

# The distance spectra of Cayley graphs of Coxeter groups

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## ABSTRACT

The absolute (respectively, weak) order graph on a Coxeter group is the graph underlying the absolute (respectively, weak) order poset. We investigate the distance spectra of many of these graphs and pose several open problems.

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## 1. Introduction

Let  $(W, S)$  be a finite Coxeter system<sup>1</sup> and let  $T = \{wsw^{-1} : w \in W, s \in S\}$  be the set of all reflections of  $W$ . Let  $X$  be a subset of generators of  $W$  satisfying  $x \in X \Rightarrow x^{-1} \in X$  and  $1 \notin X$ , and let  $\Gamma(W, X)$  be the Cayley graph with vertices  $V(\Gamma) = \{w : w \in W\}$  and edges  $E(\Gamma) = \{\{w, wx\} : w \in W, x \in X\}$ . We can define two natural classes of Cayley graphs on Coxeter groups: the *weak order graph*  $\Gamma(W, S)$ , the graph underlying the weak Bruhat order on  $W$ , and the *absolute order graph*  $\Gamma(W, T)$ , the graph underlying the absolute order on  $W$ .<sup>2</sup>

Of particular interest are the Cayley graphs on the symmetric group  $\mathfrak{S}_n := W(A_{n-1})$ , which arise in many different fields of computer science and mathematics, notably in probability and statistics [14]. In the computer science literature (see, e.g., [2]) one encounters the weak order graph on  $\mathfrak{S}_n$  (known as the *bubble sort graph* when  $S$  is chosen to be the set of adjacent transpositions), the absolute order graph on  $\mathfrak{S}_n$  (known as the *transposition graph*), the *star graph*  $\Gamma(\mathfrak{S}_n, \{(1, k) \in \mathfrak{S}_n : 2 \leq k \leq n\})$ , and the *pancake graph*  $\Gamma(\mathfrak{S}_n, \{(k, k-1, \dots, 1, k+1, k+2, \dots, n) \in \mathfrak{S}_n : 2 \leq k \leq n\})$ . Another Cayley graph on  $\mathfrak{S}_n$  that arises in extremal combinatorics [17,32] is the *derangement graph*  $\Gamma(\mathfrak{S}_n, \{\sigma \in \mathfrak{S}_n : \sigma(i) \neq i, i \in [n]\})$ .

The adjacency spectra of these graphs (the eigenvalues of their adjacency matrices) are of interest for their own sake, as well as for various applications such as card shuffling (random walks on the symmetric group). The spectra of the transposition graph, the star graph, and the derangement graph were determined by Diaconis and Shahshahani [15], Flatto et al. [18], and this author [35], respectively. In each of these cases the computation was made possible by the fact that the generating sets have an especially nice structure. The generating sets of the transposition graph and the derangement graph

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<sup>1</sup> For standard facts about Coxeter groups, see, e.g., [8,9,27,31].

<sup>2</sup> For an extensive discussion of the absolute order on Coxeter groups, see e.g., [4].

are closed under conjugation; the corresponding graphs are said to be *normal*.<sup>3</sup> Although the star graph is not normal, the sum of the elements of its generating set, which is the relevant construct for computing the spectrum of the corresponding Cayley graph, is a difference of conjugacy class sums; more specifically the sum is a Jucys–Murphy element, whose spectrum has an elegant combinatorial interpretation (see, e.g., [41]).

The generating sets of  $\Gamma(\mathfrak{S}_n, S)$  and the pancake graph do not enjoy similarly felicitous properties, which may explain why their spectra have not yet been determined in full. Partial results are known for the Laplacian spectra of some of the weak order graphs, and hence for their adjacency spectra.<sup>4</sup> Using the technique of equitable partitions [20], Bacher [5] was able to obtain a large class of eigenvalues of the Laplacian of  $\Gamma(\mathfrak{S}_n, S)$ , but the eigenvalues of the pancake graph appear to be unknown. Akhiezer demonstrated [3] that the Laplacian spectrum of  $\Gamma(W, S)$  contains the eigenvalues of the Cartan matrix corresponding to  $W$ , each of which appears with multiplicity at least  $|S|$ .

Another natural graph invariant that has received less attention is the distance spectrum. Let  $d_\Gamma(u, v)$  be the length of a shortest path between  $u$  and  $v$  in  $\Gamma$ . It is easy to see that  $d_\Gamma(u, v)$  is symmetric, nonnegative definite, and satisfies the triangle inequality, so it is a metric on  $\Gamma$ . The *distance spectrum* of  $\Gamma$  is the spectrum of the matrix  $d_\Gamma$ , whose  $(u, v)$  entry is  $d_\Gamma(u, v)$ . The *distance polynomial*  $p_\Gamma(q)$  is the characteristic polynomial of  $d_\Gamma$ . The distance matrix of a graph plays an important role in chemistry (e.g., the Wiener index) [33], and it appears in a wide variety of social science fields [6,10]. The distance polynomials of trees have been studied in connection with a problem in data communications [22], and distance polynomials can be used to distinguish some adjacency isospectral graphs (although the distance polynomial is far from a complete invariant [6]).<sup>5</sup>

In this paper we take the first steps toward the determination of the distance spectra of the weak and absolute order graphs of the Coxeter groups. We are able to calculate these spectra exactly in a large number of cases, which is unusual because the computation of distance spectra is generally considered more difficult than the computation of adjacency spectra [6]. In the absolute order case we discover that the eigenvalues are all integral. Even more striking, we find that in the weak order case there are generally only a few distinct eigenvalues, suggesting the presence of additional underlying structures, some features of which are revealed by our analysis.

For the graphs associated to Coxeter groups, the distance function has a well known group theoretic interpretation. Every  $w \in W$  may be written as a word in the reflections  $T$  (respectively, simple reflections  $S$ ), and the minimum number of such reflections that must be used is the *absolute length*  $\ell_T(w)$  (respectively, *length*  $\ell_S(w)$ ) of  $w$ . The length functions satisfy the properties  $\ell(w) = \ell(w^{-1})$ , and  $\ell(w) = 0$  if and only if  $w = 1$ , where henceforth  $\ell$  is generic for  $\ell_S$  or  $\ell_T$ .

Given  $\ell_S$  (respectively,  $\ell_T$ ) one defines two order relations on  $W$ , called *left* and *right weak* (respectively, *absolute*) *order*. Right order on  $W$  is defined by

$$u \leq_R v \iff \ell(u) + \ell(u^{-1}v) = \ell(v),$$

while left order is defined by  $u \leq_L v \iff u^{-1} \leq_R v^{-1}$ . These two orders are identical in the absolute case, and distinct but isomorphic under the map  $w \mapsto w^{-1}$  in the weak case (see, e.g., [4]). Therefore we may define the weak and absolute order graphs using either left or right order. For consistency with our Cayley graph definition above, let us choose left order. Then, by an argument similar to that given in [8, Proposition 3.1.6] applied to both the weak and absolute orders, if  $u \leq_L v$  the intervals  $[u, v]_L$  and  $[1, uv^{-1}]_L$  are (order) isomorphic. It follows that  $u \leq_L v$  implies  $d(u, v) = d(1, uv^{-1}) = \ell(uv^{-1})$ . But, by our definitions, right multiplication by elements of  $W$  is an automorphism of both the weak and absolute order graphs, so  $d(u, v) = d(uw, vw) = d(uv^{-1}, 1) = d(1, uv^{-1}) = \ell(uv^{-1})$  holds for any pair of group elements  $\{u, v\}$ .

The tools used to obtain the distance spectra of the weak and absolute order graphs have very different flavors, attributable to the fact that the absolute length is a conjugacy class invariant. This permits the use of simple character theoretic techniques in the absolute order case. A wider variety of tools must be brought to bear in the weak order case, yet one can obtain simple closed formulae for the distance spectra of the weak order graphs of types A, D, E, and I, and perhaps for many of the others.

We begin with a general discussion of the fundamental relationship between the distance matrices of interest here and the group algebras of the Coxeter groups. This is followed by an analysis of the distance spectra of the absolute order graphs, then by a discussion of the distance spectra of the weak order graphs. Various conjectures and problems are posed, some of which are partially answered along the way.

## 2. Distance matrices and the group algebra

Define

$$\mathcal{L} := \sum_{w \in W} \ell(w) \quad w \in CW.$$

<sup>3</sup> Unfortunately, the word ‘normal’ carries two distinct meanings in the context of Cayley graphs. One meaning is that the generating set is closed under conjugation, while the other is that  $W$  is a normal subgroup of  $\text{Aut } \Gamma(W, X)$ , the group of automorphisms of  $\Gamma(W, X)$ . In general it is possible to have one without the other. For example,  $\Gamma(W, S)$  is normal in the latter sense (see, e.g., [1, Exercise 3.35]) but not in the former sense.

<sup>4</sup> As Cayley graphs are uniform, the Laplacian and adjacency matrices differ by a constant multiple of the identity.

<sup>5</sup> For some other work on distance polynomials, see, e.g., [12,26,28,33,43].

**Table 1**  
Distance polynomials of absolute order graphs for some Coxeter types.

Type	Distance polynomial $p_{\Gamma(W,T)}(q)$
$A_2$	$(q - 7)(q - 1)(q + 2)^4$
$A_3$	$(q - 46)(q - 2)^9(q + 2)^5(q + 6)^9$
$A_4$	$(q - 326)(q - 6)^{37}(q - 2)^{25}(q + 4)^{16}(q + 6)^{25}(q + 24)^{16}$
$B_3$	$q^9(q - 100)(q - 4)^{14}(q + 4)^9(q + 8)^{15}$
$H_3$	$(q - 268)(q - 8)^{18}(q - 4)^{25}(q + 2)^{32}(q + 8)^{25}(q + 12)^{18}(q + 32)$
$D_4$	$(q - 544)(q - 16)(q - 8)^{36}(q - 4)^{64}q^{27}(q + 8)^{20}(q + 16)^{27}(q + 32)^{16}$
$F_4$	$q^{160}(q - 3600)(q - 240)(q - 48)^2(q - 32)^{36}(q - 24)^{16}(q - 16)^{162}(q - 12)^{256}(q - 8)^{144}(q + 16)^{117}$ $(q + 24)^{128}(q + 48)^{105}(q + 96)^{24}$
$I_2(5)$	$(q - 13)(q - 3)(q + 2)^8$
$I_2(6)$	$(q - 16)(q - 4)(q + 2)^{10}$
$I_2(7)$	$(q - 19)(q - 5)(q + 2)^{12}$

Let  $\rho$  be the (left) regular representation of  $W$  extended linearly to  $\mathbb{C}W$ . Then the matrix representation of  $\mathcal{L}$  with respect to the basis elements  $w \in W$  is precisely  $d$ :

$$\rho(\mathcal{L})v = \sum_w \ell(w)\rho(w)v = \sum_w \ell(w)wv = \sum_u \ell(uv^{-1})u = \sum_u d(u, v)u.$$

Thus, to investigate the distance spectrum of the absolute order graphs, we may study the properties of  $\mathcal{L}_T$ , which is constructed from  $\ell_T$ , and *mutatis mutandis* for the weak order graphs.

### 3. The absolute order graphs

#### 3.1. A formula for the spectrum

In Table 1 we have calculated the distance polynomials of the absolute order graphs for some Coxeter types.<sup>6</sup> In all the cases computed, the spectra are observed to be integral; this turns out to hold in general, as we shall see in Section 3.2. Using character theoretic techniques we have the following result.

**Theorem 1.** Let  $d_T$  be the distance matrix of the absolute order graph  $\Gamma(W, T)$ . Then the eigenvalues of  $d_T$  are given by

$$\eta_\chi = \frac{1}{f} \sum_K |K| \ell_T(w_K) \chi(w_K),$$

where  $K$  runs through the conjugacy classes of  $W$ ,  $w_K$  is any element of  $K$ ,  $\chi$  ranges over all the irreducible characters of  $W$ , and  $f = \chi(1)$ . Moreover, the multiplicity of  $\eta_\chi$  is  $f^2$ .

**Proof.** By a lemma of Carter [11],  $\ell_T(w) = \text{codim } V^w$ , where  $V^w$  is the fixed point space of  $w$  in the reflection representation  $V$ . Hence,  $\ell_T$  is constant on conjugacy classes. The result now follows by applying Lemma 5 of [15] to  $\rho(\mathcal{L}_T)$ , recalling that each irreducible representation appears  $f$  times in the regular representation.  $\square$

#### 3.2. Integrality

If  $\ell_T$  were only constant on conjugacy classes there would be no necessary reason for the eigenvalues given by Theorem 1 to be integral. But  $\ell_T$  satisfies a stronger property that does ensure integrality. The following argument is essentially due to Isaacs [29].<sup>7</sup>

Let  $G$  be a finite group, and let  $K(g)$  denote the conjugacy class of  $g$ . The rational class  $Kr(g)$  of  $g$  is a disjoint union of conjugacy classes:

$$Kr(g) = \bigcup_{\langle z \rangle = \langle g \rangle} K(z) = \bigcup_{s \in R_r} K(g^s),$$

where  $\langle g \rangle$  is the cyclic group generated by  $g$ ,  $r$  is the order of  $g$ , and  $R_r$  is a reduced residue system modulo  $r$ .

**Lemma 2.** Let  $V$  be a  $G$ -module, and let  $\psi(g) = \dim V^g$ . Then  $\psi$  is constant on rational classes.

<sup>6</sup> These polynomials (and the ones in Table 2) were computed in MAPLE with the help of John Stembridge's package COXETER. The author is grateful to Stembridge for making his package freely available.

<sup>7</sup> Isaacs omits many details, asserting that the main result of this section, Theorem 6, holds for "quite trivial reasons". At the risk of dwelling on trivialities, the author felt that it was worthwhile to elucidate some of the relevant details.

**Proof.** If the spectrum of  $g$  on  $V$  is  $\{\lambda_i\}_{i=1}^n$ , then the spectrum of  $g^s$  is  $\{\lambda_i^s\}_{i=1}^n$ . If  $g$  has order  $r$  then  $\lambda_i$  is an  $r$ th root of unity, so if  $s$  is coprime to  $r$  then  $\lambda_i = 1 \Leftrightarrow \lambda_i^s = 1$ . Hence  $g$  and  $g^s$  have precisely the same number of eigenvalues equal to unity, and  $\psi(g) = \psi(g^s)$ . The result now follows because  $\psi$  is constant on conjugacy classes.  $\square$

**Lemma 3.** For every irreducible character  $\chi$  of  $G$  and for any  $g \in G$ ,

$$\sum_{h \in \text{Kr}(g)} \chi(h) \in \mathbb{Z}.$$

**Proof.** Let  $g$  have order  $r$ . Then

$$\varphi(g) := \sum_{h \in \text{Kr}(g)} \chi(h) = \sum_{s \in R_r} \chi(g^s) |K(g^s)| = |K(g)| \sum_{s \in R_r} \chi(g^s),$$

where we used the fact that  $(tgt^{-1})^s = tg^s t^{-1}$  implies  $|K(g^s)| = |K(g)|$ .

Let  $\zeta$  be a primitive  $r$ th root of unity. Let  $\mathbb{Q}_r = \mathbb{Q}[\zeta]$  be the  $r$ th cyclotomic field, and let  $\mathcal{G} := \text{Gal}(\mathbb{Q}_r/\mathbb{Q})$  be the corresponding Galois group.  $\mathcal{G}$  acts on elements of  $\mathbb{Q}_r$  by sending  $\zeta$  to  $\zeta^m$  for some  $m$  relatively prime to  $r$ . Let  $V$  be the  $G$ -module affording  $\chi$ , and let the spectrum of  $g$  on  $V$  be  $\{\lambda_i\}_{i=1}^n$ . Then for any  $s$ ,

$$\chi(g^s) = \sum_i \lambda_i^s \in \mathbb{Q}_r.$$

It follows that for any  $\sigma \in \mathcal{G}$ ,  $\chi(g^s)^\sigma = \chi(g^k)$  for some  $k$  relatively prime to  $r$ . Hence  $\varphi(g)^\sigma = \varphi(g)$ . But an algebraic integer is rational if and only if it is fixed by  $\mathcal{G}$ .  $\square$

**Lemma 4.** Let  $\psi : G \rightarrow \mathbb{Z}$  be an integral-valued function on  $G$ . If  $\psi$  is constant on rational classes, then  $|G|\psi$  is a virtual character of  $G$ .

**Proof.**  $\psi$  is certainly constant on conjugacy classes, so it can be written as a linear combination of irreducible characters:

$$\psi = \sum a_\chi \chi.$$

Let  $T$  be a set of representatives from each rational class. Let  $[\cdot, \cdot]$  denote the usual inner product on characters. As the rational classes partition  $G$ ,

$$\begin{aligned} |G|a_\chi &= [\chi, |G|\psi] = \sum_{g \in G} \chi(g)\psi(g) = \sum_{t \in T} \sum_{h \in \text{Kr}(t)} \chi(h)\psi(h) \\ &= \sum_{t \in T} \psi(t) \sum_{h \in \text{Kr}(t)} \chi(h) \in \mathbb{Z}, \end{aligned}$$

by Lemma 3.  $\square$

**Remark 5.** The main result of [29] is actually a somewhat stronger (and much less “trivial”) fact, namely that if  $\psi(g) = \dim V^g$  then  $e\psi$  is a virtual character where  $e$  is the least common multiple of the orders of the elements of  $G$ .<sup>8</sup>

**Theorem 6.** The distance spectra of the absolute order graphs are integral.

**Proof.** By Theorem 1 we can write  $\eta_\chi = (|W|/f)[\ell_T, \chi]$ . Because central characters (expressions of the form  $\chi(w_K)|K|/f$ ) are algebraic integers (e.g., [30, Theorem 3.7]),  $\eta_\chi$  is an algebraic integer. Carter’s lemma and Lemma 2 imply that  $\ell_T$  is constant on rational classes of  $W$ , so by Lemma 4,  $|W|\ell_T$  is a virtual character. Hence  $\eta_\chi$  must be a rational integer.  $\square$

### 3.3. Eigenvalues for various Coxeter types

For every Coxeter group  $W$  and for each irreducible character  $\chi$  of  $W$ , define a Poincaré-type polynomial by

$$P_\chi(q) := \frac{1}{f} \sum_{w \in W} \chi(w)q^{\ell_T(w)}.$$

From Theorem 1 we get

**Corollary 7.** Let  $\chi$  be an irreducible character of  $W$  of degree  $f$ . Then

$$\eta_\chi = \left. \frac{dP_\chi(q)}{dq} \right|_{q=1},$$

is an eigenvalue of the distance matrix of the absolute order graph  $\Gamma(W, T)$  with multiplicity  $f^2$ , and these eigenvalues constitute the entire distance spectrum.

<sup>8</sup>  $e$  is often called the exponent of  $G$ , but as this word has a different meaning for reflection groups, we eschew its use here.

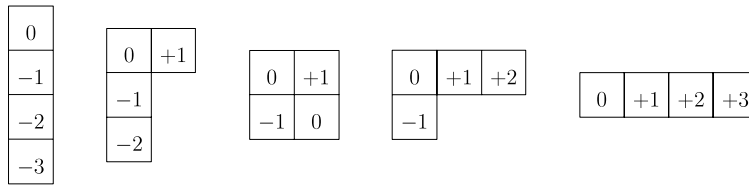


Fig. 1. Ferrers diagrams and their contents for  $n = 4$ .

Using Corollary 7 we may compute the distance spectra of some of the absolute order graphs more explicitly. First we consider the graphs of type A. Recall [21] that the irreducible characters  $\chi_\lambda$  of  $\mathfrak{S}_n$  are indexed by partitions  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$  of  $n$  (written  $\lambda \vdash n$ ) with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t > 0$ . The degree of  $\chi_\lambda$  is  $f_\lambda$ . Viewing the partition  $\lambda$  as a Ferrers diagram with  $\lambda_i$  cells in the  $i$ th row, the content  $c(u)$  of the cell  $u$  in the  $i$ th row and  $j$ th column is  $c(u) := j - i$ , and the hook length  $h(u)$  is  $\lambda_i + \lambda'_j - i - j + 1$ , where  $\lambda'$  is the conjugate partition obtained by flipping the diagram of  $\lambda$  about the diagonal.<sup>9</sup>

**Theorem 8.** *The eigenvalues of the distance matrix of the absolute order graph  $\Gamma(\mathfrak{S}_n, T)$  are given by*

$$\eta_\lambda = \sum_{u \in \lambda} c(u) \prod_{u' \neq u} (1 + c(u')).$$

In particular, the eigenvalues are integral. Each one occurs with multiplicity  $f_\lambda^2$ , where

$$f_\lambda = \frac{n!}{\prod_u h(u)}.$$

**Proof.** For any permutation  $\sigma \in \mathfrak{S}_n$ ,  $\dim V^\sigma = k(\sigma) - 1$  where  $k(\sigma)$  is the number of cycles in the cycle decomposition of  $\sigma$ . Hence,  $\ell_T(\sigma) = n - k(\sigma)$ , because  $\dim V = n - 1$ . We also have ([39, Cor. 7.21.6 and Ex. 7.50]; see also [34]),

$$\frac{1}{f_\lambda} \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) q^{k(\sigma)} = \prod_{u \in \lambda} (q + c(u)).$$

Thus

$$P_\lambda(q) = q^n \prod_{u \in \lambda} (q^{-1} + c(u)) = \prod_{u \in \lambda} (1 + qc(u)).$$

Now differentiate with respect to  $q$  at  $q = 1$ . The multiplicity assertion follows from the hook length formula.  $\square$

To illustrate Theorem 8, let us take  $n = 4$ . The Ferrers diagrams together with their contents are given in Fig. 1. The dimensions of the corresponding irreducible representations are 1, 3, 2, 3, and 1, respectively. From Theorem 8 we obtain the eigenvalues  $-2, +2, -2, -6$ , and 46, respectively. Hence, the distance polynomial of the absolute order graph of type  $A_3$  is

$$p_{\Gamma(\mathfrak{S}_4, T)}(q) = (q - 46)(q - 2)^9(q + 2)^5(q + 6)^9,$$

as indicated in Table 1.

In [34] Molchanov derives expressions for the ‘‘Poincaré polynomials’’

$$R_\chi(q) = \frac{1}{f} \sum_{w \in W} \chi(w) q^{\dim V^w}$$

for the finite Coxeter groups of types  $A_{n-1}, B_n, D_n$ , and  $I_2(m)$ . Using his formulae one can obtain results analogous to those of Theorem 8 for the absolute order graphs corresponding to these types. The computations for types  $B_n$  and  $D_n$  are similar to those for type  $A_{n-1}$  but more involved, and are left to the reader. Type  $I_2(m)$  is sufficiently simple that we can write a closed form expression for the distance polynomial in this case.

**Theorem 9.** *The distance polynomial of the absolute order graph  $\Gamma(I_2(m), T)$  is given by*

$$p_{\Gamma(W(I_2(m)), T)}(q) = (q - 3m + 2)(q - m + 2)(q + 2)^{2m-2}.$$

**Proof.** From Carter’s lemma we have

$$P_\chi(q) = q^n R_\chi(q^{-1}).$$

<sup>9</sup> We employ English-style diagrams where the upper leftmost cell is (1, 1) and  $i$  increases down and  $j$  increases to the right.

Using the expressions for  $R_\chi(q)$  in type  $I_2(m)$  given in [34], we obtain

$$P_\chi(q) = \begin{cases} 1 + mq + (m - 1)q^2, & \text{(trivial representation),} \\ 1 - mq + (m - 1)q^2, & \text{(alternating representation),} \\ 1 - q^2, & \text{(all others).} \end{cases}$$

Combining these results with Corollary 7 yields the assertion.  $\square$

**Remark 10.** Shephard and Todd [37] and Solomon [38] showed that the Poincaré polynomial of the trivial character is

$$P_1(q) = \prod_i (1 + (d_i - 1)q),$$

where the  $d_i$  are the invariant degrees of  $W$ . As  $|W| = P_1(1) = \prod_i d_i$ , Corollary 7 gives the following.

**Corollary 11.**

$$\eta_1 = |W| \sum_i \frac{d_i - 1}{d_i}.$$

Examination of the results of Table 1 suggests a pattern.

**Conjecture 12.**  $\eta_1$  is the largest eigenvalue of the distance matrix of the absolute order graphs.

**Remark 13.** The methods discussed in this section can be applied to derive simple expressions for the determinants of the  $q$ -deformed operators:

$$\sum_w q^{\ell_T(w)} w.$$

For example, the theorem below follows easily from the proof of Theorem 8 and a result of Frobenius [19] on group determinants.

**Theorem 14.**

$$\det \left( q^{\ell_T(\sigma\pi^{-1})} \right)_{\sigma, \pi \in \mathfrak{S}_n} = \prod_{\lambda \vdash n} \left[ \prod_{u \in \lambda} (1 + qc(u)) \right]^{f_\lambda^2}.$$

Similar but more involved expressions can be written down for types B, D, and  $I_2$  using Molchanov’s results.

#### 4. The weak order graphs

As mentioned in the introduction, computation of the distance spectrum of the weak order graphs is complicated by the fact that the length function  $\ell_S$  is not a conjugacy class invariant. Yet examination of the distance polynomials in this case (Table 2) reveals that there appear to be far fewer nontrivial eigenvalues in the weak order case than in the absolute order case. The data indicate that, in most cases, the majority of the eigenvalues are zero, revealing that the distance matrices are highly degenerate. Furthermore, in some cases the spectra are integral, with only three nontrivial eigenvalues. Intriguingly, these cases seem to correspond precisely to the simply laced Coxeter groups, namely those of types A, D, and E. In Section 4.8 we explicitly compute the distance spectra of the weak order graphs of types A, D, and E, and, in the process, provide an explanation for the observed spectral structure.

##### 4.1. The dihedral case

The weak order graphs of type  $I_2(m)$  are particularly simple: they are just cycles of length  $2m$  [8, p. 66]. Therefore, we can easily obtain the distance matrix  $d_S$ :

$$d_S = \text{circ}(0, 1, 2, \dots, m - 2, m - 1, m, m - 1, m - 2, \dots, 2, 1).$$

Here  $\text{circ } a$  denotes the circulant matrix with first row  $a$  (and subsequent entries right-shifted). The spectra of circulant matrices are well known (e.g., [7, 13.2]).

**Theorem 15.** Let  $C = \text{circ}(a_0, a_1, \dots, a_{2m-1})$ . Define  $\zeta := e^{i\pi/m}$  and  $\xi_k = \zeta^k$ ,  $0 \leq k \leq 2m - 1$ . Then the eigenvectors of  $C$  are of the form

$$(1, \xi_k, \xi_k^2, \dots, \xi_k^{2m-1})^T,$$

with corresponding eigenvalues

$$\lambda_k = a_0 + a_1 \xi_k + a_2 \xi_k^2 + \dots + a_{2m-1} \xi_k^{2m-1}.$$

**Table 2**  
Distance polynomials of weak order graphs for some Coxeter types.

Type	Distance polynomial $p_{\Gamma(W,S)}(q)$
$A_2$	$q^2(q - 9)(q + 1)(q + 4)^2$
$A_3$	$q^{17}(q - 72)(q + 4)^3(q + 20)^3$
$A_4$	$q^{109}(q - 600)(q + 20)^6(q + 120)^4$
$B_3$	$q^{38}(q - 216)(q + 8)^3(q^2 + 64q + 384)^3$
$D_4$	$q^{179}(q - 1152)(q + 224)^4(q + 32)^8$
$D_5$	$q^{1899}(q - 19\,200)(q + 2880)^5(q + 320)^{15}$
$H_3$	$q^{104}(q - 900)(q^2 + 248q + 3856)^3(q + 24)^4(q + 12)^5$
$F_4$	$q^{1127}(q - 13\,824)(q + 192)^{16}(q^2 + 2688q + 313\,344)^4$
$E_6$	$q^{51803}(q - 933\,120)(q + 112\,320)^6(q + 8640)^{30}$
$I_2(5)$	$q^4(q - 25)(q + 1)(q^2 + 12q + 16)^2$
$I_2(6)$	$q^5(q - 36)(q + 2)^2(q^2 + 16q + 16)^2$
$I_2(7)$	$q^6(q - 49)(q + 1)(q^3 + 24q^2 + 80q + 64)^2$

Applying Theorem 15 to the matrix  $d_S$  gives the distance spectra (and therefore the distance polynomials) of the dihedral groups.<sup>10</sup>

**Theorem 16.** The eigenvalues of the distance matrix  $d_S$  in type  $I_2(m)$  are given by

$$\lambda_k = \begin{cases} -\operatorname{csc}^2(k\pi/2m), & k \text{ odd and nonzero,} \\ 0, & k \text{ even and nonzero,} \\ m^2, & k = 0, \end{cases}$$

where  $0 \leq k \leq 2m - 1$ .

**Proof.** When  $k = 0$  we find

$$\lambda_0 = \left( 2 \sum_{j=0}^m j \right) - m = m^2.$$

For  $k \neq 0$  we get

$$\lambda_k = \sum_{j=0}^{m-1} j (\zeta^{jk} + \zeta^{-jk}) + m\zeta^{mk}.$$

Now

$$\sum_{j=0}^{m-1} jz^j = z \frac{d}{dz} \sum_{j=0}^{m-1} z^j = z \frac{d}{dz} \left( \frac{1 - z^m}{1 - z} \right) = \frac{-mz^m}{1 - z} + z \frac{1 - z^m}{(1 - z)^2},$$

so substituting  $z = \zeta^k$  gives

$$\begin{aligned} \lambda_k &= (-1)^{k+1}m \left( \frac{1}{1 - \zeta^k} + \frac{1}{1 - \zeta^{-k}} \right) + (1 - (-1)^k) \left( \frac{\zeta^k}{(1 - \zeta^k)^2} + \frac{\zeta^{-k}}{(1 - \zeta^{-k})^2} \right) + (-1)^k m \\ &= (1 - (-1)^k) \frac{2}{(\zeta^{k/2} - \zeta^{-k/2})^2} = -\frac{(1 - (-1)^k)}{2} \operatorname{csc}^2 \frac{k\pi}{2m}. \quad \square \end{aligned}$$

#### 4.2. Computations in the group algebra

In this section we obtain some general results valid for all Coxeter types. Recall from Section 2 that the distance spectrum coincides with the spectrum of the operator  $\mathcal{L}_S$  in  $\mathbb{C}W$ .

**Theorem 17.** Let  $w_0$  be the element of maximum length in  $W$ . Define  $v(w) := (1 + w_0)(1 - w)$ , where  $w \neq 1, w_0$ . Then  $v(w) \in \ker \mathcal{L}_S$ .

**Proof.** It is well known that  $w_0$  is an involution satisfying, for any  $w \in W$ ,

$$\ell_S(w w_0) = \ell_S(w_0 w) = \ell_S(w_0) - \ell_S(w).$$

<sup>10</sup> The distance spectrum of a cycle has undoubtedly appeared many times in the literature. The earliest reference of which we are aware is [23].

**Table 3**  
Eigenvalue of the trivial idempotent by Coxeter type.

$A_{n-1}$	$B_n$	$D_n$	$E_6$	$E_7$	$E_8$
$\frac{1}{2} \binom{n}{2} n!$	$2^{n-1} n^2 n!$	$2^{n-1} \binom{n}{2} n!$	$2^8 3^6 5$	$2^9 3^6 5 7^2$	$2^{16} 3^6 5^3 7$

$F_4$	$G_2$	$H_3$	$H_4$	$I_2(m)$
$2^9 3^3$	36	$2^2 3^2 5^2$	$2^7 3^3 5^3$	$m^2$

Thus

$$\mathcal{L}_S(1 + w_0) = \mathcal{L}_S + \sum_u \ell_S(uw_0)u = \mathcal{L}_S + \sum_u (\ell_S(w_0) - \ell_S(u))u = \ell_S(w_0)\mathfrak{1},$$

where

$$\mathfrak{1} := \sum_u u$$

is (up to a constant) the trivial idempotent of  $W$ . But every group algebra element of  $W$  of the form  $1 - w$  is annihilated by the trivial idempotent.  $\square$

We remark that not all of the elements  $v(w)$  given in the theorem are linearly independent. Indeed, at most half of them are, because

$$v(w) = v(w_0w).$$

We could multiply from the right by another element  $u \neq 1$  to get additional zero modes, but it becomes a delicate matter to determine the number of these that are linearly independent. Instead, we will proceed differently below.

It is easy to construct two more eigenvectors of  $\mathcal{L}_S$ .

**Theorem 18.** Let  $\mathfrak{1} = \sum_u u$  be the trivial idempotent (up to a constant). Then

$$\mathcal{L}_S \mathfrak{1} = \frac{1}{2} N |W| \mathfrak{1},$$

where  $N$  is the number of positive roots and  $|W|$  is the cardinality of  $W$ .

**Proof.**

$$\mathcal{L}_S \mathfrak{1} = \sum_w \ell_S(w)w \sum_v v = \sum_w \ell_S(w) \sum_u u = \left( \sum_w \ell_S(w) \right) \mathfrak{1} =: \lambda_1 \mathfrak{1}.$$

Let  $W(q) := \sum_{w \in W} q^{\ell_S(w)}$  denote the Poincaré polynomial, or length generating function, of the Coxeter group  $W$ . It is a classical fact that

$$W(q) = \prod_{i=1}^n \frac{q^{d_i} - 1}{q - 1} = \prod_{i=1}^n [d_i]_q,$$

where  $\{d_1, d_2, \dots, d_n\}$  are the invariant degrees of  $W$  and  $[k]_q := 1 + q + q^2 + \dots + q^{k-1}$ . Clearly,

$$\lambda_1 = \left. \frac{dW(q)}{dq} \right|_{q=1}.$$

But

$$[k]_1 = k \quad \text{and} \quad \left. \frac{d[k]_q}{dq} \right|_{q=1} = \sum_{j=1}^{k-1} j = \frac{1}{2} k(k-1).$$

Moreover,

$$\prod_i d_i = |W| \quad \text{and} \quad \sum_i (d_i - 1) = N,$$

where  $|W|$  is the size of  $W$  and  $N$  is the number of positive roots. So

$$\lambda_1 = \sum_{i=1}^n \frac{1}{2} d_i (d_i - 1) \prod_{j \neq i} d_j = \frac{1}{2} \sum_i (d_i - 1) |W| = \frac{1}{2} N |W|. \quad \square$$

The values of  $\lambda_1$  for the various Coxeter types are given in Table 3.



**Theorem 19.** Let  $\mathcal{A} := \sum_w (-1)^w w$  be the alternating idempotent (again, up to a constant), where  $(-1)^w$  means  $(-1)^{\ell_S(w)}$ . Then

$$\mathcal{L}_S \mathcal{A} = \lambda_2 \mathcal{A},$$

where

$$\lambda_2 = \begin{cases} -1, & \text{if } W = W(A_2) \text{ or } W(I_2(m)) \text{ for } m \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.**

$$\begin{aligned} \mathcal{L}_S \mathcal{A} &= \sum_w \ell_S(w) w \sum_v (-1)^v v = \sum_w \ell_S(w) \sum_v (-1)^v w v \\ &= \sum_w \ell_S(w) \sum_u (-1)^{w^{-1}u} u = \sum_w (-1)^w \ell_S(w) \sum_u (-1)^u u \\ &= \left( \sum_w (-1)^w \ell_S(w) \right) \mathcal{A} =: \lambda_2 \mathcal{A}, \end{aligned}$$

where we used the properties of the length function  $\ell_S(w) = \ell_S(w^{-1})$  and  $(-1)^{uw} = (-1)^u (-1)^v$  (e.g., [27, Sec. 1.6]). In terms of the Poincaré polynomial, we have

$$\lambda_2 = q \frac{dW(q)}{dq} \Big|_{q=-1} = - \sum_{i=1}^n \frac{d}{dq} [d_i]_q \Big|_{q=-1} \prod_{j \neq i} [d_j]_{-1}.$$

But

$$[k]_{-1} = \frac{d[k]_q}{dq} \Big|_{q=-1} = \begin{cases} 0 & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

The degrees of  $W(A_2)$  are 2 and 3, and the degrees of  $W(I_2(m))$  are 2 and  $m$ , whereas every other Coxeter group has at least two even degrees.  $\square$

The other eigenvectors are more difficult to determine, because the noncommutativity of the group algebra becomes a serious impediment to computation. Instead, we follow a different line of attack, which enables us to compute a part of the distance spectrum of all weak order graphs, and the entire spectrum of the weak order graphs of types A, D, and E.

### 4.3. A variant of the distance matrix

Let  $V$  be the vector space affording the reflection representation of  $W$ , equipped with the usual inner product  $(\cdot, \cdot)$ . Let  $\Phi$  be the root system associated to  $W$ . As the length of the root vectors is immaterial here, we choose  $(\alpha, \alpha) = 2$  for  $\alpha \in \Phi$ . The positive roots are denoted by  $\Pi$ , and we write  $\alpha > 0$  (respectively,  $\alpha < 0$ ) if  $\alpha \in \Pi$  (respectively,  $\alpha \in -\Pi$ ). The length of any element  $w \in W$  can be expressed as the number of positive roots sent to negative roots by  $w$ :

$$\ell_S(w) = |w\Pi \cap (-\Pi)| = |w\Pi \cap w_0\Pi|,$$

where, as before,  $w_0$  is the longest element of  $W$ . Hence, the distance matrix can be written

$$d_S(u, v) = \ell_S(u^{-1}v) = |u^{-1}v\Pi \cap w_0\Pi| = |v\Pi \cap uw_0\Pi|.$$

It turns out to be more convenient to define a new function  $\tilde{d}_S(u, v)$ , whose spectrum is simply related to that of  $d_S(u, v)$ .

**Theorem 20.** Let  $\tilde{d}_S(u, v) := |u\Pi \cap v\Pi|$ . Then

$$\text{spec } d_S = \{\lambda_1, \lambda_2, \dots, \lambda_{|W|}\} \Leftrightarrow \text{spec } \tilde{d}_S = \{\lambda_1, -\lambda_2, \dots, -\lambda_{|W|}\}.$$

**Proof.** We have

$$\begin{aligned} \tilde{d}_S(u, v) &= |u\Pi \cap v\Pi| = |u^{-1}v\Pi \cap \Pi| = |w_0u^{-1}v\Pi \cap w_0\Pi| = \ell_S(w_0u^{-1}v) \\ &= N - \ell_S(u^{-1}v) = N - d_S(u, v), \end{aligned}$$

where we used the fact that  $\ell_S(w_0) = N$ . Viewing  $d_S$  and  $\tilde{d}_S$  as matrices, this equation reads

$$\tilde{d}_S = Nj - d_S,$$

where  $J$  is the all ones matrix.

Now we use **Theorem 18**, which says that the all ones vector  $\mathbf{j}$  (corresponding to the trivial idempotent  $\mathbf{1}$ ) is an eigenvector of  $d_S$  with eigenvalue  $N|W|/2$ . Hence

$$\tilde{d}_S \mathbf{j} = N\mathbf{j} - d_S \mathbf{j} = (N|W| - \frac{1}{2}N|W|)\mathbf{j} = \frac{1}{2}N|W|\mathbf{j} = \lambda_1 \mathbf{j}.$$

That is,  $\mathbf{j}$  is an eigenvector of  $\tilde{d}_S$  with eigenvalue  $\lambda_1$ . Furthermore, any other eigenvector  $\mathbf{v}$  of  $d_S$  (with eigenvalue  $\lambda$ , say) is orthogonal to  $\mathbf{j}$ , so it is annihilated by  $J$ . Thus

$$\tilde{d}_S \mathbf{v} = -d_S \mathbf{v} = -\lambda \mathbf{v}. \quad \square$$

4.4. The permutation representation

Let  $\Psi$  be the vector space direct sum of the one dimensional subspaces spanned by the root vectors in  $\Phi$ . To distinguish root vectors as elements of  $\Psi$  as opposed to elements of  $V$  we use Dirac’s bra–ket notation. Thus, vectors in  $\Psi$  are denoted by  $|\psi\rangle$  and dual vectors by  $\langle\psi|$ . The standard inner product on  $\Psi$  is given by  $\langle\alpha|\beta\rangle = \delta_{\alpha\beta}$ , where  $\alpha, \beta \in \Phi$  and  $\delta_{\alpha\beta}$  is the Kronecker delta. Let  $\rho$  be the (unitary) permutation representation of  $W$  afforded by  $\Psi$ , so that  $\rho(w)|\alpha\rangle = |w\alpha\rangle$ . Define  $|\psi_e\rangle := \sum_{\alpha>0} |\alpha\rangle$  and  $|\psi_w\rangle := \rho(w)|\psi_e\rangle$ . By construction, we have

$$\tilde{d}_S(u, v) = \langle\psi_u|\psi_v\rangle.$$

That is, we may view  $\tilde{d}_S$  as a Gram matrix of size  $|W|$ .

But the vectors  $|\psi\rangle$  live in a much smaller space  $\Psi$  of dimension  $2N$ . It is a standard fact of linear algebra [25, Theorem 1.3.20] that, for any matrix  $A$ , the matrices  $AA^T$  and  $A^T A$  have identical spectra up to zeros. We can exploit this fact here.

**Theorem 21.** Define a linear operator on  $\Psi$  given by

$$\mathfrak{D} = \sum_w |\psi_w\rangle\langle\psi_w|.$$

Then, up to zeros, the spectra of  $\tilde{d}_S$  and  $\mathfrak{D}$  coincide.

**Proof.** Define a  $|W| \times 2N$  matrix  $A$  by

$$|\psi_w\rangle = \sum_{\alpha \in \Phi} A(w, \alpha)|\alpha\rangle.$$

On the one hand we have

$$\begin{aligned} \tilde{d}_S(u, v) &= \langle\psi_u|\psi_v\rangle = \sum_{\alpha, \beta \in \Phi} A(u, \beta)A(v, \alpha)\langle\beta|\alpha\rangle \\ &= \sum_{\alpha \in \Phi} A(u, \alpha)A(v, \alpha) \\ &= (AA^T)_{uv}. \end{aligned}$$

On the other hand, identifying  $\mathfrak{D}$  with its matrix representation in the orthonormal basis  $\{|\alpha\rangle\}$  of roots gives

$$\begin{aligned} \mathfrak{D}_{\alpha\beta} &= \langle\alpha|\mathfrak{D}|\beta\rangle = \sum_{w \in W} \langle\alpha|\psi_w\rangle\langle\psi_w|\beta\rangle \\ &= \sum_{w \in W} A(w, \alpha)A(w, \beta) \\ &= (A^T A)_{\alpha\beta}. \quad \square \end{aligned}$$

The advantage of studying  $\mathfrak{D}$  is that it is of size  $2N$ , which is generally considerably smaller than  $|W|$ , the size of  $\tilde{d}_S$ . Moreover, it has a particularly elegant formulation.

**Corollary 22.** Let  $\{\alpha, \beta\} \subset \Phi$ . Then

$$\mathfrak{D}_{\alpha\beta} = |\{w \in W : w\alpha > 0 \text{ and } w\beta > 0\}|.$$

**Proof.** Note that  $A(w, \alpha) = 1$  if some positive root is sent to  $\alpha$  by  $w$ , and 0 otherwise. Thus,  $\mathfrak{D}_{\alpha\beta}$  is the number of group elements that map a pair of positive roots to  $\alpha$  and  $\beta$ . Equivalently, by considering inverses, we see that  $\mathfrak{D}_{\alpha\beta}$  is the number of group elements that map the roots  $\alpha$  and  $\beta$  to a pair of positive roots.  $\square$

For future use, we observe that  $\mathfrak{D}$  is equivariant:

**Theorem 23.**  $\mathfrak{D} : \Psi \rightarrow \Psi$  is a  $W$ -module homomorphism.

**Table 4**  
Root data for type  $A_{n-1}$ .

$\theta_{\alpha\beta}$	$(\alpha, \beta)$	$\mathfrak{D}_{\alpha\beta}$	$m_{\alpha\beta}$	$\pi/m_{\alpha\beta}$
0	2	$n!/2$	1	$\pi$
$\pi/3$	1	$n!/3$	$3/2$	$2\pi/3$
$\pi/2$	0	$n!/4$	2	$\pi/2$
$2\pi/3$	-1	$n!/6$	3	$\pi/3$
$\pi$	-2	0	$\infty$	0

**Proof.**  $\rho$  is unitary, so  $\rho^\dagger = \rho^{-1}$ , where ‘ $\dagger$ ’ denotes ‘Hermitian adjoint’. Hence,

$$\begin{aligned} \rho(u)\mathfrak{D}\rho^{-1}(u) &= \sum_w \rho(u)\rho(w)|\psi_e\rangle\langle\psi_e|\rho^{-1}(w)\rho^{-1}(u) \\ &= \sum_w \rho(uw)|\psi_e\rangle\langle\psi_e|\rho^{-1}(uw) \\ &= \sum_{w'} \rho(w')|\psi_e\rangle\langle\psi_e|\rho^{-1}(w') \\ &= \mathfrak{D}. \quad \square \end{aligned}$$

In particular, by Schur’s lemma,  $\mathfrak{D}$  is constant on the irreducible submodules of  $\Psi$ . We will use this fact in Section 4.8.

$W$  preserves angles (because it acts by orthogonal transformations in the reflection representation  $V$ ), so the entry  $\mathfrak{D}_{\alpha\beta}$  will depend on the angle  $\theta_{\alpha\beta}$  between the two roots. For  $\alpha, \beta \in \Phi$  define  $m_{\alpha\beta}$  by

$$(\alpha, \beta) = -2 \cos \frac{\pi}{m_{\alpha\beta}},$$

so that  $\theta_{\alpha\beta} = \pi - (\pi/m_{\alpha\beta})$ . When restricted to simple roots, the  $m_{\alpha\beta}$  are the edge weights of the Coxeter–Dynkin diagrams representing the various Coxeter groups, but it is useful here to extend the definition to all roots. This brings us to the following key result.

**Theorem 24.** For all  $\{\alpha, \beta\} \in \Phi$  and for all Coxeter types,

$$\mathfrak{D}_{\alpha\beta} = \frac{|W|}{2m_{\alpha\beta}} = \frac{|W|}{2\pi} \left( \frac{\pi}{m_{\alpha\beta}} \right).$$

In particular,  $\mathfrak{D}_{\alpha\beta}$  is proportional to the supplement of the angle between the roots.

In Section 4.5 we prove Theorem 24 for the symmetric groups, while in Section 4.6 we prove it in general.

#### 4.5. The $\mathfrak{D}$ matrix of type $A$

**Theorem 25.** Let  $W = W(A_{n-1}) = \mathfrak{S}_n$ . Then, for any pair  $\alpha, \beta \in \Phi$ ,

$$\mathfrak{D}_{\alpha\beta} = \frac{n!}{2m_{\alpha\beta}} = \frac{n!}{2\pi} \left( \frac{\pi}{m_{\alpha\beta}} \right),$$

where the  $m_{\alpha\beta}$  are given in Table 4.

**Proof.** The positive roots of type  $A_{n-1}$  are of the form  $e_i - e_j$ ,  $1 \leq i < j \leq n$ , where  $e_i$  is the  $i$ th unit vector in  $\mathbb{R}^n$ . For shorthand (and with an eye towards generalization to type D), we denote  $|e_i - e_j\rangle$  by  $|i, \bar{j}\rangle$  so that

$$\Psi_{A_{n-1}} = \{|i, \bar{j}\rangle : 1 \leq i, j \leq n\}.$$

Consider the root  $|i, \bar{j}\rangle$ . There are  $n!/2$  ways to map it to a positive root, because half of the permutations  $\sigma$  satisfy  $\sigma(i) > \sigma(j)$  and half satisfy  $\sigma(i) < \sigma(j)$ , independent of  $i$  and  $j$ . Hence  $\mathfrak{D}_{\alpha\alpha} = n!/2$  for all  $\alpha$ . Next, consider the off-diagonal terms of  $\mathfrak{D}$ . The pair  $\{\alpha, \beta\}$  could be one of the following generic forms: (a)  $\{|i, \bar{j}\rangle, |j, \bar{i}\rangle\}$ , (b)  $\{|i, \bar{j}\rangle, |i, \bar{k}\rangle\}$ , (c)  $\{|i, \bar{j}\rangle, |k, \bar{j}\rangle\}$ , (d)  $\{|i, \bar{j}\rangle, |j, \bar{k}\rangle\}$ , or (e)  $\{|i, \bar{j}\rangle, |k, \bar{\ell}\rangle\}$ . Case (a): No permutation  $\sigma$  carries a pair of the form  $\{|i, \bar{j}\rangle, |j, \bar{i}\rangle\}$  to a pair of positive roots, because either  $\sigma(i) > \sigma(j)$  or  $\sigma(j) > \sigma(i)$ , so one of the images is always a negative root. Case (b): We want  $\sigma(i) < \sigma(j)$  and  $\sigma(i) < \sigma(k)$ . Given a triple  $(a, b, c)$  of numbers with  $a < b < c$  chosen from a set of  $n$  numbers, we may set  $\sigma(i) = a$ ,  $\sigma(j) = b$ , and  $\sigma(k) = c$ , or  $\sigma(i) = a$ ,  $\sigma(j) = c$ , and  $\sigma(k) = b$ . There are  $2 \binom{n}{3}$  ways to do this, and  $(n-3)!$  choices for the remaining numbers, for a total of  $2 \binom{n}{3} (n-3)! = n!/3$  choices. Case (c): This is the reverse of case b, so the value here is also  $n!/3$ . Case (d): Any triple  $(a, b, c)$  satisfies  $a < b < c$ , so we specify  $\sigma$  by choosing a triple in  $\binom{n}{3}$  ways and the rest of the numbers

in  $(n - 3)!$  ways, for a total of  $n!/6$  choices. Case (e): We want  $\sigma(i) < \sigma(j)$  and  $\sigma(k) < \sigma(\ell)$ . First pick a quartet of numbers in  $\binom{n}{4}$  ways, then pick pairs in  $\binom{4}{2} = 6$  ways, and finally pick the rest of the numbers in  $(n - 4)!$  ways, for a total of  $6 \binom{n}{4} (n - 4)! = n!/4$  choices. Now correlate the various cases with the angles between the roots to obtain the results in Table 4.  $\square$

4.6. The dihedral case revisited

The roots of  $W$  may be identified with roots of unity, appropriately scaled to agree with our convention on root lengths. Let  $\zeta = e^{i\pi/m}$ . Then

$$\Phi = \{\sqrt{2}\zeta^j : 0 \leq j \leq 2m - 1\};$$

We choose the first  $m$  of these to be positive. For brevity, we write  $|a\rangle$  for  $|\sqrt{2}\zeta^a\rangle$  and  $|a\rangle > 0$  if  $\sqrt{2}\zeta^a \in \Pi$ . In this notation,  $|\psi_e\rangle = \sum_{0 \leq a \leq m-1} |a\rangle$  and  $|\psi_w\rangle = \sum_{0 \leq a \leq m-1} |wa\rangle$ .

**Theorem 26.** Let  $W = I_2(m)$ . Then for any two roots  $|a\rangle$  and  $|b\rangle$  with  $b - a = k$ ,  $k = 0, 1, \dots, m$ ,

$$\mathfrak{D}_{ab} = \langle a|\mathfrak{D}|b\rangle = m - k = \frac{|W|}{2\pi} \left( \frac{\pi}{m_{ab}} \right),$$

where  $m_{ab} = m/(m - k)$  for  $k \neq m$  and  $m_{ab} = \infty$  if  $k = m$ . In particular, the entries of  $\mathfrak{D}$  are proportional to the supplements of the angles between the roots.

**Proof.** Let  $r \in W$  be a rotation through  $2\pi/m$ , and let  $t$  be the reflection about the  $y$ -axis. Then

$$r^\pm |a\rangle = |a \pm 2\rangle \quad \text{and} \quad t|a\rangle = |m - a\rangle,$$

where arithmetic within kets is performed modulo  $2m$ . Label the group elements with the numbers from 0 to  $2m - 1$  by setting

$$u_j := \begin{cases} r^{j/2}, & \text{if } j \text{ is even,} \\ r^{(j-1)/2}t, & \text{if } j \text{ is odd.} \end{cases}$$

Then, for any  $i$ ,

$$u_j |i\rangle = \begin{cases} |j + i\rangle, & \text{if } j \text{ is even,} \\ |m + j - i - 1\rangle, & \text{if } j \text{ is odd.} \end{cases}$$

It follows that

$$|\psi_{u_j}\rangle = \sum_{0 \leq a \leq m-1} |a + j\rangle.$$

Recall that we defined  $A(w, \alpha)$  in Section 4.4 by

$$|\psi_w\rangle = \sum_{\alpha} A(w, \alpha)|\alpha\rangle.$$

Labeling the rows by  $\{u_j\}$  and the columns by  $|k\rangle$  (starting with  $j = k = 0$ ) we find that the matrix  $A$  is of the form

$$A = \text{circ}(\overbrace{1, 1, 1, \dots, 1}^{m \text{ times}}, 0, 0, 0, \dots, 0).$$

Circulant matrices are normal (e.g., [7, p. 352]), so

$$\tilde{d}_S = AA^T = A^T A = \mathfrak{D}.$$

In particular, we have

$$\tilde{d}_S = \mathfrak{D} = \text{circ}(m, m - 1, m - 2, \dots, 1, 0, 1, 2, \dots, m - 1).$$

(Observe that we regain the results of Section 4.1, because  $d_S = N - \tilde{d}_S = m - \tilde{d}_S$ .)

Lastly,

$$(a, b) = 2(\zeta^a, \zeta^b) = 2 \operatorname{Re} \zeta^{b-a} = 2 \operatorname{Re} \zeta^k = 2 \cos \frac{k\pi}{m} = -2 \cos \frac{\pi}{m_{ab}},$$

where  $m_{ab} = m/(m - k)$ , and  $\pi/m_{ab}$  is the supplement of the angle between the roots  $a$  and  $b$ .  $\square$

**Proof of Theorem 24.** Let  $\Phi_{\alpha\beta}$  be the root system of the dihedral subgroup  $G_{\alpha\beta}$  generated by the reflections through the hyperplanes  $\ker \alpha$  and  $\ker \beta$ . If  $w \in W$  satisfies  $w\alpha = \gamma$  and  $w\beta = \delta$ , then by linearity  $w\Phi_{\alpha\beta} = \Phi_{\gamma\delta}$ . Partition

$$U := \{u \in W : u\Phi_{\alpha\beta} = \Phi_{\gamma\delta}\}$$

into right cosets  $U_i$  of  $G_{\gamma\delta}$ . By Theorem 26,  $|G_{\gamma\delta}|/2m_{u\alpha, u\beta} = |U_i|/2m_{\alpha\beta}$  elements of  $U_i$  map  $\{\alpha, \beta\}$  to  $\Pi \cap \Phi_{\gamma\delta}$ , so  $|U|/2m_{\alpha\beta}$  elements of  $U$  map  $\{\alpha, \beta\}$  to a pair of positive roots. Summing over all subsets of the form  $U$  gives the desired result.  $\square$

#### 4.7. The spectrum of $\mathfrak{D}$

The determination of the spectrum of  $\mathfrak{D}$  requires two steps. First we must construct  $\mathfrak{D}$  by finding the angles between all the elements of  $\Phi$ , then we must diagonalize the matrix. In this section we present some general results applicable to all Coxeter groups, while in Section 4.8 we consider the specific cases of types A, D, and E.

**Lemma 27.** The trace of  $\mathfrak{D}$  is  $N|W|$ .

**Proof.** The dimension of  $\Psi$  is  $2N$ , and by Theorem 24,  $\mathfrak{D}_{\alpha\alpha} = |W|/2$ .  $\square$

**Lemma 28.** Let  $w \in W$  and  $\{\alpha, \beta\} \subset \Phi$ . Then

$$\mathfrak{D}_{\alpha, \beta} + \mathfrak{D}_{\alpha, -\beta} = \frac{1}{2}|W|.$$

**Proof.** This follows immediately from Theorem 24. It can also be seen directly. If  $w\alpha > 0$  then  $ws_\alpha\alpha < 0$ . As  $w \rightarrow ws_\alpha$  is a bijection of  $W$ , we get  $|\{w \in W : w\alpha > 0\}| = |W|/2$ . Let  $w$  be such that  $w\alpha > 0$ . Then either  $w\beta > 0$  or  $w\beta < 0$ , but not both.  $\square$

Choose  $0 < \alpha \in \Phi$  and define

$$|\psi_\alpha\rangle := |\alpha\rangle + |-\alpha\rangle$$

and

$$|\iota\rangle := \sum_{\gamma \in \Phi} |\gamma\rangle.$$

**Lemma 29.**

$$\mathfrak{D}|\psi_\alpha\rangle = \frac{1}{2}|W| |\iota\rangle.$$

**Proof.** By Lemma 28,

$$\mathfrak{D}|\psi_\alpha\rangle = \sum_{\gamma \in \Phi} (\mathfrak{D}_{\gamma, \alpha} + \mathfrak{D}_{\gamma, -\alpha})|\gamma\rangle = \frac{1}{2}|W| \sum_{\gamma \in \Phi} |\gamma\rangle. \quad \square$$

We can now recover the results of Table 3.

**Lemma 30.** With the definitions above, we have

$$\mathfrak{D}|\iota\rangle = \frac{1}{2}N|W| |\iota\rangle.$$

**Proof.** This follows from Lemma 29 and the fact that there are  $N$  positive roots.  $\square$

For any pair  $\{\alpha, \beta\}$  of distinct roots, define

$$|\psi_{\alpha\beta}\rangle := |\psi_\alpha\rangle - |\psi_\beta\rangle$$

and let  $Y$  be the subspace of  $\Psi$  spanned by all vectors of this type.

**Lemma 31.**

$$Y \subseteq \ker \mathfrak{D}.$$

Furthermore, the dimension of  $Y$  is  $N - 1$ .

**Proof.** It is clear from Lemma 29 that  $\mathfrak{D}|\psi_{\alpha\beta}\rangle = 0$  for all  $\{\alpha, \beta\} \subset \Phi$ . There are  $N$  linearly independent vectors of the form  $|\psi_\alpha\rangle$ , but only  $N - 1$  linearly independent vectors of the form  $|\psi_{\alpha\beta}\rangle$ , because  $\langle \iota | \psi_{\alpha\beta} \rangle = 0$ . (For example, one linearly independent set consists of all the vectors of the form  $|\psi_{\alpha\beta}\rangle$  with  $\alpha$  fixed and  $\beta \neq \alpha$ .)  $\square$

From Theorem 21 and Lemma 31 it follows that the multiplicity of zero as an eigenvalue of  $d$  is at least  $|W| - N - 1$ . Examination of the results in Table 2 reveals that equality holds in all the cases computed. This prompts the following.

**Conjecture 32.**  $Y = \ker \mathfrak{D}$  for all Coxeter types.

**Remark 33.** Conjecture 32 holds for types I, A, D, and E by virtue of Theorems 16 and 37.

4.8. The distance spectra in types A, D, and E

In this section we obtain the spectrum of the matrix  $\mathfrak{D}$  for types A, D, and E, and hence, by Theorems 20 and 21, the distance spectra of the corresponding weak order graphs.

4.8.1. The eigenspaces  $\Psi_c$  and  $U$

The possible angles between root pairs are  $0, \pi/3, \pi/2, 2\pi/3$ , and  $\pi$ . Select  $\{\alpha, \beta, \gamma\} \subset \Phi$  to be distinct coplanar roots with  $\alpha + \beta + \gamma = 0$ , so that each pair has an angular separation of  $2\pi/3$ . Define

$$|\psi_{\alpha\beta\gamma}\rangle := |\alpha\rangle + |\beta\rangle + |\gamma\rangle - |-\alpha\rangle - |-\beta\rangle - |-\gamma\rangle,$$

and let  $\Psi_c$  be the subspace of  $\Psi$  spanned by all vectors of this form.

**Theorem 34.** Let  $W$  be of type A, D, or E. Then, with the above notation,  $|\psi_{\alpha\beta\gamma}\rangle$  is an eigenvector of the matrix  $\mathfrak{D}$  with eigenvalue  $|W|/6$ .

**Proof.** We have

$$\langle \alpha | \mathfrak{D} | \psi_{\alpha\beta\gamma} \rangle = \frac{|W|}{2\pi} \left( \pi + \frac{\pi}{3} + \frac{\pi}{3} - 0 - \frac{2\pi}{3} - \frac{2\pi}{3} \right) = \frac{|W|}{6}.$$

Similarly,

$$\langle \beta | \mathfrak{D} | \psi_{\alpha\beta\gamma} \rangle = \langle \gamma | \mathfrak{D} | \psi_{\alpha\beta\gamma} \rangle = \frac{|W|}{6}$$

and

$$\langle -\alpha | \mathfrak{D} | \psi_{\alpha\beta\gamma} \rangle = \langle -\beta | \mathfrak{D} | \psi_{\alpha\beta\gamma} \rangle = \langle -\gamma | \mathfrak{D} | \psi_{\alpha\beta\gamma} \rangle = -\frac{|W|}{6}.$$

Next, suppose  $\sigma \notin \{\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma\}$ . By construction

$$0 = \langle \sigma, \alpha + \beta + \gamma \rangle = 2(\cos \theta_{\sigma\alpha} + \cos \theta_{\sigma\beta} + \cos \theta_{\sigma\gamma}),$$

so the only two possibilities are

$$\{\theta_{\sigma\alpha}, \theta_{\sigma\beta}, \theta_{\sigma\gamma}\} = \{\pi/3, \pi/2, 2\pi/3\}$$

or

$$\{\theta_{\sigma\alpha}, \theta_{\sigma\beta}, \theta_{\sigma\gamma}\} = \{\pi/2, \pi/2, \pi/2\}.$$

In either case,

$$\langle \sigma | \mathfrak{D} | \psi_{\alpha\beta\gamma} \rangle = 0.$$

Hence,

$$\mathfrak{D} | \psi_{\alpha\beta\gamma} \rangle = \frac{|W|}{6} | \psi_{\alpha\beta\gamma} \rangle. \quad \square$$

**Theorem 35.** In type  $A_{n-1}$ , the dimension of  $\Psi_c$  is  $\binom{n-1}{2}$ .

**Proof.** Let  $\alpha = e_i - e_j, \beta = e_j - e_k$ , and  $\gamma = e_k - e_i$ , where  $i < j < k$  or  $i > j > k$ . Then in type  $A_{n-1}$ ,

$$|\psi_{\alpha\beta\gamma}\rangle = |\psi_{ijk}\rangle := |i, \bar{j}\rangle + |j, \bar{k}\rangle + |k, \bar{i}\rangle - |j, \bar{i}\rangle - |k, \bar{j}\rangle - |i, \bar{k}\rangle.$$

Temporarily identifying  $|i, \bar{j}\rangle$  with  $-|j, \bar{i}\rangle$ , we may view the vector  $|\psi_{ijk}\rangle$  as an oriented 3-cycle in the complete graph  $K_n$  on  $n$  vertices by associating  $|i, \bar{j}\rangle$  (respectively,  $|j, \bar{i}\rangle$ ) with the oriented edge  $(i, j)$  (respectively,  $(j, i)$ ). The cycle space of  $K_n$  is spanned by these 3-cycles, and the cycle rank, the dimension of the cycle space, is the number of linearly independent 3-cycles. The cycle rank of any graph is  $m - n + c$ , where  $m$  is the number of edges,  $n$  the number of vertices, and  $c$  the number of components. For  $K_n, m = \binom{n}{2}$ , so the number of linearly independent vectors of the form  $|\psi_{ijk}\rangle$  is  $\binom{n}{2} - n + 1 = \binom{n-1}{2}$ .  $\square$

It is more difficult to obtain the dimension of  $\Psi_c$  in types D and E in this manner, because the linear dependencies are more involved. For example, the root system of  $D_n$  consists of all vectors of the form  $\pm e_i \pm e_j$  where  $1 \leq i < j \leq n$ . Denote  $|e_i + e_j\rangle$  by  $|i, j\rangle$  and  $|-(e_i + e_j)\rangle$  by  $|\bar{i}, \bar{j}\rangle$ , and adopt the convention that  $|i, \bar{j}\rangle = |\bar{j}, i\rangle$ . Then in type D, in addition to eigenvectors of the form  $|\psi_{ijk}\rangle$  given above,  $\Psi_c$  also contains eigenvectors of the form

$$|i, \bar{j}\rangle + |j, k\rangle + |\bar{k}, \bar{i}\rangle - |\bar{i}, j\rangle - |\bar{j}, \bar{k}\rangle - |k, i\rangle,$$

as well as others of a similar nature. Mimicking the proof of Theorem 35 would necessitate the introduction of graphs whose edges are decorated with signs as well as arrows. Although a proof along these lines may be possible, it would not generalize easily to other Coxeter types. Instead, we proceed differently.<sup>11</sup>

**Theorem 36.** *The dimension of  $\Psi_c$  in types A, D, and E is  $N - n$ .*

**Proof.** Consider the linear map  $\pi : \Psi \rightarrow V$  given by  $|\alpha\rangle \mapsto \alpha$ . It is clearly surjective, so  $\dim \ker \pi = 2N - n$ . Define  $\Psi^+ := \text{span}\{|\alpha\rangle + |-\alpha\rangle\}$  for  $\alpha > 0$ . Obviously,  $\Psi^+ \subseteq \ker \pi$ . But we also have  $\Psi_c \subseteq \ker \pi$ . Moreover, the two subspaces  $\Psi^+$  and  $\Psi_c$  are orthogonal relative to the standard inner product. The objective is to show that  $\ker \pi = \Psi^+ \oplus \Psi_c$ . The theorem then follows from the observation that  $\dim \Psi^+ = N$ .

The essential fact is that for crystallographic root systems, every non-simple positive root can be written as the sum of two positive roots [36, Proposition V.5.]. For each non-simple positive root  $\gamma$  fix an expression of the form  $\gamma = \alpha_\gamma + \beta_\gamma$ . Now restrict to types A, D, and E. Necessarily, the angle between each pair of  $\{\alpha_\gamma, \beta_\gamma, -\gamma\}$  is  $2\pi/3$ , so if  $\Omega$  is the span of the  $N - n$  elements of the form

$$|\gamma\rangle + |-\alpha_\gamma\rangle + |-\beta_\gamma\rangle - |-\gamma\rangle - |\alpha_\gamma\rangle - |\beta_\gamma\rangle,$$

then  $\Omega \subseteq \Psi_c$ .

Any root is a linear combination of simple roots. If  $\gamma = \sum_{\alpha \in S} c_\alpha \alpha$  then we can write

$$|\gamma\rangle = \sum_{\alpha \in S} c_\alpha |\alpha\rangle + |\xi\rangle$$

for some  $|\xi\rangle \in \Psi^+ \oplus \Omega$ . (Either  $\gamma$  is simple, in which case we may choose  $|\xi\rangle = 0$ , or  $\gamma$  is non-simple, in which case we may choose  $|\xi\rangle = |\gamma\rangle - |\alpha_\gamma\rangle - |\beta_\gamma\rangle \in \Psi^+ \oplus \Omega$ .) It follows that  $\Psi / (\Psi^+ \oplus \Omega)$  has dimension at most  $n$ , which implies that  $\Omega = \Psi_c$  and  $\ker \pi = \Psi^+ \oplus \Psi_c$ .  $\square$

**Theorem 37.** *Let  $U := \Psi / \ker \pi$ . Then  $\mathfrak{D}$  has four eigenspaces in types A, D, and E, namely  $\text{span } |\iota\rangle, Y, \Psi_c$  and  $U$ , with respective dimensions  $1, N - 1, N - n$ , and  $n$ .*

**Proof.** By Theorem 23, the eigenspaces of  $\mathfrak{D}$  are  $W$ -invariant subspaces of  $\Psi$ .  $\pi$  is  $W$ -equivariant, so  $V$  and  $U$  are isomorphic as  $W$ -modules. As  $V$  is irreducible,  $U$  must be an eigenspace of  $\mathfrak{D}$ . But  $\Psi^+ = Y \oplus \text{span } |\iota\rangle$ , so the four eigenspaces of  $\mathfrak{D}$  are  $\text{span } |\iota\rangle, Y, \Psi_c$ , and  $U$  (with dimensions  $1, N - 1, N - n$ , and  $n$ , respectively).  $\square$

**Theorem 38.** *The four distinct eigenvalues of  $\mathfrak{D}$  in types A, D, and E corresponding to the four eigenspaces  $\text{span } |\iota\rangle, Y, \Psi_c$  and  $U$  are  $N|W|/2, 0, |W|/6$ , and  $|W|(h + 1)/6$ , respectively, where  $h$  is the Coxeter number of  $W$ .*

**Proof.** The first three eigenvalues were obtained in Lemmata 30 and 31, and Theorem 34, respectively. From Lemma 27 and Theorem 37, the sum of the eigenvalues is

$$N|W| = \frac{1}{2}N|W| + \frac{1}{6}|W|(N - n) + \lambda n,$$

whence the result follows. (Recall that  $h = 2N/n$ .)  $\square$

**Theorem 39.** *Let  $W$  be of type A, D, or E, and let  $\Gamma(W, S)$  be the corresponding weak order graph. Then the distance polynomials  $p_\Gamma(q)$  of these graphs are of the form*

$$p_\Gamma(q) = q^{|W|-N-1} \left( q - \frac{N|W|}{2} \right) \left( q + \frac{|W|}{6} \right)^{N-n} \left( q + \frac{|W|(h+1)}{6} \right)^n.$$

**Proof.** Combine Theorems 20 and 21 with Theorem 38.  $\square$

Using the values of  $|W|, N, n$ , and  $h$  for the various Coxeter groups given in, for example, [27], we can write out the polynomials of Theorem 39 explicitly. The results are presented in Table 5. The reader should compare Tables 2 and 5.

<sup>11</sup> In a prior version of this paper the author put forward a detailed representation theoretic argument to calculate the dimension of  $\Psi_c$  in type D. An anonymous referee suggested the simpler and more uniform approach that we use here (Theorems 36 and 37). We are grateful to the referee for allowing us to use his argument.

**Table 5**  
Distance polynomials of weak order graphs in types A, D, and E.

Type	Distance polynomial $p_{r(w,s)}(q)$
$A_{n-1}$	$q^{n!-\binom{n}{2}-1} \left(q - \frac{n!}{2} \binom{n}{2}\right) \left(q + \frac{n!}{6} \binom{n-1}{2}\right) \left(q + \frac{(n+1)!}{6}\right)^{n-1}$
$D_n$	$q^{2^{n-1}n!-(n^2-n+1)} \left(q - 2^{n-1} \binom{n}{2} n!\right) \left(q + \frac{2^{n-1}n!}{6}\right)^{n(n-2)} \left(q + \frac{2^{n-1}n!(2n-1)}{6}\right)^n$
$E_6$	$q^{51803} (q - 933\,120)(q + 8640)^{30}(q + 112\,320)^6$
$E_7$	$q^{2^{10} \cdot 3^4 \cdot 5 \cdot 7 - 64} (q - 2^9 \cdot 3^6 \cdot 5 \cdot 7^2)(q + 2^9 \cdot 3^3 \cdot 5 \cdot 7)^{56}(q + 2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 19)^7$
$E_8$	$q^{2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 - 121} (q - 2^{16} \cdot 3^6 \cdot 5^3 \cdot 7)(q + 2^{13} \cdot 3^4 \cdot 5^2 \cdot 7)^{112}(q + 2^{13} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 31)^8$

4.8.2. A basis for the eigenspace  $U$  in types A and D

For types A and D is possible to write down a simple basis of eigenvectors of  $\mathfrak{D}$  that span  $U$ . Define

$$|\psi_i^{*+}\rangle := \sum_{j \neq i} (|\bar{i}, \bar{j}\rangle + |i, j\rangle) \quad \text{and} \quad |\psi_i^{*-}\rangle := \sum_{j \neq i} (|\bar{i}, j\rangle + |\bar{i}, \bar{j}\rangle),$$

where we agree that the terms of the form  $|i, j\rangle$  and  $|\bar{i}, \bar{j}\rangle$  are absent from the sums in type A. Set

$$|\psi_i^*\rangle := |\psi_i^{*+}\rangle - |\psi_i^{*-}\rangle,$$

**Theorem 40.** Let  $W$  be of type A or D. Then, with the above notation, the vectors  $|\psi_i^*\rangle$  form a basis of eigenvectors of  $U$ .

**Proof.** Note that, from Theorem 24, for any  $\{\alpha, \beta\} \subset \Phi$ ,

$$\langle \alpha | \mathfrak{D} | \beta \rangle - \langle \alpha | \mathfrak{D} | -\beta \rangle = 2\langle \alpha | \mathfrak{D} | \beta \rangle - \frac{|W|}{2}.$$

It follows that

$$\langle \alpha | \mathfrak{D} | \psi_i^* \rangle = 2\langle \alpha | \mathfrak{D} | \psi_i^{*+} \rangle - x \binom{n-1}{2} |W|,$$

where  $x = 1$  in type  $A_{n-1}$  and  $x = 2$  in type  $D_n$ . Also, by Lemma 28 (and the symmetry of  $\mathfrak{D}$ ),

$$\langle -\alpha | \mathfrak{D} | \beta \rangle = \frac{|W|}{2} - \langle \alpha | \mathfrak{D} | \beta \rangle,$$

so

$$\langle -\alpha | \mathfrak{D} | \psi_i^* \rangle = -\langle \alpha | \mathfrak{D} | \psi_i^* \rangle.$$

Begin with type  $A_{n-1}$ . In what follows assume  $k \neq i$ . Then

$$\langle i, \bar{k} | \mathfrak{D} | \psi_i^{*+} \rangle = \frac{n!}{2\pi} \left( \pi + (n-2) \frac{2\pi}{3} \right) = n! \left( \frac{2n-1}{6} \right),$$

which gives

$$\langle i, \bar{k} | \mathfrak{D} | \psi_i^* \rangle = 2n! \left( \frac{2n-1}{6} \right) - n! \left( \frac{n-1}{2} \right) = \frac{(n+1)!}{6}.$$

Also, for any  $\ell \neq i$ ,

$$\langle k, \bar{\ell} | \mathfrak{D} | \psi_i^{*+} \rangle = \frac{n!}{2\pi} \left( \frac{\pi}{3} + \frac{2\pi}{3} + (n-3) \frac{\pi}{2} \right) = n! \left( \frac{n-1}{4} \right),$$

so

$$\langle k, \bar{\ell} | \mathfrak{D} | \psi_i^* \rangle = 0.$$

Combining the above results with the fact that  $h = n$  in type  $A_{n-1}$  shows that

$$\mathfrak{D} |\psi_i^*\rangle = \frac{(n+1)!}{6} |\psi_i^*\rangle = \frac{|W|}{6} (h+1) |\psi_i^*\rangle.$$

There are  $n$  eigenvectors of the form  $|\psi_i^*\rangle$ , but only  $n - 1$  of these are linearly independent in type  $A_{n-1}$  because

$$\sum_i |\psi_i^*\rangle = 0.$$



Now consider type  $D_n$ . To evaluate  $\langle \alpha | \mathfrak{D} | \psi_i^{*+} \rangle$  we must consider four cases. Case (i):  $\langle \alpha | = \langle i, k |$ . We have

$$\langle i, k | \mathfrak{D} | \psi_i^{*+} \rangle = \frac{2^{n-1}n!}{2\pi} \left( \pi + \frac{\pi}{2} + 2(n-2)\frac{2\pi}{3} \right) = 2^{n-1}n! \left( \frac{8n-7}{12} \right),$$

so

$$\langle i, k | \mathfrak{D} | \psi_i^* \rangle = 2^n n! \left( \frac{8n-7}{12} \right) - 2^{n-1}n!(n-1) = 2^{n-1}n! \left( \frac{2n-1}{6} \right).$$

Case (ii):  $\langle \alpha | = \langle i, \bar{k} |$ . Same as Case (i). Case (iii):  $\langle \alpha | = \langle k, \ell |$ ,  $\ell \neq i$ .

$$\langle k, \ell | \mathfrak{D} | \psi_i^{*+} \rangle = 2 \cdot \frac{2^{n-1}n!}{2\pi} \left( \frac{\pi}{3} + \frac{2\pi}{3} + (n-3)\frac{\pi}{2} \right) = 2^{n-1}n! \left( \frac{n-1}{2} \right),$$

so

$$\langle k, \bar{\ell} | \mathfrak{D} | \psi_i^* \rangle = 0.$$

Case (iv):  $\langle \alpha | = \langle k, \bar{\ell} |$ ,  $\ell \neq i$ . Same as Case (iii). Now combine the cases and use the fact that  $h = 2n - 2$  in type  $D_n$  to get

$$\mathfrak{D} | \psi_i^* \rangle = 2^{n-1}n! \left( \frac{2n-1}{6} \right) | \psi_i^* \rangle = |W| \left( \frac{h+1}{6} \right) | \psi_i^* \rangle.$$

There are  $n$  eigenvectors of the form  $| \psi_i^* \rangle$ , and all are linearly independent in type  $D_n$ .  $\square$

## 5. Questions and comments

We close with a few questions.

1. Are there simple descriptions of the eigenspaces of  $\mathfrak{D}$  for the Coxeter groups with non-simply laced diagrams?
2. As observed in Section 2, the left action of the operator

$$\sum_w \ell_S(w) w$$

on  $\mathbb{C}W$  yields the distance spectra of the weak order graphs. A closely related operator that arises in the study of  $q$ -statistics [42], noncommutative symmetric functions [16], and hyperplane arrangements [40] (see also [13,24]) is

$$\sum_w q^{\ell_S(w)} w.$$

Obviously, the former operator is a  $q$ -derivative of the latter. Is there any utility in this connection?

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