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Existence of Good and Best Approximations on Unbounded Domains by Exponential Sums in Several Independent Variables*

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In this paper we consider the problem of using exponential sums to approximate a given complex-valued function f defined on the possibly unbounded domain $\mathscr{D}$ in $\mathbb{R}^m$. We establish the existence of a best approximation from the set of exponential sums having order at most n and formulate a Weierstrass-type density theorem. In so doing we extend previously known results which apply only in the special cases where $\mathscr{D}$ is bounded or where $m = 1$.

1. INTRODUCTION

Let $\mathscr{D}$ be a nonvoid open subset of $\mathbb{R}^m$ and for $1 \leqslant p \leqslant \infty$ let $L_p(\mathscr{D})$ be defined in the usual manner with $\| \ \|_p$ being the associated norm. Let $C_0(\mathscr{D})$ denote the space of those functions $f \in C(\mathscr{D})$ having the property that given any $\epsilon > 0$ there exists a compact set $K \subset \mathscr{D}$ such that $|f(\mathbf{t})| < \epsilon$ whenever $\mathbf{t} \in \mathscr{D} \backslash K$. A function $y \in C^\infty(\mathbb{R}^m)$ will be called an exponential sum of order n provided that the linear space $\mathscr{L}[y]$ spanned by the functions

$$[D_1^{j_1} \cdots D_m^{j_m}]\, y(\mathbf{t}), \quad j_1, \ldots, j_m = 0, 1, \ldots, \qquad D_i = \partial/\partial t_i\,, \quad i = 1, \ldots, m$$

has dimension n, cf. [2, p. 143]. Given $\mathbf{S} \subseteq \mathbb{C}^m$ and $n = 0, 1, \ldots$, we define $V_n(\mathbf{S})$ to be the set of all exponential sums y of order at most n which can be expressed in the form

$$y(\mathbf{t}) = \sum_{j=1}^{l} p_j(\mathbf{t}) \exp(\boldsymbol{\lambda}_j \cdot \mathbf{t})$$

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where $p_1(\mathbf{t}),\ldots,p_l(\mathbf{t})$ are polynomials in the components $t_1,\ldots,t_m$ of $\mathbf{t}$, where $\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_l\in\mathbf{S}$, and where $\boldsymbol{\lambda}_j\cdot\mathbf{t}=\lambda_{j1}t_1+\cdots+\lambda_{jm}t_m$, cp. [2, p. 144]. We also define

$$V_\infty(\mathbf{S})=\bigcup_{n=1}^{\infty}V_n(\mathbf{S}).$$

In this paper we shall establish a Weierstrass-type density theorem by showing that $V_\infty(\mathbf{S})$ is a dense subset of $L_p(\mathscr{D})$ if $1\leqslant p<\infty$ and of $C_0(\mathscr{D})$ if $p=\infty$ provided that $\mathscr{D}$ and $\mathbf{S}$ satisfy mild hypotheses. We also establish the existence of a best $\|\ \|_p$-approximation to a given f from the set $V_n(\mathbf{S})$ when $\mathbf{S}$ is closed. In so doing we extend corresponding results from [3] which apply in the special case where $m=1$ and $\mathscr{D}$ is a semi-infinite interval and results from [2] which apply when $m=1$ and $\mathscr{D}$ is bounded.

2. The Spectral Set of $\mathscr{D}$

Given a nonvoid open set $\mathscr{D}\subseteq\mathbb{R}^m$ and $1\leqslant p\leqslant\infty$ we define the corresponding spectral set $U_p(\mathscr{D})$ to be the set of those $\boldsymbol{\lambda}\in\mathbb{C}^m$ for which the exponential sum $y(\mathbf{t})=\exp[\boldsymbol{\lambda}\cdot\mathbf{t}]$ lies in $L_p(\mathscr{D})$. For example, for the positive cone

$$\mathscr{D}=\{\mathbf{t}\in\mathbb{R}^m : t_i>0,\ i=1,\ldots,m\}$$

we find

$$U_p(\mathscr{D})=\{\boldsymbol{\lambda}\in\mathbb{C}^m : \operatorname{Re}\lambda_i<0,\ i=1,\ldots,m\}\qquad\text{if}\quad 1\leqslant p<\infty,$$
$$U_\infty(\mathscr{D})=\{\boldsymbol{\lambda}\in\mathbb{C}^m : \operatorname{Re}\lambda_i\leqslant 0,\ i=1,\ldots,m\}.$$

In general, $U_p(\mathscr{D})$ is convex. Indeed when $p=\infty$ the convexity is immediate, and when $1\leqslant p<\infty$ we may use Hölder's inequality to show that $\boldsymbol{\lambda}_1/p_1+\boldsymbol{\lambda}_2/p_2\in U_p(\mathscr{D})$ whenever $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2\in U_p(\mathscr{D})$, $p_1>1$, $p_2>1$, and $1/p_1+1/p_2=1$. Moreover, we also have

$$U_p(\mathscr{D}_1\cup\mathscr{D}_2)=U_p(\mathscr{D}_1)\cap U_p(\mathscr{D}_2)\qquad\text{if}\quad \mathscr{D}_1,\mathscr{D}_2\subseteq\mathbb{R}^m,$$
$$U_p(\alpha\mathscr{D}+\mathbf{t})=(1/\alpha)\,U_p(\mathscr{D})\qquad\text{if}\quad \alpha>0,\ \mathscr{D}\subseteq\mathbb{R}^m,\text{ and }\mathbf{t}\in\mathbb{R}^m,$$

and

$$U_p(\mathscr{D})=(1/p)\,U_1(\mathscr{D})\qquad\text{if}\quad\mathscr{D}\subseteq\mathbb{R}^m\text{ and }1\leqslant p<\infty.$$

If $\mathscr{D}$ is bounded we obviously have $U_p(\mathscr{D})=\mathbb{C}^m$. On the other hand, if $U_p(\mathscr{D})=\mathbb{C}^m$ and $1\leqslant p<\infty$ then $\mathscr{D}$ must have finite measure in $\mathbb{R}^m$ but need not be bounded, e.g., as is the case when $m=2$ and $\mathscr{D}$ is the "Gaussian star"

$$\mathscr{D}=\{\mathbf{t}\in\mathbb{R}^2 : t_1^2<\exp[-t_2^2]\text{ or }t_2^2<\exp[-t_1^2]\}.$$

In view of the following lemma, the interior, $U_p^0(\mathscr{D})$, of the spectral set will be of importance in the subsequent analysis.

LEMMA 1. *Let $\mathscr{D}$ be a nonvoid open subset of $\mathbb{R}^m$ and let $1 \leqslant p \leqslant \infty$. Then $V_\infty(U_p^0(\mathscr{D})) \subset L_p(\mathscr{D})$.*

Proof. It is sufficient to show that when $\boldsymbol{\lambda} \in U_p^0(\mathscr{D})$ and $k_1, \ldots, k_m$ are nonnegative integers with sum $k \geqslant 0$ the exponential sum

$$y(\mathbf{t}) = t_1^{k_1} \cdots t_m^{k_m} \cdot \exp[\boldsymbol{\lambda} \cdot \mathbf{t}]$$

lies in $L_p(\mathscr{D})$. Accordingly, let $\delta > 0$ be chosen so small that for each $i = 1, \ldots, m$ and $\sigma = \pm 1$ the exponential sum

$$y_{i\sigma}(\mathbf{t}) = \exp[\boldsymbol{\lambda} \cdot \mathbf{t} + \delta\sigma t_i]$$

lies in $L_p(\mathscr{D})$. For $i = 1, \ldots, m$ and $\sigma = \pm 1$ we define the cone

$$H_{i\sigma} = \{\mathbf{t} \in \mathbb{R}^m : \max[| t_1 |, \ldots, | t_m |] = \sigma t_i\}.$$

We let $\chi_{i\sigma}$ denote the characteristic function of $H_{i\sigma}$ so that

$$\begin{aligned} | y(\mathbf{t})\, \chi_{i\sigma}(\mathbf{t})| &= | t_1^{k_1} \cdots t_m^{k_m} \cdot \exp[-\delta\sigma t_i] \cdot y_{i\sigma}(\mathbf{t}) \cdot \chi_{i\sigma}(\mathbf{t})| \\ &\leqslant M \cdot | y_{i\sigma}(\mathbf{t})|, \qquad \mathbf{t} \in \mathbb{R}^m, \end{aligned}$$

where

$$M = \max\{\tau^k \cdot \exp[-\delta\tau] : \tau \geqslant 0\} = [k/(\delta e)]^k.$$

Using this pointwise bound we find

$$\begin{aligned} \| y \|_p &= \Big\| \sum_{i,\sigma} y \cdot \chi_{i\sigma} \Big\|_p \\ &\leqslant \sum_{i,\sigma} \| y \cdot \chi_{i\sigma} \|_p \\ &\leqslant M \cdot \sum_{i,\sigma} \| y_{i\sigma} \|_p < \infty \end{aligned}$$

so that $y \in L_p(\mathscr{D})$. ∎

We note that it is possible for $U_p(\mathscr{D})$ to have no interior points, e.g., as is the situation when $m = 2$ and

$$\mathscr{D} = \{\mathbf{t} \in \mathbb{R}^2 : | t_1 | < (1 + t_2^2)^{-1} \text{ or } | t_2 | < (1 + t_1^2)^{-1}\}$$

in which case

$$U_p(\mathscr{D}) = \{\boldsymbol{\lambda} \in \mathbb{C}^2 : \operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = 0\}.$$

3. Existence of Good Approximations

Before presenting a density theorem we first prepare two lemmas.

Lemma 2. *Let $f \in C_0[0, \infty)$ and $\epsilon > 0$ be given. Then there exists some even polynomial p such that*

$$|f(t) - p(t)e^{-t}| < \epsilon \qquad \text{for} \quad 0 \le t < \infty. \tag{1}$$

If $f(0) = 0$, then (1) also holds for some odd polynomial p.

Proof. Using Pollard's solution of the Bernstein approximation problem [4, Theorem 1, p. 403] (with $\Phi(t) = e^{-|t|}$ and with the sequence of partial sums from the Maclaurin series for cosh t) we see that the set of finite linear combinations of the functions

$$t^{\mu}e^{-|t|} \qquad \mu = 0, 1, \ldots$$

is dense in $C_0(\mathbb{R})$. This being the case there exists some polynomial q such that

$$|f(|t|) - q(t)e^{-|t|}| < \epsilon \qquad \text{for} \quad -\infty < t < \infty$$

and it follows that (1) holds with the even polynomial

$$p(t) = [q(t) + q(-t)]/2.$$

A similar construction shows that (1) holds for an odd polynomial p provided $f(0) = 0$. ∎

Lemma 3. *For each $i = 1, \ldots, m$ let $f_i \in C_0[0, \infty)$ have a compact support, and let the separable function*

$$f(\mathbf{t}) = f_1(t_1) \cdots f_m(t_m)$$

be defined for all $\mathbf{t}$ *in the nonnegative cone*

$$\mathbb{R}_+^m = \{\mathbf{t} \in \mathbb{R}^m : t_i \ge 0 \text{ for } i = 1, \ldots, m\}.$$

Let the parity constant $\pi_i = \pm 1$ be chosen subject to the constraint that $\pi_i = +1$ if $f_i(0) \neq 0$, $i = 1, \ldots, m$, and let $\epsilon > 0$, $\delta > 0$ be given. Then there exist polynomials $p_1, \ldots, p_m$ such that

$$p_i(-t_i) = \pi_i \cdot p_i(t_i), \qquad -\infty < t_i < \infty, \tag{2}$$

$i = 1, \ldots, m$ and such that the separable exponential sum

$$y(\mathbf{t}) = [p_1(t_1)\, e^{-\delta t_1}] \cdots [p_m(t_m)\, e^{-\delta t_m}] \tag{3}$$

uniformly approximates f on $\mathbb{R}_+^m$ *so well that*

$$|f(\mathbf{t}) - y(\mathbf{t})| < \epsilon \qquad \text{for all} \quad \mathbf{t} \in \mathbb{R}_+^m. \tag{4}$$

Proof. Let $|\ |_\infty$ denote the sup norm on $C_0[0, \infty)$, let

$$B = \max\{|f_1|_\infty, \ldots, |f_m|_\infty\},$$

and for each $i = 1, \ldots, m$ let a polynomial p_i satisfying the parity constraint (2) be selected in such a manner that the function

$$\epsilon_i(t_i) = f_i(t_i) - p_i(t_i)\, e^{-\delta t_i}, \qquad t_i \geqslant 0 \tag{5}$$

has norm

$$|\epsilon_i|_\infty < \beta \tag{6}$$

where $\beta > 0$ is chosen so small that

$$(B + \beta)^m - B^m < \epsilon. \tag{7}$$

Such polynomials exist by virtue of Lemma 1. Let y be defined by (3). Using Eqs. (3) and (5)–(7) we find

$$\begin{aligned} |f(\mathbf{t}) - \mathbf{y}(\mathbf{t})| &= \left| \prod_{i=1}^m f_i(t_i) - \prod_{i=1}^m [f_i(t_i) - \epsilon_i(t_i)] \right| \\ &\leqslant \prod_{i=1}^m [|f_i(t_i)| + |\epsilon_i(t_i)|] - \prod_{i=1}^m |f_i(t_i)| \\ &\leqslant (B + \beta)^m - B^m \\ &< \epsilon \end{aligned}$$

whenever $\mathbf{t} \in \mathbb{R}_+^m$ so (4) holds. ∎

THEOREM 1. *Let* $\mathscr{D}$ *be a nonvoid open subset of* $\mathbb{R}^m$, *let* $1 \leqslant p \leqslant \infty$, *and assume that the point* $\boldsymbol{\lambda} \in \mathbb{C}^m$ *lies in the interior of the spectral set* $U_p(\mathscr{D})$. *Then* $V_\infty(\{\boldsymbol{\lambda}\})$ *is dense in* $L_p(\mathscr{D})$ *if* $1 \leqslant p < \infty$ *and in* $C_0(\mathscr{D})$ *if* $p = \infty$.

Proof. Let f be arbitrarily chosen from $L_p(\mathscr{D})$ if $1 \leqslant p < \infty$ and from $C_0(\mathscr{D})$ if $p = \infty$. We must show that we may $\|\ \|_p$-approximate f as closely as we please with the elements of $V_\infty(\{\boldsymbol{\lambda}\})$. Since the space $\mathscr{S}$ of continuous functions having compact support is dense in $L_p(\mathscr{D})$, $1 \leqslant p < \infty$, and in $C_0(\mathscr{D})$ we may assume (with no loss of generality) that $f \in \mathscr{S}$. Moreover, since the subalgebra, $\mathscr{A}$, of finite linear combinations of separable functions is $\|\ \|_p$-dense in $\mathscr{S}$ (as can be seen with the aid of the Stone–Weierstrass theorem [1, p. 191]) we may further assume that $f \in \mathscr{A}$ or equivalently, that f has the representation

$$f(\mathbf{t}) = \varphi(\mathbf{t}) \exp(\boldsymbol{\lambda} \cdot \mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^m \tag{8}$$

where

$$\varphi(\mathbf{t}) = \varphi_1(t_1) \cdots \varphi_m(t_m), \qquad \mathbf{t} \in \mathbb{R}^m, \tag{9}$$

and where $\varphi_1, \ldots, \varphi_m$ are continuous functions with compact support. Finally, since each φ_i may be replaced by the sum of its even and odd parts, we may still further assume that each φ_i has definite parity $\pi_i = \pm 1$, i.e.,

$$\varphi_i(-t_i) = \pi_i \cdot \varphi_i(t_i), \qquad t_i \in \mathbb{R},\ i = 1, \ldots, m. \tag{10}$$

By hypothesis $\boldsymbol{\lambda}$ lies in the interior of $U_p(\mathscr{D})$ and thus there exists some $\delta > 0$ such that each of the exponential sums

$$y_j(\mathbf{t}) = \exp[\boldsymbol{\lambda} \cdot \mathbf{t} + \delta \boldsymbol{\sigma}_j \cdot \mathbf{t}], \qquad j = 1, \ldots, 2^m$$

lies in $L_p(\mathscr{D})$ where $\boldsymbol{\sigma}_j$, $j = 1, \ldots, 2^m$, is an enumeration of the 2^m vectors $(\pm 1, \ldots, \pm 1)$ from $\mathbb{R}^m$. We define

$$s(\mathbf{t}) = |t_1| + \cdots + |t_m|, \qquad \mathbf{t} \in \mathbb{R}^m$$

noting that the function

$$\psi(\mathbf{t}) = \exp[\boldsymbol{\lambda} \cdot \mathbf{t} + \delta s(\mathbf{t})]$$

also lies in $L_p(\mathscr{D})$ since

$$\| \psi \|_p \leqslant \sum_j \| y_j \|_p < \infty$$

and that $\| \psi \|_p > 0$ since $\mathscr{D}$ is nonvoid.

Now let $\epsilon > 0$ be selected. In view of Lemma 3 there exists some separable polynomial

$$p(\mathbf{t}) = p_1(t_1) \cdots p_m(t_m)$$

such that p_i and φ_i have the same parity π_i, $i = 1, \ldots, m$, and such that

$$\sup\{| E(\mathbf{t})| : t_i \geqslant 0 \text{ for } i = 1, \ldots, m\} < \epsilon / \| \psi \|_p$$

where

$$E(\mathbf{t}) = [\varphi(\mathbf{t}) - p(\mathbf{t})] \exp[-\delta s(\mathbf{t})], \qquad \mathbf{t} \in \mathbb{R}^m.$$

Since p_i and φ_i have the same parity it follows that

$$\| E \|_\infty < \epsilon / \| \psi \|_p .$$

This being the case the exponential sum

$$y(\mathbf{t}) = p(\mathbf{t}) \exp(\boldsymbol{\lambda} \cdot \mathbf{t})$$

from $V_\infty(\{\boldsymbol{\lambda}\})$ satisfies

$$\| f - y \|_p = \| E\psi \|_p \leqslant \| E \|_\infty \cdot \| \psi \|_p < \epsilon$$

and since $\epsilon > 0$ is arbitrary, the proof is complete. ∎

4. Existence of Best Approximations

The following result is an extension of the existence theorem presented in [2] for the case where $\mathscr{D}$ is bounded.

THEOREM 2. *Let $\mathscr{D}$ be a nonvoid open subset of $\mathbb{R}^m$, let $\mathbf{S} \subsetneq \mathbb{C}^m$ be closed, let $1 \leqslant p \leqslant \infty$, and let $n = 1, 2, \ldots$. Then every $f \in L_p(\mathscr{D})$ has a best $\| \ \|_p$-approximation from $V_n(\mathbf{S})$.*

Proof. Let $\mathscr{D}_1 \subsetneq \mathscr{D}_2 \subsetneq \cdots$ be an expanding sequence of nonvoid bounded open sets in $\mathbb{R}^m$ with union $\mathscr{D}$, and for each $\mu = 1, 2, \ldots$ let the seminorm $\| \ \|_{p,\mu}$ be defined on $L_p(\mathscr{D})$ by

$$\| f \|_{p,\mu} = \| f \cdot \chi_\mu \|_p \tag{11}$$

where

$$\begin{aligned} \chi_\mu(\mathbf{t}) &= 1 \quad \text{if} \quad t \in \mathscr{D}_\mu, \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{12}$$

Let $f \in L_p(\mathscr{D})$ be selected, and let the minimizing sequence $y_1, y_2, \ldots$ be chosen from $V_n(\mathbf{S})$ in such a manner that

$$\lim \| f - y_\nu \|_p = \inf\{\| f - y \|_p : y \in V_n(\mathbf{S})\}.$$

This sequence is $\| \ \|_p$-bounded and thus $\| \ \|_{p,\mu}$-bounded for each fixed $\mu = 1, 2, \ldots$. This being the case, we see by using the lemma in [2] that after passing to a subsequence, if necessary, we may effect a decomposition

$$y_\nu = v_\nu + x_\nu \qquad \text{where} \quad v_\nu, x_\nu \in V_n(\mathbf{S}),\ \nu = 1, 2, \ldots \tag{13}$$

and find some $v \in V_n(\bar{\mathbf{S}}) = V_n(\mathbf{S})$ such that

$$\lim \| v_\nu - v \|_{p,\mu} = 0, \qquad \mu = 1, 2, \ldots \tag{14}$$

$$\liminf \| g + x_\nu \|_{p,\mu} \geqslant \| g \|_{p,\mu} \qquad \text{for every} \quad g \in L_p(\mathscr{D}),\ \mu = 1, 2, \ldots. \tag{15}$$

This being the case

$$\begin{aligned} \|f - v\|_{p,\mu} &\leq \liminf \|f - v - x_\nu\|_{p,\mu} \\ &= \liminf \|f - y_\nu\|_{p,\mu} \\ &\leq \liminf \|f - y_\nu\|_p \\ &= \inf\{\|f - y\|_p : y \in V_n(\mathbf{S})\} \end{aligned}$$

for each $\mu = 1, 2,\ldots$, and since $\mathscr{D} = \bigcup \mathscr{D}_\mu$ we have

$$\|f - v\|_p \leq \inf\{\|f - y\|_p : y \in V_n(\mathbf{S})\}.$$

Since $v \in V_n(\mathbf{S})$ equality must hold, i.e., v is a best $\|\ \|_p$-approximation to f from $V_n(\mathbf{S})$. ∎

Note. In the preceding theorem the blanket hypothesis that $\mathscr{D}$ is a nonvoid open set can be weakened to the hypothesis that $\mathscr{D}$ is a measurable set with a nonvoid interior and with a boundary having zero measure. When $\mathscr{D}$ is bounded, the closure of $\mathbf{S}$ is a necessary and sufficient condition for every $f \in L_p(\mathscr{D})$ to have a best $\|\ \|_p$-approximation from $V_n(\mathbf{S})$, but when $\mathscr{D}$ is unbounded this closure hypothesis is not the best possible. For example, when $m = 1$ or $n = 1$, a necessary and sufficient condition for existence is that $\mathbf{S}$ be closed in $U_p(\mathscr{D})$, cp. [3, Theorem 3]. Unfortunately, when $n \geq 2$ and $m \geq 2$ this is no longer the case, and no such optimum closure hypothesis for $\mathbf{S}$ is known in this situation.

THEOREM 3. *Let $\mathscr{D}$ be a nonvoid open subset of $\mathbb{R}^m$, let $1 \leq p < \infty$, and let $f \in L_p(\mathscr{D})$. Let $n = 1, 2,\ldots$ and let $\mathbf{S}$ be a closed subset of $\mathbb{C}^m$. Let $\mathscr{D}_1 \subseteq \mathscr{D}_2 \subseteq \cdots$ be an expanding sequence of nonvoid bounded open subsets of $\mathbb{R}^m$ with union $\mathscr{D}$, and for each $\nu = 1, 2,\ldots$ let y_ν be a best $\|\ \|_{p,\nu}$-approximation to f from $V_n(\mathbf{S})$ where the seminorm $\|\ \|_{p,\nu}$ is defined by* (11) *and* (12). *Let some subsequence of $\{y_\nu\}$ and some $v \in V_n(\mathbf{S})$ be selected so that* (13)–(15) *hold. Then v is a best $\|\ \|_p$-approximation to f from $V_n(\mathbf{S})$.*

Proof. Let y be a best $\|\ \|_p$-approximation to f from $V_n(\mathbf{S})$. Then for each fixed $\mu = 1, 2,\ldots$ we have

$$\begin{aligned} \|f - v\|_{p,\mu} &\leq \liminf \|f - v - x_\nu\|_{p,\mu} \\ &= \liminf \|f - y_\nu\|_{p,\mu} \\ &\leq \liminf \|f - y_\nu\|_{p,\nu} \\ &\leq \liminf \|f - y\|_{p,\nu} \\ &= \|f - y\|_p\,, \end{aligned}$$

so that

$$\|f - v\|_p \leq \|f - y\|_p\,,$$

i.e., v is a best approximation. ∎

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