Full length article

Entropy numbers of embeddings of some 2-microlocal Besov spaces

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Received 19 March 2010; received in revised form 30 September 2010; accepted 11 December 2010
Available online 21 December 2010

Communicated by Przemyslaw Wojtaszczyk

Abstract

We investigate compactness and asymptotic behaviour of the entropy numbers of embeddings

\[ B_{p_1,q_1}^{s_1,s_1'}(\mathbb{R}^n, U) \hookrightarrow B_{p_2,q_2}^{s_2,s_2'}(\mathbb{R}^n, U). \]

Here \( B_{p,q}^{s,s'}(\mathbb{R}^n, U) \) denotes a 2-microlocal Besov space with a weight given by the distance to a fixed set \( U \subset \mathbb{R}^n \).

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Keywords: 2-microlocal spaces; Compact embeddings; \( d \)-sets; Entropy numbers

1. Introduction

Let \( X \) and \( Y \) be Banach spaces and \( P : X \rightarrow Y \) a linear and continuous operator. The entropy numbers \( e_k(P : X \rightarrow Y), k = 1, 2, \ldots \) describe the compactness of \( P \) in a qualitative way (cf. Section 3 for the definition). In particular, \( P \) is compact, if and only if,

\[ \lim_{k \rightarrow \infty} e_k(P : X \rightarrow Y) = 0. \]
Let $X = Y$ and let $P \in \mathcal{L}(X)$ be a compact map. Let $(\lambda_k)_{k=1}^{\infty}$ be the sequence of all non-zero eigenvalues of $P$, repeated according to their algebraic multiplicity and ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0$. Then Carl’s inequality

$$|\lambda_k(P : X \rightarrow X)| \leq \sqrt{2}e_k(P : X \rightarrow X)$$

links the behaviour of entropy numbers and eigenvalues. It opens the door to several applications of the following type: given a pseudodifferential operator $P$ with known mapping properties, i.e., $P \in \mathcal{L}(X)$, check the asymptotic behaviour of its entropy numbers and derive by means of Carl’s inequality information about the distribution of eigenvalues of $P$. Many times the behaviour of $e_k(P : X \rightarrow X)$ can be traced to the behaviour of $e_k(\text{id} : \tilde{X} \rightarrow \tilde{Y})$, where $\text{id}$ denotes the identity operator and $\tilde{X}$ and $\tilde{Y}$ are appropriate weighted Besov or Lizorkin–Triebel spaces, cf. [4, Chapt. 5].

Here we concentrate on some 2-microlocal Besov spaces $B^{s,mloc}_{p,q}(\mathbb{R}^n, w)$ with one special type of weights—$w_j(x) = (1 + 2^j \text{dist}(x, U))^{s'}$ and denoted by $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$.

The asymptotic behaviour of the entropy numbers will depend on the ratio of the weights instead of the weights itself. In a first step we shall investigate weights of the same type with respect to some bounded subset of $\mathbb{R}^n$ such that

$$\frac{w_{1,j}(x)}{w_{2,j}(x)} = (1 + 2^j \text{dist}(x, U))^{s'}, \quad x \in \mathbb{R}^n, \ s' > 0.$$

The definition of such spaces was firstly given by Peetre 1975 [22] $B^{s}_{p,q}(a)$ with $(1 + 2^j |x|)^a$.

Later Bony 1984 [1] introduced spaces $H^{s,s'}_{x_0}$ to describe local regularity of functions near a singularity in $x_0$. In connection with wavelet methods Jaffard 1991 [10] introduced and also studied spaces $C^{s,s'}_{x_0}$ again with weight $(1 + 2^j |x - x_0|)^{s'}$, that means with $U = \{x_0\}$. Further considerations can also be found in [11,29,20]. Later Moritoh and Yamada [21] extended the definition for homogeneous spaces to weights $(1 + 2^j \text{dist}(x, U))^{s'}$.

Such spaces were studied systematically in the thesis of Kempka in 2008 [12], see also [14]. He introduced a large class of admissible weight sequences and proved a lot of properties for these spaces and in particular a characterization by wavelets. This characterization is the starting point of our considerations. It enables us to deal with weighted sequence spaces instead of weighted function spaces. Then the techniques, developed and used in [16,17], can be applied.

The paper is organized as follows. In Section 2 we introduce 2-microlocal Besov spaces and discuss a few properties. In particular, we recall a characterization by wavelets proved by Kempka—see [12] or [14]. First we can show in Section 3 that the restriction to a bounded set $U$ is a natural condition if we are interested in compact embeddings. Section 4 deals with the investigation of compactness and the entropy numbers of embeddings of weighted sequence spaces. Finally, in Section 5 we shift these results from the sequence space level to the function spaces.

NOTATIONS. The symbol id always refers to the identity operator. Sometimes we do not indicate the spaces where id is considered and similarly for other operators. Let $T$ be a linear operator which maps the Banach space $A$ into the Banach space $B$. If no confusion is possible, we feel free to write $\|T\|$ instead of the more exact versions $\|T|\mathcal{L}(A, B)\|$ or $\|T|A \rightarrow B\|$. We denote $a \sim b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1}a \leq b \leq ca.$$

All unimportant constants will be denoted by $c$, sometimes with additional indices.
2. 2-microlocal Besov spaces

In this section we recall the definition and a few of the properties of 2-microlocal Besov spaces.

2.1. Definition and preliminaries

Let \( \varphi_0 \in S(\mathbb{R}^n) \) with
\[
\varphi_0(\xi) = 1 \quad \text{for } |\xi| \leq 1 \text{ and } \text{supp} \varphi_0 \subset \{ \xi : |\xi| \leq 2 \}. 
\]
(1)

For \( j \geq 1 \) define
\[
\varphi_j(\xi) := \varphi_0(2^{-j} \xi) - \varphi_0(2^{-j+1} \xi). 
\]
(2)

Then we have
\[
f = \varphi_0(D)f + \sum_{j=1}^{\infty} \varphi_j(D)f, \quad f \in S'(\mathbb{R}^n)
\]
where
\[
\varphi_j(D)f(x) := (2\pi)^{-n} \int \int e^{i\xi(x-y)} \varphi_j(\xi)f(y)dy d\xi.
\]

The 2-microlocal space \( H^{s,s'}_{\varphi_0}(\mathbb{R}^n) \) was introduced for real \( s \) and \( s' \) in [1] as the set of all tempered distributions such that
\[
\|2^{js}(1 + 2^j|x - x_0|)^{s'} \varphi_j(D)f|_{L_2} \| \leq c_j \quad \text{with} \quad \sum_{j=0}^{\infty} c_j^2 < \infty.
\]

Moreover, the space \( C^{s,s'}_{\varphi_0}(\mathbb{R}^n) \) in [10] is the collection of tempered distributions such that
\[
|2^{js}(1 + 2^j|x - x_0|)^{s'} \varphi_j(D)f| \leq C \quad \text{holds for all } j.
\]

Remark 1. Please note that, if \( s > 0 \) and \( s' > 0 \), then \( f \in C^{s,s'}_{\varphi_0} \) if, and only if, \( f \in C^s(\mathbb{R}^n) \) and
\[
\|f|C^{s+s'}(\Gamma_\rho)\| \leq C\rho^{-s'}
\]
where \( \Gamma_\rho = \{ x : \rho \leq |x - x_0| \leq 3\rho \} \). This means that we require not only \( f \) to belong to \( C^{s+s'} \) outside of \( x_0 \), but also some decrease of the corresponding norms with respect to the annuli.

For our purposes it will be convenient to work with the following class of weight sequences defined in [12]—see also [14].

Definition 1. Let \( w = (w_j)_{j\in \mathbb{N}_0} \) be a sequence of non-negative measurable functions. The sequence is called an admissible weight sequence belonging to \( \mathcal{W}^{\alpha}_{\alpha_1, \alpha_2} \), \( \alpha, \alpha_1, \alpha_2 \) real, if there is a constant \( C > 0 \) with:
\[
0 < w_j(x) \leq C w_j(y)(1 + 2^j|x - y|)^{\alpha} \quad \text{for all } x, y \in \mathbb{R}^n
\]
(3)
and
\[
2^{\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \quad \text{holds for all } j \in \mathbb{N}_0.
\]
(4)
For each fixed \( j \) the first condition coincides exactly with the usual one for weighted Besov spaces—see for example [25,4] or [16]. For a single weight which fulfills this condition we say that it belongs to \( \mathcal{W}^s_{\text{scalar}} \). The second condition compares weights for different indices.

By \( \mathcal{W}_{\text{scalar}} \) we will denote here the union of \( \mathcal{W}^s_{\text{scalar}} \) and by \( \mathcal{W} \) the union of \( \mathcal{W}^s_{\alpha_1\alpha_2} \) over all parameters. This is now a set of weight sequences instead of single weights as in [25] or [16].

**Remark 2.** Typical examples for such weight sequences are:

\[
\begin{align*}
  w_j(x) &= (1 + 2^j|x - x_0|)^{s'}, \quad x_0 \in \mathbb{R}^n, \ s' \in \mathbb{R} \\
  w_j(x) &= 2^{js} (1 + 2^j|x - x_0|)^{s'}, \quad x_0 \in \mathbb{R}^n, \ s, \ s' \in \mathbb{R} \\
  w_j(x) &= (1 + 2^j \text{dist}(x, U))^{s'}, \quad U \subset \mathbb{R}^n, \ s' \in \mathbb{R} \\
  w_j(x) &= 2^{js} w(x) \quad \text{for fixed } w \in \mathcal{W}_{\text{scalar}} \text{ and } s \in \mathbb{R} \\
  w_j(x) &= \beta_j \quad \text{where } (\beta_j)_{j \in \mathbb{N}_0} \text{ fulfills } d_0\beta_j \leq \beta_{j+1} \leq d_1\beta_j \text{ for all } j.
\end{align*}
\]

It is known that for the last example, embeddings are compact only if the function spaces are defined on bounded domains as for the classical Besov spaces \( B^s_{p,q} \). For some results about entropy numbers of the embeddings we refer to [18,3]. In the case of one scalar weight in many different situations the behaviour of the entropy numbers is known—see for example [4,8,9,7,15–17]. So we will concentrate on the first three examples which are included in some sense in the third one.

Let \( \varphi_0 \in S(\mathbb{R}^n) \) with (1) and \( \varphi_j \) be defined by dilation as in (2). Then once again we have a suitable resolution of unity so

\[
f = \varphi_0(D)f + \sum_{j=1}^{\infty} \varphi_j(D)f \quad f \in S'(\mathbb{R}^n).
\]

Using such a resolution Kempka defined in [12,14] generalized 2-microlocal spaces as follows.

**Definition 2.** Let \( w = (w_j(x))_{j \in \mathbb{N}_0} \in \mathcal{W}^s_{\alpha_1\alpha_2} \), \( 0 < p, q \leq \infty \) and \( s \) be real. We put

\[
B^{s,\text{mloc}}_{p,q}(\mathbb{R}^n, w) = \{ f \in S'(\mathbb{R}^n) : \| f|B^{s,\text{mloc}}_{p,q}(\mathbb{R}^n, w)\|_w < \infty \},
\]

with

\[
\| f|B^{s,\text{mloc}}_{p,q}(\mathbb{R}^n, w)\|_w = \left( \sum_{j=0}^{\infty} 2^{jsq} \| w_j \varphi_j(D)f |L^p(\mathbb{R}^n)\|^q \right)^{1/q}.
\]

There is an analogous definition for \( F^{s,\text{mloc}}_{pq}(\mathbb{R}^n, w) \). In [13] these definitions were extended even to variable parameters \( p \).

We mention some obvious relations of these spaces to some well-known definitions.

- In the case of \( w_j(x) = 1 \) they are the usual Besov spaces.
- In the case of \( w_j(x) = (1 + |x|)^{\alpha} \) independent of \( j \) we obtain the usual weighted Besov spaces with polynomial weights.
- In the case of \( w_j(x) = \beta_j \) where \( d_0\beta_j \leq \beta_{j+1} \leq d_1\beta_j \) and \( (\beta_j)_{j \in \mathbb{N}_0} \) is an admissible sequence of real numbers we obtain Besov spaces of generalized smoothness, see for example [5].
With \( w_j(x) = (1 + 2^j |x - x_0|)^s \) we have \( C^{s,s'}_{x_0} (\mathbb{R}^n) = B^{s,mloc}_{\infty,\infty} (\mathbb{R}^n, w) \) and \( H^{s,s'}_{x_0} (\mathbb{R}^n) = B^{s,mloc}_{2,2} (\mathbb{R}^n, w) \).

- In the case of \( w_j(x) = (1 + 2^j \text{dist}(x, U))^s \) the corresponding spaces were denoted by \( B^{s,s'}_{p,q} (\mathbb{R}^n, U) \). The homogeneous version of the spaces was considered by Moritoh and Yamada in [21] in case when \( U \) is an open set.

A lot of properties of these spaces were proved in [12,14]. For example the independence of the spaces from the choice of the function \( \varphi \), theorems about Fourier multipliers, the existence of lift-operators, embeddings, maximal functions and also some far-reaching properties like characterizations by local means, by atoms and molecules and by wavelets were considered there.

### 2.2. Wavelet decomposition of 2-microlocal Besov spaces

We follow the notation in [28,14]. Let \( \psi_M \in C^k(\mathbb{R}) \) and \( \psi_F \in C^k(\mathbb{R}) \) be real compactly supported Daubechies wavelets with

\[
\int_{\mathbb{R}} x^\beta \psi_M(x) \, dx = 0 \quad \text{for} \ |\beta| < k.
\]

By a tensor product procedure these can be generalized to the \( n \)-dimensional case—see [28, Section 4.2.1.].

Let \( G = (G_1, \ldots, G_n) \in G^j = \{F, M\}^n \) where \( n^* \) indicates that at least one of the components of \( G \) must be an \( M \) and let \( G^0 = \{F, M\}^n \). The cardinality of \( \{F, M\}^n \) is \( 2^n - 1 \) and the cardinality of \( \{F, M\}^n \) is \( 2^n \), respectively. We define

\[
\psi^{j}_{G,m}(x) := 2^{j\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^{j}x_r - m_r)
\]

where \( j \in \mathbb{N}_0 \), \( G \in G^j \) and \( m \in \mathbb{Z}^n \). Then \( \{\psi^{j}_{G,m}(x) : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\} \) is an orthonormal basis in \( L_2(\mathbb{R}^n) \)—see [28, Section 4.2.1.]. For a characterization of the 2-microlocal Besov spaces we need the following sequence spaces—see [14, Definition 7].

**Definition 3.** Let \( w \in \mathcal{W} \), \( s \in \mathbb{R} \) and \( 0 < p, q \leq \infty \).

\[
\tilde{b}^{s,mloc}_{p,q}(w) = \{\lambda^{j}_{G,m} : ||\lambda||_{\tilde{b}^{s,mloc}_{p,q}(w)} < \infty\}
\]

and

\[
||\lambda||_{\tilde{b}^{s,mloc}_{p,q}(w)} = \left( \sum_{j=0}^{\infty} 2^{j(s-nq/p)} \sum_{G \in G^j} \left( \sum_{m \in \mathbb{Z}^n} |\lambda^{j}_{G,m}|^{p} w_j^{p}(2^{-j}m) \right)^{q/p} \right)^{1/q}.
\]

Also the following proposition can be found there, see [14, Theorem 4].

**Proposition 1.** Let \( f \in B^{s,mloc}_{p,q} (\mathbb{R}^n, w) \), \( k \) large enough and

\[
\lambda^{j}_{G,m}(f) := 2^{j \frac{n}{2}} \langle f, \psi^{j}_{G,m} \rangle = 2^{j \frac{n}{2}} \int f(x) \psi^{j}_{G,m}(x) \, dx.
\]

Then

\[
I : f \rightarrow \left( 2^{j \frac{n}{2}} \langle f, \psi^{j}_{G,m} \rangle \right)_{j,G,m}
\]

is an isomorphic map from \( B^{s,mloc}_{p,q} (\mathbb{R}^n, w) \) onto \( \tilde{b}^{s,mloc}_{p,q}(w) \).
This characterization was specified to the spaces $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$, see [14, Corollary 2], in the following way:

A function $f$ belongs to $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m}^j(f) 2^{-js} \psi_{G,m}^j$$

with $\lambda \in \tilde{b}_{p,q}^{s,s'}(U)$.

The representation is unique

$$\lambda_{G,m}^j(f) := 2^{js} \int f(x) \psi_{G,m}^j(x) \, dx$$

and

$I : f \mapsto \left(2^{js} (f, \psi_{G,m}^j)\right)_{j,G,m}$

is an isomorphic map from $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$ onto $\tilde{b}_{p,q}^{s,s'}(U)$.

Here

$$\|\lambda\|_{\tilde{b}_{p,q}^{s,s'}(U)} := \left(\sum_{j=0}^{\infty} 2^j \left(\sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} |\lambda_{G,m}^j|^p \left(1 + 2^j \text{dist}(2^{-j}m, U)\right)^{s'}\right)^{q/p}\right)^{1/q} < \infty$$

3. The characterization of continuous and compact embeddings of weighted sequence spaces

The aim of this section is to characterize all possible situations where either a continuous embedding or a compact embedding holds true. By means of the Proposition 1 we can shift the problem to the side of the sequence spaces.

Motivated by Definition 3 and Proposition 1 we introduce the following sequence spaces. For a given weight sequence $(w_j)_{j \in \mathcal{W}}$ we define

$$\ell_q(2^{js} \ell_p(w)) := \left\{ \lambda = (\lambda_{j,m})_{j,m} : \lambda_{j,m} \in \mathbb{C}, \|\lambda\|_{\ell_q(2^{js} \ell_p(w))} = \left(\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{j,m} w_j(2^{-j}m)|^p\right)^{q/p}\right)^{1/q} < \infty \right\}$$

(usual modifications if $p = \infty$ and/or $q = \infty$).

It should be clear that we can work with the spaces $\ell_q(2^{js} \ell_p(w))$ instead of $\tilde{b}_{p,q}^{s,s', \text{mloc}}(w)$ when we study the boundedness and compactness of the embeddings. Similar considerations, with one weight function instead of a sequence of weights can be found in [15–17,6,7] or [3].

Observe further that also

$$\ell_{q_1}\left(2^{j(s_1 - \frac{n}{p_1})} \ell_{p_1}(w_1)\right) \hookrightarrow \ell_{q_2}\left(2^{j(s_2 - \frac{n}{p_2})} \ell_{p_2}(w_2)\right)$$
is equivalent to
\[ \ell_{q_1}(2^j(s_1-s_2-n(\frac{1}{p_1} - \frac{1}{p_2})) \ell_{p_1}(w_1/w_2)) \hookrightarrow \ell_{q_2}(\ell_{p_2}). \]

So it will be again sufficient to consider the unweighted space as the target space. Moreover, it will be convenient to use the following well-known abbreviation
\[ \delta := s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right). \tag{5} \]

Also for a real number \(a\) we define \(a^+ := \max(a, 0)\).

**Theorem 2.** (i) The embedding \(\ell_{q_1}(2^j\delta \ell_{p_1}(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2})\) holds if, and only if,
\[ \left(2^{-j\delta}\|(w_{j,m}^{-1}m)|\ell_{p^*}\|\right)_j \in \ell_{q^*}, \]
where
\[ \frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \quad \text{and} \quad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+. \]

Moreover, it holds that
\[ \|\text{id}|\ell_{q_1}(2^j\delta \ell_{p_1}(w)) \rightarrow \ell_{q_2}(\ell_{p_2})\| = \|(2^{-j\delta}\|(w_{j,m}^{-1}m)|\ell_{p^*}\|)_j|\ell_{q^*}\|. \tag{6} \]

(ii) The embedding \(\ell_{q_1}(2^j\delta \ell_{p_1}(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2})\) is compact if, and only if,
\[ (2^{-j\delta}w_{j,m}^{-1}m)_{j,m} \in \ell_{q^*}(\ell_{p^*}), \]
and, in addition,
\[ \lim_{j \to \infty} 2^{-j\delta}\|(w_{j,m}^{-1}m)|\ell_{p^*}\| = 0 \quad \text{if} \quad q^* = \infty, \tag{7} \]
and
\[ \lim_{|m| \to \infty} w_{j,m} = \infty \quad \text{for all} \quad j \in \mathbb{N}_0 \quad \text{if} \quad p^* = \infty. \tag{8} \]

We refer to [16] or [19] for the proof in a more general setting.

**Corollary 3.** Let \(U\) be unbounded and
\[ w_{j,m} = w_j(2^{-j}m) = (1 + 2^j\text{dist}(2^{-j}m, U))^{s'}. \]

Then the embedding
\[ \ell_{q_1}(2^j\delta \ell_{p_1}(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2}) \]

can never become compact.

**Proof.** For each fixed \(j_0\) there exists a sequence \(m_i\) with \(|m_i| \to \infty\) and
\[ \text{dist}(2^{-j_0}m_i, U) \leq \sqrt{n} 2^{-j_0}. \]

Consequently we have
\[ 1 \leq w_{j_0,m_i} = w_{j_0}(2^{j_0}m_i) = (1 + 2^{j_0}\text{dist}(2^{-j_0}m_i, U))^{s'} \leq (1 + \sqrt{n})^{s'}, \]
\[ s' \geq 0. \quad \square \]
It follows from the last corollary that only bounded subsets of $\mathbb{R}^n$ are of interest for us. It should be obvious that we can restrict the attention to compact subsets. We are able to give the precise description for so-called $d$-sets, which are fractal sets in between single point sets $\{x_0\}$ and compact sets with non-empty interior.

We recall the notation. Let $U$ be a compact set and $\mu$ a Radon measure with $\text{supp} \mu = U$. The set $U$ is called a $d$-set, $0 \leq d \leq n$, if for each ball of radius $r$ and centered in $y \in U$ holds

$$\mu(B(y, r)) \sim r^d \quad \text{for } 0 < r < 1$$

cf. [27].

Let a $d$-set $U$ be covered by balls of radius $2^{-k}$, centered on $U$, such that the balls with the same center but with radius $2^{-k-\kappa}$ for fixed $\kappa \in \mathbb{N}$ are disjoint. Then it is well-known that the number of balls in such a covering is always equivalent to $2^{kd}$ where the equivalence constants depend on $\kappa$ but are independent of the radius $2^{-k}$. We are interested in estimation of a number of dyadic cubes of a fixed side length that are in a predetermined distance to the set $U$.

For $j \in \mathbb{N}_0$ and $i \in \mathbb{N}$ we denote by $N_{j,i}$ the number of cubes $Q_{j,m}$ of side length $2^{-j}$, centered in $2^{-j}m$ with

$$\sqrt{n}2^{-j+i} < \text{dist}(Q_{j,m}, U) \leq 4\sqrt{n}2^{-j+i}.$$  \hspace{1cm} (9)

**Lemma 4.** Let $U$ be a $d$-set, then

$$N_{j,i} \sim \begin{cases} 2^{in}2^{j-i}d & 0 \leq i < j, \\ 2^{in} & j \leq i. \end{cases}$$

**Proof.** Let us assume that the $d$-set $U$ is contained in the unit ball $\{x : |x| \leq 1\}$.

**Step 1.** If $i \geq j$, then the assertion follows immediately. The number of cubes $N_{j,i}$ is smaller than the number of cubes $Q_{j,m}$ contained in a ball of radius $4\sqrt{n}2^{-j+i} + 1$ and larger as the number of cubes $Q_{j,m}$ contained in an annulus with radii $\sqrt{n}2^{-j+i} + 1$ and $4\sqrt{n}2^{-j+i} - 1$. Since $2^{-j+i} \geq 1$ by volume arguments this gives

$$N_{j,i} \sim 2^{in} \quad i \geq j.$$

**Step 2.** We use the Whitney decomposition of $\mathbb{R}^n \setminus U$. Let $Q_{k,m,k,r}, k \in \mathbb{Z}, m_k,r \in \mathbb{Z}^n$, $r = 1, 2, \ldots, r(k)$ be (open) cubes in $\mathbb{R}^n$ with sides parallel with the axes of coordinates, centered in $2^{-k}m_k,r$ with side length $2^{-k}$ such that for suitable disjoint cubes $Q_{k,m,k,r}$

$$\mathbb{R}^n \setminus U = \bigcup_{k,r} Q_{k,m,k,r} \quad \text{and} \quad \text{dist}(Q_{k,m,k,r}, U) \sim 2^{-k},$$

more precisely

$$\sqrt{n}2^{-k} \leq \text{dist}(Q_{k,m,k,r}, U) \leq 4\sqrt{n}2^{-k},$$

cf. the proof of Theorem 1 in [26, page 167].

Let $\{B(x_i, 2^{-k})\}_{i=1,\ldots,m}$ be a covering of $U$ with the centers $x_i$ belonging to $U$, and such that the balls $B(x_i, 2^{-k-1})$ are pairwise disjoint. Then as we mentioned earlier $m \sim 2^{dk}$ with the constants independent of $k$. The family $\{B(x_i, 6\sqrt{n}2^{-k})\}_{i=1,\ldots,m}$ covers an “annulus”

$$\{x \in \mathbb{R}^n : \sqrt{n}2^{-k} \leq \text{dist}(x, U) \leq 5\sqrt{n}2^{-k}\}$$

due to disjointness of cubes in the Whitney decomposition we get

$$2^{-kn}r(k) \leq c2^{dk}(6\sqrt{n}2^{-k})^n.$$
On the other hand, if we blow up any cube $Q_{k,m_i}$ by the factor $\lambda = 9\sqrt{n}$, then any of the balls $B(x_i, 2^{-k-1})$, $i = 1, \ldots, m$, is contained in at least one enhanced cube, so
\[
c_{2}^{-k-1}2^{kd} \leq r(k)(9\sqrt{n}2^{-k})^n.
\]
In consequence
\[
r(k) \sim 2^{kd}.
\]

**Step 3.** We estimate now the number $N_{j,i}$ in the case $i < j$.

Estimate from below: By (10) we have at least $c2^{(j-i)d}$ cubes of side length $2^{j-i}$ with
\[
\sqrt{n}2^{-j+i} < \text{dist}(Q_{j-i,m}, U) \leq 4\sqrt{n}2^{-j+i}.
\]
Each of these cubes can be divided into $2^{ni}$ cubes of side length $2^{-j}$ and so we have at least $c2^{(j-i)d}2^{ni}$ cubes with (9).

Estimate from above: Let $Q_{j,m}$ be a cube with side length $2^{-j}$ that satisfies (9). Then it is contained in some, maybe larger, cube of the Whitney decomposition—$Q_{j-l,m}$, $0 \leq l \leq i$.

Since the last cube belongs to the Whitney decomposition we have
\[
\sqrt{n}2^{-j+l} < \text{dist}(Q_{j-l,m}, U) \leq 4\sqrt{n}2^{-j+l}.
\]
There exist at most $2^{(j-l)d}$ such cubes, cf. (10). But each cube $Q_{j-l,m}$ from the Whitney decomposition can be approximated by at most $2^{ln}$ original cubes $Q_{j,m}$. For this reason we have
\[
N_{j,i} \leq \max_{l=0,\ldots,i}2^{(j-l)d}2^{ln} \leq \max_{l=0,\ldots,i}2^{ld}2^{l(n-d)} \leq 2^{jd}2^{j(n-d)}.
\]

**Step 4.** If the $d$-set $U$ is contained in a larger ball $\{x : |x| \leq R\}$, then by dilation with $2^{-J} \leq R^{-1}$ for a suitable fixed $J$ we can reduce the considerations to the previous case. Afterwards blowing up the cubes by $2^J$ we have the same estimates, but now with new constants where $2^{-Jn}$ and $2^{Jn}$ come in. □

**Theorem 5.** Let $w_{j,m} = w_j(2^{-j}m) = (1 + 2^j\text{dist}(2^{-j}m, U))^{s'}$ and $U$ be a $d$-set. Then the embedding
\[
\ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \hookrightarrow \ell_{q_2}(\ell_{p_2})
\]
is compact if, and only if,
\[
s' > n/p^* \quad \text{and} \quad \delta > d/p^*.
\]

**Proof.** We introduce a decomposition of the identity which will be of great benefit for us also in the proof of the next theorem. Let
\[
A := \{\lambda = (\lambda_{j,m})_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} : \lambda_{j,m} \in \mathbb{C}, \: j \in \mathbb{N}_0, \: m \in \mathbb{Z}^n\}.
\]
Let $I_{j,i} \subset \mathbb{N}_0 \times \mathbb{Z}^n$ s.t.
\[
I_{j,i} := \{(j, m) : \sqrt{n}2^{-j+i-1} < \text{dist}(2^{-j}m, U) \leq \sqrt{n}2^{-j+i}\}, \quad i \in \mathbb{N}, \quad j \in \mathbb{N}_0
\]
\[
I_{j,0} := \{(j, m) : \text{dist}(2^{-j}m, U) \leq \sqrt{n}2^{-j}\}, \quad j \in \mathbb{N}_0.
\]

Further, let $P_{j,i} : A \rightarrow A$ be the canonical projection with respect to $I_{j,i}$, i.e., for $\lambda \in A$ we put
\[
(P_{j,i}\lambda)_{u,v} := \begin{cases} 
\lambda_{u,v} & (u, v) \in I_{j,i}, \\
0 & \text{otherwise}
\end{cases}, \quad u \in \mathbb{N}_0, \: v \in \mathbb{Z}^n.
\]
Observe

\[ \text{id}_A = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} P_{j,i}. \]

The benefit of the decomposition is that we have for all \((j, m)\) in the same index set \(I_{j,i}\) an uniform estimate for the weight

\[ w_j(2^{-j} m) \sim (1 + 2^{j} 2^{-j+i})^{s'} \sim 2^{is'} \quad \text{if } (j, m) \in I_{j,i}. \] (11)

The cardinality of \(I_{j,i}\) is denoted by \(M_{j,i}\). In the case of \(0 < i < j\) this cardinality depends on the structure of \(U\) and is fundamental in the sequel. If \(U\) is a \(d\)-set, then an easy observation and the result of Lemma 4 give

\[ M_{j,i} \leq N_{j,i+2} + N_{j,i+3} \sim \begin{cases} 2^{in} 2^{(j-i)d} & 0 \leq i < j, \\ 2^{in} & j \leq i \end{cases} \] (12)

and

\[ M_{j,i+1} + M_{j,i+2} \geq N_{j,i} \sim \begin{cases} 2^{in} 2^{(j-i)d} & 0 \leq i < j, \\ 2^{in} & j \leq i. \end{cases} \] (13)

Using the second part of Theorem 2 we can prove now necessary and sufficient conditions for the compactness of the embedding. By the estimation (11) we have

\[ \|2^{-j\delta}(w_{j,m})^{-1}|\ell_q^*(\ell_p^*)\| \sim \left( \sum_{j=0}^{\infty} 2^{-j\delta q^*} \left( \sum_{i=0}^{M_{j,i}} 2^{-is' p^*} \right) \right)^{1 \over q^*}. \]

\[ \leq C \left( \sum_{j=0}^{\infty} 2^{-j\delta q^*} \left( \sum_{i=0}^{j-1} 2^{in+(j-i)d} 2^{-is' p^*} + \sum_{i=j}^{\infty} 2^{in} 2^{-is' p^*} \right) \right)^{1 \over q^*}. \]

We can estimate this norm from below by the same series

\[ \left( \sum_{j=0}^{\infty} 2^{-j\delta q^*} \left( \sum_{i=0}^{M_{j,i}} 2^{-is' p^*} \right) \right)^{1 \over q^*}. \]

\[ \geq \left( \sum_{j=0}^{\infty} 2^{-j\delta q^*} \left( \sum_{i=0}^{M_{j,i+1} + M_{j,i+2}} 2^{-(i+1)s' p^*} \min(1, 2-s' p^*) \right) \right)^{1 \over q^*}. \]

\[ \geq C \left( \sum_{j=0}^{\infty} 2^{-j\delta q^*} \left( \sum_{i=0}^{j-1} 2^{in+(j-i)d} 2^{-is' p^*} + \sum_{i=j}^{\infty} 2^{in} 2^{-is' p^*} \right) \right)^{1 \over q^*}. \]
In the case of $p^* < \infty$ the sums over $i$ converge if, and only if, $n < s'p^*$. So in that case

\[
\left( \sum_{j=0}^{\infty} 2^{-j\delta q^*} \left( \sum_{i=0}^{j-1} 2^{in+(j-i)d} 2^{-is'p^*} + \sum_{i=j}^{\infty} 2^{in} 2^{-is'p^*} \right) \right) \frac{1}{q^*} \\
\sim \left( \sum_{j=0}^{\infty} 2^{-j\delta q^*} (2jd + 2jn 2^{-js'p^*}) \frac{q^*}{p^*} \right) \frac{1}{q^*} \sim \left( \sum_{j=0}^{\infty} 2^{-j\delta q^*} 2jdq^*/p^* \right) \frac{1}{q^*}.
\]

Thus in the case $p^* < \infty$ and $q^* < \infty$ the norm is finite if, and only if, $s' > n/p^* \text{ and } \delta > d/p^*$. If either $p^* = \infty$ or $q^* = \infty$, then one should consider in addition (7) or (8). □

4. Entropy numbers of embeddings of special weighted sequence spaces

We are interested in measuring the compactness of the embedding of $\ell_{q_1}(2^{j(s_1 - n_{1})} \ell_{p_1}(w_1))$ into $\ell_{q_2}(2^{j(s_2 - n_{2})} \ell_{p_2}(w_2))$.

4.1. Preliminaries

Let us briefly recall the definition of entropy numbers.

**Definition 4.** Let $X, Y$ be two complex Banach spaces and let $P$ be a linear and continuous operator from $X$ into $Y$. Let $k \in \mathbb{N}$. The $k$-th entropy number $e_k(P : X \to Y)$ is the infimum of all numbers $\varepsilon > 0$ such that there exist $2^{k-1}$ balls in $Y$ of radius $\varepsilon$ which cover the image of the unit ball $U := \{ x \in X : \|x\|_X \leq 1 \}$ under the mapping $P$.

In particular, $P$ is compact if, and only if, $\lim_{k \to \infty} e_k(P : X \to Y) = 0$. For details and basic properties we refer for example to the monographs [2,4].

Let $w_j(x) = \frac{w_{1,j}(x)}{w_{2,j}(x)}$ and consequently $\ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) = \ell_{q_1}(2^{j\delta} \ell_{p_1}(w_{1/w_2}))$.

Then the mapping $I$ defined by

\[
\lambda_{j,m} \mapsto \lambda_{j,m} 2^{j(s_2 - n_{2})} w_2(2^{-j} m), \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n,
\]

yields an isometry of $\ell_{q_1}(2^{j(s_1 - n_{1})} \ell_{p_1}(w_1))$ onto $\ell_{q_1}(2^{j\delta} \ell_{p_1}(w))$. Furthermore, $I^{-1}$ yields an isometry of $\ell_{q_2}(\ell_{p_2})$ onto $\ell_{q_2}(2^{j(s_2 - n_{2})} \ell_{p_2}(w_2))$. As a consequence of the definition of the entropy numbers and the properties of $I, I^{-1}$ we obtain the identity for $k = 1, 2, \ldots$

\[
e_k \left( \text{id : } \ell_{q_1}(2^{j(s_1 - n_{1})} \ell_{p_1}(w_1)) \to \ell_{q_2}(2^{j(s_2 - n_{2})} \ell_{p_2}(w_2)) \right)
= e_k \left( \text{id : } \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right).
\]

Hence, we may concentrate on $e_k \left( \text{id : } \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right)$. The abstract concept of an operator ideal, see [23,24], has been proved to be a useful tool in various situations. Here it simplifies again the estimates of the entropy numbers. We refer to [16] where this concept was used extensively.
**Definition 5.** Let $X, Y$ be quasi-Banach spaces, $\ell_{r,\infty}$ Lorentz sequence space and let $T \in L(X, Y)$. Then we put

$$L_{r,\infty}^{(e)}(T) := \|e_k(T)\|_{\ell_{r,\infty}}$$

and

$$L_{r,\infty}^{(e)}(X, Y) := \left\{ T \in L(X, Y) : L_{r,\infty}^{(e)}(T) < \infty \right\}.$$ 

**Remark 3.** $L_{r,\infty}^{(e)}(X, Y)$ is a complete quasi-normed space, i.e., a quasi-Banach space and there exists an equivalent $\varrho$-norm on it.

We have

$$L_{r,\infty}^{(e)}(T) = \sup_{k} \{ e_k(T)k^{1/r} \} < c \quad \text{if, and only if,} \quad e_k(T) \leq ck^{-\frac{1}{r}}. \quad (14)$$

For details see [16].

**4.2. Estimates for the entropy numbers**

One of the tools in proving the next theorem will be the characterization of the asymptotic behaviour of the entropy numbers of the embeddings $\ell^N_{p_1} \hookrightarrow \ell^N_{p_2}$. For $0 < p_1 \leq p_2 \leq \infty$ and for all $k \in \mathbb{N}$ we have

$$e_k(\text{id} : \ell^N_{p_1} \rightarrow \ell^N_{p_2}) \sim \begin{cases} 1 & \text{if } 1 \leq k \leq \log 2N, \\ \left( \frac{\log (1 + \frac{N}{k})}{k} \right)^{-\frac{1}{p_1} - \frac{1}{p_2}} & \text{if } \log 2N \leq k \leq 2N, \\ 2^{-\frac{k}{p_2}N^{\frac{1}{p_2}} - \frac{1}{p_1}} & \text{if } 2N \leq k, \end{cases} \quad (15)$$

and in the case $0 < p_2 < p_1 \leq \infty$ it holds

$$e_k(\text{id} : \ell^N_{p_1} \rightarrow \ell^N_{p_2}) \sim 2^{-\frac{k}{p_2}N^{\frac{1}{p_2}} - \frac{1}{p_1}} \quad \text{for all } k \in \mathbb{N}. \quad (16)$$

This implies

$$L_{r,\infty}^{(e)}(\text{id} : \ell^N_{p_1} \rightarrow \ell^N_{p_2}) \sim N^{-\frac{1}{r} - \left( \frac{1}{p_1} - \frac{1}{p_2} \right)} \quad \text{if } \frac{1}{r} > \max \left( 0, \frac{1}{p_1} - \frac{1}{p_2} \right). \quad (17)$$

For details and references see once again [16].

**Theorem 6.** Let $U$ be a d-set, $0 \leq d \leq n$ and $w_{j,m} = w_j(2^{-j}m) = (1 + 2^j \text{dist}(2^{-j}m, U))^{s'}$ a sequence of weights. Let

$$\delta = s_1 - s_2 - n \left( \frac{1}{p_1} - \frac{1}{p_2} \right) > \frac{d}{p^*} = d \left( \frac{1}{p_2} - \frac{1}{p_1} \right) \quad \text{and} \quad s' = s'_1 - s'_2 > \frac{n}{p^*}. \quad$$

Then

$$e_k(\text{id} : \ell_{q_1}(2^{j^2}\ell_{p_1}(w)) \rightarrow \ell_{q_2}(\ell_{p_2})) \sim k^{-\frac{1}{p_1} + \frac{1}{p_2} - \min\left( \frac{d}{p^*}, \frac{s'}{p^*} \right)}.$$
Proof. Step 1. Preparations. We use the same decomposition of identity as in the proof of Theorem 5 and employ the same notations as before. Recall that

\[ w_j(2^{-j} m) \sim 2^is \quad \text{if} \ (j, m) \in I_{j,i}. \]  

(18)

Monotonicity arguments and elementary properties of the entropy numbers yield

\[ e_k\left(P_{j,k} : \ell_q(2^{1/8} \ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2})\right) \leq c 2^{-j 8} 2^{-is'} e_k\left(\text{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}\right), \]  

(19)

with a constant \( c \) independent of \( k, j \) and \( i \).

Step 2. Now the operator ideal comes into play. Using (14) and (19) we find

\[ L_{r,\infty}(P_{j,i}) \leq c 2^{-j 8} 2^{-is'} L_{r,\infty}(\text{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}). \]  

(20)

To shorten notations let \( 1/p = 1/p_1 - 1/p_2 \). Under the assumption \( 1/r > \max(0, 1/p) \) we conclude from Lemma 4, (12) and (17) that

\[ L_{r,\infty}(\text{id} : \ell_{p_1}^{M_{j,i}} \to \ell_{p_2}^{M_{j,i}}) \leq \begin{cases} 2^{(in+d(j-i))(\frac{1}{r}-\frac{1}{p})}, & 0 \leq i < j \\ 2^{in(\frac{1}{r}-\frac{1}{p})}, & 0 < j \leq i \end{cases} \]  

(21)

and consequently

\[ L_{r,\infty}(P_{j,i}) \leq c 2^{-j 8} 2^{-is'} \begin{cases} 2^{(in+d(j-i))(\frac{1}{r}-\frac{1}{p})}, & 0 \leq i < j \\ 2^{in(\frac{1}{r}-\frac{1}{p})}, & 0 < j \leq i \end{cases} \]  

(22)

Now, for given \( M \in \mathbb{N}_0 \) let

\[ P^1 := \sum_{j=0}^{M} \sum_{i=0}^{j-1} P_{j,i}, \quad P^2 := \sum_{j=M+1}^{\infty} \sum_{i=0}^{j-1} P_{j,i}, \]

\[ Q^1 := \sum_{j=0}^{M} \sum_{i=j}^{M} P_{j,i}, \quad Q^2 := \sum_{j=0}^{\infty} \sum_{i=M+1}^{\infty} P_{j,i}, \quad Q^3 := \sum_{j=M+1}^{\infty} \sum_{i=j}^{\infty} P_{j,i}. \]

Substep 2.1. First we estimate \( L_{r,\infty}(P^1) \). Recall, for any \( r > 0 \) there exists an equivalent \( \varrho \)-norm on \( L_{r,\infty} \) with \( 0 < \varrho \leq 1 \). Hence we have

\[ L_{r,\infty}(P^1) \leq \sum_{j=0}^{M} \sum_{i=0}^{j-1} L_{r,\infty}(P_{j,i}) \leq c_1 \sum_{j=0}^{M} \sum_{i=0}^{j-1} 2^{-j 8} 2^{-is'} 2^{(in+d(j-i))\varrho(\frac{1}{r}-\frac{1}{p})} \]

\[ \leq c_2 \sum_{j=0}^{M} 2^{-j 8} 2^{d(\frac{1}{r}-\frac{1}{p})} \sum_{i=0}^{j-1} 2^{-is'} 2^{(n-d)\varrho(\frac{1}{r}-\frac{1}{p})} \]

\[ \leq c_3 \sum_{j=0}^{M} 2^{-j 8} 2^{d(\frac{1}{r}-\frac{1}{p})} 2^{-js'} 2^{(n-d)\varrho(\frac{1}{r}-\frac{1}{p})} \]

\[ \leq c_4 2^{-M 8} 2^{-Ms'} 2^{Mn\varrho(\frac{1}{r}-\frac{1}{p})} \]

if \( r \) is chosen such that

\[ \left(\frac{1}{r} - \frac{1}{p}\right)(n-d) > s' \quad \text{and} \quad d\left(\frac{1}{r} - \frac{1}{p}\right) > \delta. \]  

(23)
These imply
\[ n \left( \frac{1}{r} - \frac{1}{p} \right) > s' + \delta \]
in the last summation. If \( 0 < d < n \), then one can choose \( r \) according to (23) and in view of (14) this gives
\[ e_{2Mn} \left( P_1^1 : \ell q_1 \left( 2j \delta \ell p_1(w) \right) \rightarrow \ell q_2 \left( \ell p_2 \right) \right) \leq c_4 2^{nM \left( -\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n} \right)}. \] (24)

On the other hand, in case of \( d = 0 \), we have to choose \( r \) in such a way that again
\[ n \left( \frac{1}{r} - \frac{1}{p} \right) > s' + \delta \]
holds and we obtain the same result.

If we consider the case \( d = n \) we obtain in a similar way
\[ e_{2Mn} \left( P_1^1 : \ell q_1 \left( 2j \delta \ell p_1(w) \right) \rightarrow \ell q_2 \left( \ell p_2 \right) \right) \leq c_5 2^{nM \left( -\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n} \right)}. \] (25)

Now we estimate \( L_{r,\infty}^{(e)} (Q^1) \).
\[
L_{r,\infty}^{(e)} (Q^1)^e \leq \sum_{j=0}^{M} \sum_{i=j}^{M} L_{r,\infty}^{(e)} (P_{j,i})^e \leq c_1 \sum_{j=0}^{M} \sum_{i=j}^{M} 2^{-j\delta} 2^{x'2^{\infty q^e} \left( \frac{1}{p} - \frac{1}{p} \right)}
\leq c_2 \sum_{j=0}^{M} \sum_{i=j}^{M} 2^{-j\delta} 2^{x'2^{\infty q^e} \left( \frac{1}{p} - \frac{1}{p} \right)} \leq c_3 \sum_{j=0}^{M} 2^{-j\delta} 2^{-Ms'2^{\infty q^e} \left( \frac{1}{p} - \frac{1}{p} \right)}
\leq c_4 2^{-Ms'2^{\infty q^e} \left( \frac{1}{p} - \frac{1}{p} \right)}
\]

if \( r \) is chosen such that
\[ n \left( \frac{1}{r} - \frac{1}{p} \right) > s'. \] (26)

This implies
\[ e_{2Mn} \left( Q^1 : \ell q_1 \left( 2j \delta \ell p_1(w) \right) \rightarrow \ell q_2 \left( \ell p_2 \right) \right) \leq c_4 2^{nM \left( -\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n} \right)}. \] (27)

Substep 2.2. To estimate \( L_{r,\infty}^{(e)} (Q^2) \), \( L_{r,\infty}^{(e)} (P^2) \) and \( L_{r,\infty}^{(e)} (Q^3) \) we assume that
\[ n \left( \frac{1}{r} - \frac{1}{p} \right) < s' \quad \text{and} \quad d \left( \frac{1}{r} - \frac{1}{p} \right) < \delta. \] (28)

Because of
\[ s' > n/p^* = n \max (0, -1/p) \quad \text{and} \quad \delta > d/p^* = d \max (0, -1/p) \]
we have
\[ n \max \left( 0, \frac{1}{p} \right) < s' + \frac{n}{p} \quad \text{and} \quad d \max \left( 0, \frac{1}{p} \right) < \delta + \frac{d}{p}. \]
Hence, there exists for $0 \leq d \leq n$ an appropriate $r$ with

$$\max\left(0, \frac{1}{p}\right) < \frac{1}{r}$$

and

$$n\left(\frac{1}{r} - \frac{1}{p}\right) < s' \quad \text{and} \quad d\left(\frac{1}{r} - \frac{1}{p}\right) < \delta.$$

We proceed now as before and obtain for $0 \leq d \leq n$ and with condition (28)

$$L_{r,\infty}^{(e)}(Q^2)^\varrho \leq c_2 \sum_{j=0}^{M} 2^{-j\varrho} \sum_{i=M+1}^{\infty} 2^{-is'\varrho} \varrho^{i\varrho\left(\frac{1}{r} - \frac{1}{p}\right)}$$

$$\leq c_3 \sum_{j=0}^{M} 2^{-j\varrho} 2^{-(M+1)s'\varrho} \varrho^{(M+1)n\varrho\left(\frac{1}{r} - \frac{1}{p}\right)}$$

$$\leq c_4 2^{-Ms'\varrho} \varrho^{n\varrho\left(\frac{1}{r} - \frac{1}{p}\right)}.$$  

This gives

$$e^{2M\varrho} \left( Q^2 : \ell_{q_1}(2^{j\varrho}\ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right) \leq c_4 2^{nM\left(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{p}\right)}.$$

Now we estimate $L_{r,\infty}^{(e)}(P^2)$. We assume for $0 < d \leq n$ again condition (28)

$$L_{r,\infty}^{(e)}(P^2)^\varrho \leq c_2 \sum_{j=M+1}^{\infty} 2^{-j\varrho} \sum_{i=0}^{j-1} 2^{-is'\varrho} \varrho^{i\varrho\left(\frac{1}{r} - \frac{1}{p}\right)}$$

$$\leq c_3 \sum_{j=M+1}^{\infty} 2^{-j\varrho} \varrho^{j\varrho\left(\frac{1}{r} - \frac{1}{p}\right)}$$

$$\leq c_4 2^{-Ms'\varrho} \varrho^{M\varrho\left(\frac{1}{r} - \frac{1}{p}\right)}.$$  

This gives

$$e^{2Md} \left( P^2 : \ell_{q_1}(2^{j\varrho}\ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right) \leq c_4 2^{Md\left(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{p}\right)}, \quad d \neq 0.$$  

In the case $d = 0$ we choose instead of condition (28) the parameter $r$ such that

$$s' + \delta > n\left(\frac{1}{r} - \frac{1}{p}\right) > s'.$$

holds and obtain in this case

$$e^{2Mn} \left( P^2 : \ell_{q_1}(2^{j\varrho}\ell_{p_1}(w)) \to \ell_{q_2}(\ell_{p_2}) \right) \leq c_4 2^{nM\left(-\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{p}\right)}.$$

To estimate $L_{r,\infty}^{(e)}(Q^3)$ we assume (28) for $0 \leq d \leq n$

$$L_{r,\infty}^{(e)}(Q^3)^\varrho \leq c_2 \sum_{j=M+1}^{\infty} 2^{-j\varrho} \sum_{i=j}^{\infty} 2^{-is'\varrho} \varrho^{i\varrho\left(\frac{1}{r} - \frac{1}{p}\right)}.$$
Lemma 4

with now the estimate of the

with respect to below

We use once more the characterization from

Hence we obtain

Here

\( T_{j,i} \) and
\( S_{j,i} \) are defined by

where \( \varphi \) denotes a bijection of \( I_{j,i} \) onto \( \{1, 2, \ldots, M_{j,i}\} \). Observe

Hence we obtain

We use once more the characterization from (15) and (16) and now the estimate of the \( M_{j,i} \)
from below (13). In the case \( i = 0 \) either \( M_{j,1} \) or \( M_{j,2} \) can be estimated from below by \( N_{j,0}/2 \),
respectively, that is by Lemma 4, by \( c2^{j_d} \) with a constant \( c > 0 \). Therefore without loss of

generality we can assume \( 2^{j_d} \sim M_{j,2} \) where the estimate from above is justified again by (12)
and Lemma 4. Now we find from (29) with \( k = 2^{j_d} \)

In the same way, using \( M_{0,i+1} \) or \( M_{0,i+2} \) we get from (29) with \( k = 2^{j_n} \)

The estimate for the remaining \( k \)'s follows by monotonicity of the entropy numbers. This finishes
the proof. \( \square \)
5. Entropy numbers of embeddings of 2-microlocal Besov spaces

The next theorem, which is the main theorem of the paper, follows now immediately from Proposition 1, Theorems 5 and 6.

**Theorem 7.** Let \( U \) be a \( d \)-set, \( 0 \leq d \leq n \) and \( w_{i,j}(x) = (1 + 2^j \text{dist}(x, U))^s_i \), \( i = 1, 2 \). Let
\[
\frac{1}{p^*} := \left( \frac{1}{p_2} - \frac{1}{p_1} \right)_+ \quad \text{and} \quad \delta = s_1 - s_2 - n \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \quad \text{and} \quad s' = s'_1 - s'_2.
\]

The embedding
\[
B_{p_1,q_1}^{s_1,s'_1} (\mathbb{R}^n, U) \hookrightarrow B_{p_2,q_2}^{s_2,s'_2} (\mathbb{R}^n, U)
\]
(31)
is compact if, and only if, \( \delta > d/p^* \) and \( s' > n/p^* \).

Moreover, for the entropy numbers of the embedding we have
\[
e_k(\text{id} : B_{p_1,q_1}^{s_1,s'_1} (\mathbb{R}^n, U) \to B_{p_2,q_2}^{s_2,s'_2} (\mathbb{R}^n, U)) \sim k^{-\frac{1}{p_1} + \frac{1}{p_2} - \min(\frac{\delta}{n}, \frac{s'}{n})}.
\]

**Corollary 8.** If \( U = \{x_0\} \) then the embedding (31) is compact if, and only if, \( \delta > 0 \) and \( s' > n/p^* \). Moreover
\[
e_k(\text{id} : B_{p_1,q_1}^{s_1,s'_1} (\mathbb{R}^n, \{x_0\}) \to B_{p_2,q_2}^{s_2,s'_2} (\mathbb{R}^n, \{x_0\})) \sim k^{-\frac{1}{p_1} + \frac{1}{p_2} - \frac{s'}{n}}.
\]

In particular the embedding \( H_{x_0}^{s_1,s'_1} (\mathbb{R}^n) \hookrightarrow C_{x_0}^{s_2,s'_2} (\mathbb{R}^n) \) is compact if, and only if, \( s_1 - s_2 > n/2 \) and \( s'_1 > s'_2 \). Furthermore
\[
e_k(\text{id} : H_{x_0}^{s_1,s'_1} (\mathbb{R}^n) \to C_{x_0}^{s_2,s'_2} (\mathbb{R}^n)) \sim k^{-\frac{1}{2} - \frac{s'_1 - s'_2}{n}}.
\]

**Remark 4.** Finally we want to compare our main theorem with the known results about entropy numbers of embeddings of Besov spaces with polynomial weights. We will consider the case \( U = \{0\} \). For the weighted spaces we have
\[
e_k(\text{id} : B_{p_1,q_1}^{s_1} (\mathbb{R}^n, (x)^{\alpha}) \hookrightarrow B_{p_2,q_2}^{s_2} (\mathbb{R}^n)) \sim k^{-(\frac{1}{p_1} - \frac{1}{p_2}) - \min(\frac{\delta}{n}, \frac{s}{n})}
\]
if \( \min(\delta, \alpha) > n/p^* \) and \( \delta \neq \alpha \) where again \( \delta = s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2}) \)—see for example [16].

In the case \( U = \{0\} \) the weights of the admissible weight sequence become \( w_j(x) = (1 + 2^j |x|)^s_j \), that is for each fixed \( j \) they are equivalent to the polynomial weight \( w(x) = (1 + |x|^s)^j = (x)^s \).

The spaces \( B_{p,q}^s (\mathbb{R}^n, (x)^{\alpha}) \) have a wavelet characterization with the same wavelet basis as described in Section 2.2. Therefore we can reduce the comparison of entropy numbers of the embeddings to the comparison of corresponding weights. Moreover we can shift all considerations to the case where the target space is \( B_{p_2,q_2}^{n/p_2} (\mathbb{R}^n) \) since
\[
e_k(\text{id} : B_{p_1,q_1}^{s_1} (\mathbb{R}^n, \{0\}) \to B_{p_2,q_2}^{s_2} (\mathbb{R}^n, \{0\})) \sim e_k(\text{id} : B_{p_1,q_1}^{s+n/p_1} (\mathbb{R}^n, \{0\}) \to B_{p_2,q_2}^{n/p_2} (\mathbb{R}^n))
\]
and
\[ e_k(\text{id} : B^s_{1,q_1}(\mathbb{R}^n, \langle x \rangle^\alpha) \to B^t_{2,q_2}(\mathbb{R}^n)) \\
\sim e_k(\text{id} : B^{s+n/p_1}_{p_1,q_1}(\mathbb{R}^n, \langle x \rangle^\alpha) \to B^{t/n/p_2}_{p_2,q_2}(\mathbb{R}^n)). \] (32)

If \( s' > 0 \) then by easy calculations we have
\[ 2^{-js'} w_j(2^{-j}m)^{s'} \leq (2^{-j}m)^{s'} \leq w_j(2^{-j}m)^{s'} \leq 2^{js'}(2^{-j}m)^{s'}. \]

These inequalities give for \( \alpha = s' \) the following embeddings
\[ B^{t,s'}_{p,q}(\mathbb{R}^n, \langle x \rangle^{s'}) \hookrightarrow B^{t,s'}_{p,q}(\mathbb{R}^n, \{0\}) \hookrightarrow B^t_{p,q}(\mathbb{R}^n, \langle x \rangle^{s'}) \hookrightarrow B^{t-s'}_{p,q}(\mathbb{R}^n, \{0\}). \] (33)

If \( \delta > s' \), then
\[ B^{\delta+n/p_1-s',s'}_{p_1,q_1}(\mathbb{R}^n, \{0\}) \hookrightarrow B^{n/p_2}_{p_2,q_2}(\mathbb{R}^n). \]

Hence the estimates for the entropy numbers of embeddings of the spaces with a polynomial weight follow from (33) and Corollary 8 with \( \{x_0\} = \{0\} \).

Vice versa the result for the spaces with a polynomial weight gives in this particular case the result of Corollary 8.

References


