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An Operator Not Satisfying Lomonosov's Hypothesis (1)

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An example is presented of a Hilbert space operator such that no non-scalar operator that commutes with it commutes with a non-zero compact operator. This shows that Lomonosov's invariant subspace theorem does not apply to every operator.

The invariant subspace theorem of Lomonosov [6–8] includes the following assertion: if C is an operator such that $CB = BC$ for an operator B that is not a multiple of the identity and that commutes with a non-zero compact operator, then C has a non-trivial invariant subspace. As Pearcy and Shields [7] pointed out (cf. Remark (a) at the end of this note), it is not clear that there are operators C for which there is no B satisfying the above hypothesis. Thus, it appears possible that Lomonosov's work implies that all operators have invariant subspaces. An obvious C to consider is the unilateral shift. Then the operators that commute with C are the analytic Toeplitz operators, so the question becomes: does any non-scalar analytic Toeplitz operator commute with a non-zero compact operator? Partial results suggesting that the answer to this question was negative were obtained by many authors: see [2, 4, 11] and the references given there. Recently, however, Cowen [3] found an analytic Toeplitz operator that does commute with a non-zero compact operator. This example stimulated the present authors to investigate whether weighted shifts satisfy Lomonosov's hypothesis.

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In this note we prove that certain weighted shifts do not satisfy Lomonosov's hypothesis. The basic properties of weighted shifts were developed by R. L. Kelley, A. Lambert, A.L. Shields and others. They are elegantly described in [10], by Allen Shields to whom we are greatly indebted. It is inconceivable that we would have found these examples if Shields' exposition had not been available to us. Following Shields, we consider weighted shifts as multiplication operators M_z on weighted l^2 spaces $H^2(\beta)$. For $\{\beta(n)\}_{n=0}^{\infty}$ a sequence of positive numbers with $\beta(0) = 1$, let $\beta = \{\beta(n)\}_{n=0}^{\infty}$ and define $H^2(\beta)$ to be the set of all formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2$ converges. The operator M_z takes $\sum_{n=0}^{\infty} a_n z^n$ into $\sum_{n=0}^{\infty} a_n z^{n+1}$; under appropriate conditions on β, M_z is a bounded linear transformation mapping $H^2(\beta)$ into itself. As described in [10, p. 59], every weighted shift is unitarily equivalent to an M_z on a suitable $H^2(\beta)$. The shifts that we can prove do not satisfy Lomonosov's hypothesis are what Shields calls "quasi-analytic" shifts. Their properties include ([10, p. 103]):

(1) The functions in $H^2(\beta)$ and their derivatives are analytic on the open unit disc and continuous on the closed disc.

(2) If a function $f \in H^2(\beta)$ has infinitely many zeros or a zero of infinite multiplicity in the closed disc, then f is identically 0.

(3) The operator M_z is strictly cyclic ([10, p. 103]); hence its commutant $\{M_\phi : \phi \in H^\infty(\beta)\}$ (cf. [10, p. 62]) is equal to $\{M_\phi : \phi \in H^2(\beta)\}$ [10, p. 94].

(4) If $f \in H^2(\beta)$ and f has no zeros in the closed unit disc, then $1/f \in H^2(\beta)$ [10, Corollary 1, p. 94].

(5) For each complex w with $|w| \leqslant 1$ there is a $k_w \in H^2(\beta)$ such that $(f, k_w) = f(w)$ for $f \in H^2(\beta)$ and $(M_z - w)^* k_w = 0$ [10, p. 73].

(6) If $|w| < 1$ and $f(w) = 0$ for $f \in H^2(\beta)$, then $f(z) = (z - w) f_1(z)$ and $f_1 \in H^2(\beta)$ [10, Corollary, p. 77].

One example of a quasi-analytic shift given by Shields is M_z on $H^2(\beta)$ where $\beta(n) = \exp(n^{1/2})$; the corresponding weighted shift has weights $\exp((n+1)^{1/2} - n^{1/2})$. The fact that this shift is strictly cyclic is proven in [10, p. 103], while property 2 above follows from a theorem of Carleson [1].

The only properties of the shift M_z that we require are the six listed above. Let M_ϕ be any operator in the commutant of M_z with ϕ a non-constant function (so that M_ϕ is not a multiple of the identity operator).

THEOREM. *The only compact operator that commutes with M_ϕ is* 0.

The proof of the theorem involves studying the eigenspaces of M_φ^*, which we do in several lemmas. We also need the following fact about matrices. Recall that the Schur product of the $n \times n$ matrices (a_{ij}) and (b_{ij}) is defined to be the matrix $(a_{ij} b_{ij})$; let $A \cdot B$ denote the Schur product of A and B.

LEMMA 1. *If A is an $n \times n$ matrix such that $(A \cdot B)^n = 0$ for all $n \times n$ matrices B, then at least one column of A has all entries* 0.

Proof. If no column of A has all entries 0, then for each j choose an i_j so that $a_{i_j j} \neq 0$. Let B denote the matrix (b_{ij}) with

$$\begin{aligned} b_{ij} &= 1/a_{ij} \qquad \text{for} \quad i = i_j \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

Then $A \cdot B$ is a matrix with exactly one 1 in each column and with all other entries 0. Such a matrix cannot be nilpotent, since it and all its powers send basis elements into basis elements. This contradicts the assumption that $(A \cdot B)^n = 0$, and the lemma is proven.

We are grateful to M. R. Emami for the above proof, which is a substantial simplification of our original argument.

With $\phi \in H^2(\beta)$ as above, define

$$E = \{\lambda: |\lambda| < 1, \phi(\lambda) \notin \phi(\{z: |z| = 1\}) \text{ and } \phi'(w) \neq 0 \text{ when } \phi(w) = \phi(\lambda)\}.$$

If F is the set of λ in the disc such that $\phi'(w) = 0$ for some w with $\phi(w) = \phi(\lambda)$, then F is countable since ϕ maps at most finitely many points into the same image, by property (2). Thus E is uncountable, for the union of E and F includes the open set

$$\phi^{-1}[\phi(\{z: |z| < 1\})/\phi(\{z: |z| = 1\})];$$

this open set is not empty, for property (1) implies $\phi(\{z: |z| = 1\})$ has Lebesgue measure 0.

Note that $\lambda \in E$ and $\phi(w) = \phi - \phi(w)$ has a simple zero at w.

For $\lambda \in E$ let $\mathscr{M}_\lambda$ denote the nullspace of $[M_\phi - \phi(\lambda)]^*$.

LEMMA 2. *For $\lambda \in E$, $\mathscr{M}_\lambda$ is the linear span of $\{k_w: \phi(w) = \phi(\lambda)\}$, where the functions k_w are as in property* (5) *above.*

Proof. First, if $\phi(w) = \phi(\lambda)$, then $f \in H^2(\beta)$ implies

$$(f, [M_\phi - \phi(\lambda)]^* k_w) = ([\phi - \phi(\lambda)]f, k_w) = (\phi(w) - \phi(\lambda)) f(w) = 0.$$

Thus the span of $\{k_w: \phi(w) = \phi(\lambda)\}$ is contained in $\mathscr{M}_\lambda$.

For the reverse inclusion, suppose that f is orthogonal to $\{k_w: \phi(w) = \phi(\lambda)\}$. We claim that then f must be in the range of $M_\phi - \phi(\lambda)$, (and hence orthogonal to the nullspace of $(M_\phi - \phi(\lambda))^*$, which completes the proof). For this, since $f \perp k_w$ implies $f(w) = 0$, we can write $f = pf_1$, where

$$p(z) = \prod \{z - w: \phi(w) = \phi(\lambda)\}$$

and $f_1 \in H^2(\beta)$ by property (6). Similarly write $\phi - \phi(\lambda) = p\phi_1$. Then ϕ_1 is also in $H^2(\beta)$. Moreover, the fact that $\phi - \phi(\lambda)$ has only simple zeros for λ in E implies that ϕ_1 does not vanish on $\{z: |z| \leqslant 1\}$. From property (4) above we conclude that $1/\phi_1 \in H^2(\beta)$. Hence $p[\phi - \phi(\lambda)]^{-1} = 1/\phi_1$ is in $H^2(\beta)$, and so is

$$\begin{aligned} g &= (1/\phi_1)f_1 \\ &= [\phi - \phi(\lambda)]^{-1}f. \end{aligned}$$

Clearly $f = [M_\phi - \phi(\lambda)]g$, which completes the proof.

LEMMA 3. *There is an integer N such that $K^N = 0$* whenever K is a compact operator commuting with M_ϕ.

Proof. If K is compact and commutes with M_ϕ, then K^* commutes with M_ϕ^* and thus leaves the subspaces $\mathscr{M}_\lambda$ of Lemma 2 invariant. Since $\mathscr{M}_\lambda$ is finite-dimensional for $\lambda \in E$, and since E is uncountable, there is an integer N such that uncountably many of the $\mathscr{M}_\lambda$ have dimension N. We show that $K^N = 0$. The restriction $K^* \mid \mathscr{M}_\lambda$ of K^* has an eigenvalue. By Lemma 2, if $\phi(\lambda) \neq \phi(\lambda_i)$ for $i = 1,..., m$, then $\mathscr{M}_\lambda$ intersects the linear span of $\{\mathscr{M}_{\lambda_1},..., \mathscr{M}_{\lambda_m}\}$ only in 0. Since the spectrum of K is countable and eigenspaces corresponding to non-zero eigenvalues are finite-dimensional, the spectrum of $K^* \mid \mathscr{M}_\lambda$ is $\{0\}$ for all but at most countably many λ. If $\sigma(K^* \mid \mathscr{M}_\lambda) = \{0\}$ and the dimension of $\mathscr{M}_\lambda$ is N, then $(K^* \mid \mathscr{M}_\lambda)^N = 0$. But each $\mathscr{M}_\lambda$ contains at least one kernel function k_w, so $(K^*)^N k_w = 0$ for an infinite number of w. Since any infinite collection of the k_w span $H^2(\beta)$, (by properties (2) and (5)), it follows that $(K^*)^N = 0$, and $K^N = 0$.

We can now easily finish the proof of the theorem.

Proof of Theorem. Let K_0 be a compact operator that commutes with M_ϕ, and let $\lambda \in E$. For any polynomials p and q, the operator $M_p^* K_0^* M_q^*$ is compact and commutes with M_φ^*. Note that Lemma 2 implies that $M_z^* \mid \mathscr{M}_\lambda$ has a diagonal matrix with distinct eigenvalues relative to the basis $\{k_{w_1},..., k_{w_N}\}$ of $\mathscr{M}_\lambda$. Thus for each i between 1 and N there is a polynomial p_i such that $p_i(M_z^*) \mid \mathscr{M}_\lambda$ is the projection P_i onto k_{w_i} along the span of $\{k_{w_j}: j \neq i\}$. For any $N \times N$ matrix (b_{ij}) the operator $\sum_{i,j=1}^N b_{ij} P_i K_0^* P_j$ on $\mathscr{M}_\lambda$ is the restriction of $\sum_{i,j=1}^N b_{ij}\, p_i(M_z^*)\, K_0^* p_j(M_z^*)$ to $\mathscr{M}_\lambda$, and hence, by Lemma 3, is nilpotent of order at most N. This operator is the Schur product of (b_{ij}) and $K_0^* \mid \mathscr{M}_\lambda$, so Lemma 1 implies that the matrix of $K_0^* \mid \mathscr{M}_\lambda$ with respect to $\{k_{w_1},..., k_{w_N}\}$ has a zero column. Hence $K_0^* k_{w_j} = 0$ for some k_{w_j} in $\mathscr{M}_\lambda$. Thus $K_0^* k_w = 0$ for an infinite number of w, so $K_0^* = 0$ and $K_0 = 0$.

Remark (a). Pearcy and Shields [7] asked if there are operators C which do not satisfy the following apparently more general hypothesis: $CB = BC$ for some non-scalar B which is quasi-similar to an A that commutes with a non-zero compact K. However, as pointed out in [9], this is equivalent to what we have called Lomonosov's hypothesis. For if $AX = XB$ and $YA = BY$, where X and Y are injective and have dense ranges, and if $AK = KA$, then $B(YKX) = YAKX = YKAX = (YKX)B$, so B commutes with the compact operator YKX.

Remark (b). The above theorem together with the result of Cowen [3] suggests the question: does any non-scalar analytic Toeplitz operator commute with a non-zero nilpotent compact? (As Cowen [2] points out, any compact that commutes with a non-scalar analytic Toeplitz operator must be quasinilpotent.) The answer to this question is affirmative. For if T_ϕ is an analytic Toeplitz operator which commutes with a compact operator K (an example of which is given in Cowen [3]), then the operator $T_\phi \oplus T_\phi$ commutes with the nilpotent compact $\left[\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right]$. But $T_\phi \oplus T_\phi$ is (unitarily equivalent to) the analytic Toeplitz operator $T_{\phi(z^2)}$.

Remark (c). The set of all operators that do not satisfy Lomonosov's hypothesis is strongly dense, as follows from [5] and the fact that this set is invariant under similarity.

Remark (d). It is still, of course, of interest to determine the operators that *do* satisfy Lomonosov's hypothesis. It might be noted that if $A = A_1 \oplus A_2$ is any reducible operator, for example, then A commutes with $1 \oplus 0$, and $1 \oplus 0$ commutes with $K_1 \oplus K_2$ for all compact K_1 and K_2.

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