$
$$\vee (\neg\mathrm{eq}(fx, fy) \wedge \neg\mathrm{eq}(fx', fy) \wedge \neg\mathrm{eq}(gx', fy) \wedge \neg\mathrm{eq}(gx, fy))$$
$$=_{\mathrm{BSPEC}} ((\mathrm{eq}(fx, fy) \vee \mathrm{eq}(fx', fy)) \wedge (\mathrm{eq}(gx', fy) \vee \mathrm{eq}(gx, fy)))$$
$$\vee (\neg\mathrm{eq}(fx, fy) \wedge \neg\mathrm{eq}(fx', fy))$$
$$=_{\mathrm{BSPEC}} \mathrm{eq}(fx, fy) \vee \mathrm{eq}(fx', fy) \vee (\neg\mathrm{eq}(fx, fy) \wedge \neg\mathrm{eq}(fx', fy))$$
$$=_{\mathrm{BSPEC}} \mathrm{true},$$

$\langle t_1, t_2 \rangle$ is contextually $(E - BE)$-convergent.

Case 2. $t_0 = \mathrm{del}(x_1, y)$, $u_1 = \mathrm{ins}(s, x)$, and $l = r$ is $e9$. Analogously to Case 1 we can deduce that $\langle t_1, t_2 \rangle$ is contextually $(E - BE)$-convergent.

Hence all parallel critical pairs of BE into $E-BE$ are contextually $(E-BE)$-convergent, and we conclude from Theorem 5.12 that for all $A \in \text{Log(PSPEC)}$ $(E-BE) \cup LE(A)$ is confluent and coherent w.r.t. A

Theorems 3.5 and 5.12 and Lemmata 4.6 and 5.5 imply

5.14. Consistency Theorem II. *Suppose that* $\langle \text{BPAR}, \text{Log(PSPEC)} \rangle$ *is persistent. Let* $E-BE$ *be linear, terminating, and contextually normalizing, for each* $l=r$ *in* BE, $\text{var}(r) \subseteq \text{var}(l)$ *and, for each* $l=r$ *in* $E-BE$, l *contains at least one operation symbol of* $OP-BOP$, *but no operation symbols of* $POP - \{\text{true, false}\}$. *If all critical pairs of* $E-BE$ *into* $E-BE$, *all parallel critical pairs of* BE *into* $E-BE$ *and all recursive critical pairs of* $E-BE$ *into* BE *are contextually* $(E-BE)$*-convergent, then* PAR *is consistent w.r.t.* $\langle \text{BPAR}, \text{Log(PSPEC)} \rangle$.

5.15. Example (cf. 2.7). Let $\text{PAR} = \langle \text{DATA, SET} \rangle$ and $\text{BPAR} = \langle \text{DATA, BSET} \rangle$. Using Theorem 5.14 we conclude from Example 5.13 that PAR is consistent w.r.t. $\langle \text{BPAR}, \text{Log(PSPEC)} \rangle$. By Example 4.8, PAR is complete w.r.t. $\langle \text{BPAR}, \text{Log(PSPEC)} \rangle$. Hence by Example 2.14, the decomposition lemma for persistency (2.6) implies that $\langle \text{PAR}, \text{Log(PSPEC)} \rangle$ is persistent.

Putting together all "syntactical" criteria developed in this paper we obtain

5.16. Persistency Theorem II. $\langle \text{PAR}, \text{Log(PSPEC)} \rangle$ *is persistent if* PAR *contains a "base" specification* $BPAR$ *such that*

(i) *for all* $\sigma \in BOP$, $\text{sort}(\sigma) \in PS$ *implies* $\sigma \in POP$,

(ii) *for all* $l=r$ *in* BE, $\text{var}(r) \subseteq \text{var}(l)$, *and* $\text{sort}(l) \in PS$ *implies that* $l=r$ *is in* PE,

(iii) *for all* $l=r$ *in* $E-BE$, l *contains at least one operation symbol of* $OP-BOP$, *but no operation symbols of* $POP - \{\text{true, false}\}$,

(iv) $E-BE$ *is linear, terminating and contextually normalizing,*

(v) *all critical pairs of* $E-BE$ *into* $E-BE$, *all parallel critical pairs of* BE *into* $E-BE$, *and all recursive critical pairs of* $E-BE$ *into* BE *are contextually* $(E-BE)$*-convergent.*

(*Note also the "Boolean assumptions" at the beginning of Sect.* 4.)

Acknowledgments

I am grateful to Anno Langen for many fruitful discussions, which accompanied his implementation of persistency criteria. Further thanks are due to the referee, who suggested a number of technical improvements.

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