Further Examples of
the Boundary Value Technique in
Singular Perturbation Problems

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1. Introduction

As an alternative to asymptotic methods to solve two-point singular perturbation problems, we suggested the boundary value method [3–5]. It consists of partitioning the original problem into inner and outer solutions. The outer solution problem is solved to provide the boundary conditions for the inner problem (left-hand and/or right-hand boundary layer problem). The inner problems are solved as a two-point boundary value problems and the solutions matched to the outer problem solution. The process is iterative on the thickness of the boundary layers. The thickness of the boundary layer is adjusted until the non-specified components in the boundary value problems match the outer solution components in some norm.

Most of our work has been done on second-order systems where typically the boundary layer thickness is adjusted until the derivative of the outer solution matches the scaled derivative of the inner solution. Details may be found in [3–5].

To determine the scope of applicability of the boundary value technique, we have been looking at various problems beyond the original class. Some problems, for example, may be solved more simply as initial value problems while still retaining the width of the boundary layer as the iterative variable. In other problems, it may not be possible to solve the outer solution to obtain immediately numeric results as a function of the independent variable. In still other, the matching numerically must be a compromise unless one is prepared to spend much time in making new adjustments in the choice of say, the boundary layer thickness. Furthermore, one can always gain insight into the existence and location of boun-
dary layers by studying the behavior of the original problem for various values of \( \varepsilon \), the perturbation parameter, where \( \varepsilon \) is not necessarily small.

The common theme in all these examples is the avoidance of asymptotic expressions where possible and the use of techniques available to the non-specialist in singular perturbation problems. As a point of reference we take our examples from Bender and Orszag [1].

2. Example 1 (Bender and Orszag [1, p. 446])

Consider

\[ \varepsilon y''' - y'(x) + xy(x) = 0, \ 0 \leq x < 1 \]  

(1.1)

with \( \varepsilon \) a small positive number and with the boundary conditions

\[ y(0) = y'(0) = y(1) = 1. \]  

(1.2)

To obtain the outer solution, \( y_0(x) \), set \( \varepsilon = 0 \) in (1.1) and write

\[ y'(x) = xy_0(x) \]  

(1.3)

whose solution is

\[ y_0(x) = a_0 e^{x^2/2} \]  

(1.4)

Notice that the original third-order system reduces to a first-order outer solution system.

Inspection of (1.4) shows that it cannot consistently satisfy the boundary conditions in (1.2). At \( x = 0 \),

\[ y_0(0) = a_0. \]  

(1.5)

By \( y(0) = 1 \), in the boundary conditions, we conclude

\[ a_0 = 1. \]  

(1.6)

(This value for \( a_0 \) will be superseded by a more accurate approximate which takes into account the boundary layer in (1.33)).

On the other hand at \( x = 0 \),

\[ y_0'(0) = 0 \]  

(1.7)

which does not agree with the specified initial condition in (1.2).

Similarly at \( x = 1 \),

\[ y_0(1) = a_0 e^{1/2} = e^{1/2}. \]  

(1.8)
which does not agree with \( y(1) = 1 \) in the specified boundary condition (1.2).

We conclude from (1.7) and (1.8) that the problem requires both a left-hand and a right-hand boundary layers. For the left-hand boundary layer, we rescale

\[
X = \frac{x}{\delta(e)}, 
\]

where \( \delta \) is a function of \( \varepsilon \), to be determined by the method of dominant balance [1].

If we define the left-hand boundary layer solution variable as \( Y_L(X) \) we have

\[
y(x) = Y_L(X) \quad (1.10)
\]
\[
y'(x) = \frac{Y'_L(X)}{\delta} \quad (1.11)
\]
\[
y''(x) = \frac{Y''_L(X)}{\delta^2} \quad (1.12)
\]
\[
y'''(x) = \frac{Y'''_L(X)}{\delta^3}. \quad (1.13)
\]

Substituting (1.9)–(1.13) into (1.1), we find

\[
\varepsilon \frac{\delta^3}{\delta^3} Y'''_L(X) - \frac{1}{\delta} Y'_L(X) + \delta X Y_L(X) = 0. \quad (1.14)
\]

By dominant balance, we consider the scaling

\[
\varepsilon \frac{1}{\delta^3} = \frac{1}{\delta} \quad (1.15)
\]

or

\[
\delta = \sqrt{\varepsilon}. \quad (1.16)
\]

Introducing (1.16) into (1.14) yields

\[
Y'''_L(X) - Y'_L(X) + \varepsilon X Y_L(X) = 0. \quad (1.17)
\]

As \( \varepsilon \to 0 \), (1.17) reduces to

\[
Y'''_L(X) - Y'_L(X) = 0. \quad (1.18)
\]
The solution of (1.18) may be expressed as
\[ Y_L(X) = A_0 e^X + B_0 e^{-X} + C_0, \]  
(1.19)
where \( A_0, B_0, C_0 \) are constants to be determined.

By the boundary conditions, (1.2), (1.10), (1.11), and (1.16) we write at \( x = X = 0 \),
\[ y(0) = Y_L(0) = 1 \]  
(1.20)
\[ y'(0) = \frac{Y'_L(0)}{\sqrt{\varepsilon}} = 1. \]  
(1.21)

The left-hand boundary layer problem is an initial value problem with the initial conditions
\[ Y_L(0) = 1 \]  
(1.22)
\[ Y'_L(0) = \sqrt{\varepsilon}. \]  
(1.23)

The left-hand inner solution and its derivative
\[ Y_L(X) = A_0 e^X + B_0 e^{-X} + C_0 \]  
(1.24)
\[ Y'_L(X) = A_0 e^X - B_0 e^{-X} \]  
(1.25)
may be evaluated at \( X = 0 \) as
\[ Y_L(0) = A_0 + B_0 + C_0 = 1 \]  
(1.26)
\[ Y'_L(0) = A_0 - B_0 = \sqrt{\varepsilon}. \]  
(1.27)

Here we have 2 equations in three unknowns \( A_0, B_0, C_0 \).

We argue that as \( X \to \infty \), we must have \( A_0 = 0 \), otherwise \( Y_L(X) \) becomes unbounded. Solving (1.26)-(1.27) for \( B_0 \) and \( C_0 \) yields
\[ B_0 = -\sqrt{\varepsilon} \]  
(1.28)
\[ C_0 = 1 + \sqrt{\varepsilon}. \]  
(1.29)

The left-hand boundary layer solution is thus given by
\[ Y_L(X) = 1 + \sqrt{\varepsilon}(1 - e^{-X}). \]  
(1.30)

We specify the width of the left boundary layer as \( x_{fL} \), where \( 0 \leq x_{fL} < 1 \), and where in terms of the inner solution independent variable \( X_{fL} = x_{fL}/\sqrt{\varepsilon} \) by (1.9) and (1.16).
At $x_{fL}$, the inner solution and the outer solution match

$$y_0(x_{fL}) = Y_L(x_{fL}).$$  \hspace{1cm} (1.31)

By (1.4) and (1.24), we write (1.31) as

$$a_0 e^{x_{fL}^2/2} = B_0 e^{-x_{fL} + C_0}$$  \hspace{1cm} (1.32)

and solve for $a_0$.

$$a_0 = (B_0 e^{-x_{fL} + C_0}) e^{-x_{fL}^2/2} = (B_0 e^{-x_{fL}/\sqrt{\epsilon} + C_0}) e^{-x_{fL}^2/2}.$$  \hspace{1cm} (1.33)

We note from (1.33) that $a_0$ is a function of $x_{fL}$ and $\epsilon$. We should recognize that we cannot arbitrarily set $x_{fL}$ for it must be chosen in such a way that the outer solution matches well not only the left-hand boundary layer but also the right-hand boundary layer.

Noting that $x_{fL} \to 0$, $x_{fL} \to \infty$ as $\epsilon \to 0$, then (1.33) reduces to

$$a_0 = C_0 = 1 + \sqrt{\epsilon}$$  \hspace{1cm} (1.34)

by virtue of (1.29). This is a higher order approximation for $a_0$ than that given in (1.6). Thus the outer solution may be expressed as

$$y_o(x) = (1 + \sqrt{\epsilon}) e^{x^2/2}.$$  \hspace{1cm} (1.35)

For the right-hand boundary layer, we rescale as

$$\bar{X} = \frac{1-x}{\delta(\epsilon)}.$$  \hspace{1cm} (1.36)

By (1.36) we write

$$x = 1 - \delta(\epsilon) \bar{X}.$$  \hspace{1cm} (1.37)

If we define the right-hand boundary layer as $Y_R(\bar{X})$, we have

$$y(x) = Y_R(\bar{X})$$  \hspace{1cm} (1.38)

$$y'(x) = -\frac{1}{\delta} Y_R'(\bar{X})$$  \hspace{1cm} (1.39)

$$y''(x) = \frac{1}{\delta^2} Y_R''(\bar{X})$$  \hspace{1cm} (1.40)

$$y'''(x) = -\frac{1}{\delta^3} Y_R'''(\bar{X}).$$  \hspace{1cm} (1.41)
Substituting (1.37)–(1.41) into (1.1) yields

\[-\frac{\varepsilon}{\delta^3} Y''_R(\bar{X}) + \frac{1}{\delta} Y'_R(\bar{X}) + (1 - \delta \bar{X}) Y_R(\bar{X}) = 0. \tag{1.42}\]

By the dominant balance argument, we write

\[\frac{\varepsilon}{\delta^3} = \frac{1}{\delta} \tag{1.43}\]

\[\delta = \sqrt{\varepsilon} \tag{1.44}\]

\[Y''_R(\bar{X}) - Y'_R(\bar{X}) = \sqrt{\varepsilon}(1 - \sqrt{\varepsilon \bar{X}}) Y_R(\bar{X}). \tag{1.45}\]

As \(\varepsilon \to 0\), (1.45) reduces to

\[Y''_R(\bar{X}) - Y'_R(\bar{X}) = 0 \tag{1.46}\]

whose solution is given by

\[Y_R(\bar{X}) = \tilde{A}_0 \bar{X} + B_0 e^{-\bar{X}} + C_0. \tag{1.47}\]

To establish a bounded solution we require as \(\bar{X} \to \infty\) that

\[\tilde{A}_0 = 0. \tag{1.48}\]

Since \(y(1) = 1\) by (1.2), it follows by (1.37) and (1.38) that

\[Y_R(0) = 1. \tag{1.49}\]

At \(\bar{X} = 0\), we have

\[Y_R(0) = 1 = B_0 + C_0 \tag{1.50}\]

or

\[C_0 = 1 - B_0. \tag{1.51}\]

To determine \(B_0\) and \(C_0\) we must specify the right-hand boundary layer initial point \(x_{fr}\) where the thickness of the boundary layer is \((1 - x_{fr})\). At \(x_{fr}\) (or \(\bar{x}_{fr} = (1 - x_{fr})/\sqrt{\varepsilon}\)), we match the outer solution and the right-hand boundary layer solution

\[y_0(x_{fr}) = Y_R(\bar{x}_{fr}) \tag{1.52}\]

\[a_0 \bar{x}_{fr}^{3/2} = (B_0 e^{-\bar{x}_{fr}} + C_0) = (B_0 e^{-\bar{x}_{fr}} + 1 - B_0). \tag{1.53}\]
Recalling that $a_0$ is known from (1.34) once $x_{fL}$ and $\varepsilon$ are specified, we now solve (1.53) for $\bar{B}_0$:

$$\bar{B}_0 = \frac{a_0 e^{x_{r/2}^2} - 1}{e^{x_{r/2}^2} - 1} \quad (1.54)$$

$$\bar{C}_0 = 1 - \frac{a_0 e^{x_{r/2}^2} - 1}{e^{x_{r/2}^2} - 1}. \quad (1.55)$$

To review the process:

1. Specify $\varepsilon$.
2. Calculate

$$B_0 = -\sqrt{\varepsilon} \quad (1.28)$$

$$C_0 = 1 + \sqrt{\varepsilon}. \quad (1.29)$$

3. Specify left-hand boundary layer, $x_{fL}$.
4. Specify the right-hand boundary layer initial point $x_{fR}$.
5. Compute $a_0$

$$a_0 = (B_0 e^{-x_{r/2}^2} + C_0) e^{-x_{fL}^2}. \quad (1.33)$$

6. Compute $\bar{B}_0$ and $\bar{C}_0$,

$$\bar{B}_0 = \frac{a_0 e^{x_{r/2}^2} - 1}{e^{x_{r/2}^2} - 1} \quad (1.54)$$

$$\bar{C}_0 = 1 - \bar{B}_0. \quad (1.55)$$

7. Generate left-hand boundary layer solution and derivative

$$Y_L(X) = B_0 e^{-X} + C_0, \quad 0 \leq X \leq x_{fL} = x_{fL}/\sqrt{\varepsilon} \quad (1.24)$$

$$Y'_L(X) = -B_0 e^{-X}, \quad (1.25)$$

where $A_0 = 0$ to eliminate exponential overflow.

8. Generate outer solution and derivative

$$y_0(x) = a_0 e^{x^{2/2}}, \quad x_{fL} \leq x \leq x_{fR} \quad (1.4)$$

$$y'_0(x) = a_0 xe^{x^{2/2}} \quad (1.3)$$

where $a_0 = 1 + \sqrt{\varepsilon}$, as $\varepsilon \to 0$.

1 Note. There are three ways to approximate $a_0$: a. by (1.6) from the boundary condition; b. by (1.33) where the left-hand inner solution and the outer solution are matched; c. by (1.34) where an asymptotic argument, $x_{fL} \to 0$, $X_{fL} \to \infty$, as $\varepsilon \to 0$ is applied to (1.33).
9. Generate right-hand boundary layer solution and derivative

\[ Y_R(\bar{x}) - \bar{B}_0 \bar{e}^{\bar{x}} + \bar{C}_0 \]

\[ Y_R(\bar{x}) = -\bar{B}_0 \bar{e}^{\bar{x}}, \quad 0 \leq \bar{x} \leq \frac{1-x_{fR}}{\sqrt{\varepsilon}}. \] (1.47)

10. Compute the error norms at \( x_{fL} \),

\[ \left\| y'_0(x_{fL}) - \frac{Y'_L(X_{fL})}{\sqrt{\varepsilon}} \right\|, \]

and at \( x_{fR} \),

\[ \left\| y'_0(x_{fR}) - \left( -\frac{Y'_R(X_{fR})}{\sqrt{\varepsilon}} \right) \right\|, \]

and form the overall norm, the sum of the norms at \( x_{fL} \) and \( x_{fR} \).

11. If the norms are not sufficiently small, adjust \( x_{fL}, \) \( x_{fR} \) and return to item 3.

12. Otherwise, terminate

For Example 1 we found it useful to modify the boundary value technique to focus in more directly on the choices of \( x_{fL} \) and \( x_{fR} \). As a simple

![Image](image-url)
EXAMPLES OF BOUNDARY VALUE TECHNIQUES

TABLE I

Profiles of Run BEN4, A3 and BEN5, Run 1;
\( \varepsilon = 0.01, x_{FL} = 0.10, x_{FR} = 0.80 \)

<table>
<thead>
<tr>
<th>x</th>
<th>( y(x) ) ( \text{BEN4, Run A3}^a )</th>
<th>( y(x) ) ( \text{BEN5, Run 1}^b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000000</td>
<td>0.9999480</td>
</tr>
<tr>
<td>0.01</td>
<td>1.0095163</td>
<td>1.0095155</td>
</tr>
<tr>
<td>0.05</td>
<td>1.0393469</td>
<td>1.0406756</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0632121</td>
<td>1.0687385</td>
</tr>
<tr>
<td>0.20</td>
<td>1.0792804</td>
<td>1.1089363</td>
</tr>
<tr>
<td>0.30</td>
<td>1.1066026</td>
<td>1.1461091</td>
</tr>
<tr>
<td>0.40</td>
<td>1.1460194</td>
<td>1.1898687</td>
</tr>
<tr>
<td>0.50</td>
<td>1.1987682</td>
<td>1.2433876</td>
</tr>
<tr>
<td>0.60</td>
<td>1.2665473</td>
<td>1.3055497</td>
</tr>
<tr>
<td>0.70</td>
<td>1.3516074</td>
<td>1.3681575</td>
</tr>
<tr>
<td>0.80</td>
<td>1.4568762</td>
<td>1.4055116</td>
</tr>
<tr>
<td>0.82</td>
<td>1.4410439</td>
<td>1.4050509</td>
</tr>
<tr>
<td>0.86</td>
<td>1.3980872</td>
<td>1.3896516</td>
</tr>
<tr>
<td>0.90</td>
<td>1.3340033</td>
<td>1.3462056</td>
</tr>
<tr>
<td>0.92</td>
<td>1.2909665</td>
<td>1.3095822</td>
</tr>
<tr>
<td>0.96</td>
<td>1.1741981</td>
<td>1.1942746</td>
</tr>
<tr>
<td>1.00</td>
<td>1.0000000</td>
<td>0.9999954</td>
</tr>
</tbody>
</table>

\(^a\) Code BEN4 solves (1.1)-(1.2), Bender and Orszag \([1, p. 446]\), by the boundary value method; Run A3 specifies \( \varepsilon = 0.01, x_{FL} = 0.10, x_{FR} = 0.80 \).

\(^b\) Code BEN5 solves the Bender and Orszag approximation \([1, p. 449, (9.15.17)]\); Run 1 specifies \( \varepsilon = 0.01 \).

It is expedient to define the shape of the solution of (1.1), it is convenient to solve it for various values of \( \varepsilon \) (not necessarily small) by numerical integration or by some approximation technique. In this way, we can get a feel for the relative size and location of the boundary layers. Figure 1\(^2\) (from the BEN5 Code) is an approximate solution of (1.1) by the Bender and Orszag equations (9.15.17), page 449 \([1]\). Armed with this information we can proceed to pick \( x_{FL} \) and \( x_{FR} \) more selectively. For \( \varepsilon = 0.01, x_{FL} = 0.1 \) and \( x_{FR} = 0.80 \), Table I lists the Roberts method solution as well as that of Bender and Orszag (their Eq. (9.15.17)). The results compare favorably. For \( \varepsilon = 0.001, x_{FL} = 0.10 \), and \( x_{FR} = 0.90 \), Table II lists once again the boundary

\(^2\) For (1.1)-(1.2) the boundary value method was implemented in a code, BEN4. The runs are identified as A1, A2, etc. The Bender and Orszag approximation (9.15.17), page 449 of \([1]\), was implemented in a code BEN5. The runs are identified as BEN5, Run #1, BEN5, Run #2, etc.
TABLE II
Profiles of BEN4, Run A13 and BEN5, Run 2;
$\varepsilon = 0.001, x_{fl} = 0.10, x_{fr} = 0.90$

<table>
<thead>
<tr>
<th>$x$</th>
<th>BEN4, Run A13$^a$</th>
<th>BEN5, Run 2$^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000000</td>
<td>1.0000000</td>
</tr>
<tr>
<td>0.01</td>
<td>1.0085731</td>
<td>1.0086249</td>
</tr>
<tr>
<td>0.05</td>
<td>1.0251167</td>
<td>1.0264108</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0302842</td>
<td>1.0354703</td>
</tr>
<tr>
<td>0.20</td>
<td>1.0458550</td>
<td>1.0524679</td>
</tr>
<tr>
<td>0.30</td>
<td>1.0723309</td>
<td>1.0792471</td>
</tr>
<tr>
<td>0.40</td>
<td>1.1105270</td>
<td>1.1178104</td>
</tr>
<tr>
<td>0.50</td>
<td>1.1616422</td>
<td>1.1694243</td>
</tr>
<tr>
<td>0.60</td>
<td>1.2273222</td>
<td>1.2357593</td>
</tr>
<tr>
<td>0.70</td>
<td>1.3097479</td>
<td>1.3189784</td>
</tr>
<tr>
<td>0.80</td>
<td>1.4117565</td>
<td>1.4207615</td>
</tr>
<tr>
<td>0.90</td>
<td>1.5370034</td>
<td>1.5175768</td>
</tr>
<tr>
<td>0.92</td>
<td>1.5150632</td>
<td>1.5191218</td>
</tr>
<tr>
<td>0.94</td>
<td>1.4766490</td>
<td>1.4784837</td>
</tr>
<tr>
<td>0.96</td>
<td>1.4024624</td>
<td>1.4352880</td>
</tr>
<tr>
<td>0.98</td>
<td>1.2628265</td>
<td>1.2927005</td>
</tr>
<tr>
<td>1.00</td>
<td>1.0000000</td>
<td>1.0000000</td>
</tr>
</tbody>
</table>

$^a$ Code BEN4 solves (1.1)–(1.2), Bender and Orszag [1, p. 446] by the boundary value method, Run A13 specifies $\varepsilon = 0.001, x_{fl} = 0.10, x_{fr} = 0.90$.

$^b$ Code BEN5 solves the Bender and Orszag approximation [1, p. 449, (9.15.17)]; Run 2 specifies $\varepsilon = 0.001$.

Unfortunately, the good comparisons in Tables I and II require a deeper insight than we have heretofore experienced with the method. The data in Tables I and II (Runs A3 and A13) were not captured from runs which possessed the minimum of the norms of the errors in the derivatives as previously advocated. As shown in Table III, we can find runs with smaller overall error norms at the price of choosing unlikely values for $x_{fr}$. For example, Run A7 with $\varepsilon = 0.01$ has small overall error norm of 0.477 yet its $x_{fl} = 0.10$ and $x_{fr} = 0.20$. We know from Fig. 1, that with $\varepsilon = 0.01$ that $x_{fr}$ must be in the neighborhood of 0.80. It is unreasonable that the outer solution interval should only be 0.10 in length ($x_{fr} - x_{fl}$). A plot of Run A7 is given in Fig. 2.

The runs A3 and A13 gave good results because the $x_{fl}$ and $x_{fr}$ were
### TABLE III

**BEN4 Runs—Preliminary Choices for \( x_{fL} \) and \( x_{fR} \) and the Derivative Norms**

<table>
<thead>
<tr>
<th>Run No.</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
<th>A6</th>
<th>A7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon )</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>( x_{fL} )</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>( x_{fR} )</td>
<td>0.95</td>
<td>0.90</td>
<td>0.80</td>
<td>0.70</td>
<td>0.50</td>
<td>0.30</td>
<td>0.20</td>
</tr>
<tr>
<td>( a ) ( x_{fL} )</td>
<td>0.2615</td>
<td>0.2615</td>
<td>0.2615</td>
<td>0.2615</td>
<td>0.2615</td>
<td>0.2615</td>
<td>0.2615</td>
</tr>
<tr>
<td>( a ) ( x_{fR} )</td>
<td>11.7706</td>
<td>4.8386</td>
<td>1.8805</td>
<td>1.1303</td>
<td>0.6128</td>
<td>0.3329</td>
<td>0.2161</td>
</tr>
<tr>
<td>( a ) Total</td>
<td>12.0323</td>
<td>5.1001</td>
<td>2.1420</td>
<td>1.3919</td>
<td>0.8744</td>
<td>0.5945</td>
<td>0.4776</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Run No.</th>
<th>Al3</th>
<th>A14</th>
<th>A15</th>
<th>A16</th>
<th>A17</th>
<th>A18</th>
<th>A19</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon )</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>( x_{fL} )</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>( x_{fR} )</td>
<td>0.90</td>
<td>0.80</td>
<td>0.50</td>
<td>0.20</td>
<td>0.90</td>
<td>0.80</td>
<td>0.50</td>
</tr>
<tr>
<td>( a ) ( x_{fL} )</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.2045</td>
<td>0.2045</td>
<td>0.2045</td>
</tr>
<tr>
<td>( a ) ( x_{fR} )</td>
<td>2.1338</td>
<td>1.1527</td>
<td>0.5808</td>
<td>0.2091</td>
<td>2.0856</td>
<td>1.1360</td>
<td>0.573</td>
</tr>
<tr>
<td>( a ) Total</td>
<td>2.1945</td>
<td>1.2134</td>
<td>0.6415</td>
<td>0.2698</td>
<td>2.2901</td>
<td>1.3407</td>
<td>0.7774</td>
</tr>
</tbody>
</table>

* Error norm at \( x_{fL} \), \( x_{fR} \), and sum of the norms, respectively.

*Note.* Runs A13 and A19 are tabulated in Tables I and II, respectively.

---

**Fig. 2.** Boundary value method solution of Example I—\( \varepsilon = 0.01 \), \( x_{fL} = 0.10 \), \( x_{fR} = 0.20 \) (Run BEN4, A7).
close to the correct values of the left- and right-hand boundary layers. Runs such as A6, A7, A16, while, generating, small norms had poor choices for the \( x_{fL} \) and \( x_{fR} \).

In this example we must amend our criteria for selecting the appropriate boundary layers by taking into consideration approximately correct values for \( x_{fL} \) and \( x_{fR} \). These approximate values are obtained by a preliminary analysis of the solution of (1.1)–(1.2).

3. Example 2 (Bender and Orszag [1, p. 437])

Consider

\[ \varepsilon y''(x) - x^2 y'(x) - y(x) = 0, \quad 0 \leq x \leq 1 \]  

(2.1)

with the boundary conditions

\[ y(0) = y(1) = 1. \]  

(2.2)

To obtain the outer solution differential equation, we set \( \varepsilon = 0 \) and write

\[ x^2 y_0'(x) + y_0(x) = 0, \]  

(2.3)

where solution \( y_0(x) \) is given by

\[ y_0(x) = C_0 e^{1/x}, \]  

(2.4)

where \( C_0 \) is a constant to be determined. We observe in (2.4) that as \( x \to 0 \), \( y_0(x) \to \infty \). To compensate for this unbounded growth requires

\[ C_0 = 0 \]  

(2.5)

which means the outer solution is identically zero over the interval. In view of the boundary condition (2.2), this implies the existence of both a left- and right-hand boundary layer. To generate the left-hand boundary layer differential equation we set

\[ X = x/\delta(\varepsilon), \]  

(2.6)

where \( \delta(\varepsilon) \) is to be determined, and we define the left-hand boundary layer variable \( Y_L(X) \) and its derivatives as

\[ y(x) = Y_L(X) \]  

(2.7)

\[ y'(x) = Y'_L(X)/\delta \]  

(2.8)

\[ y''(x) = Y''_L(X)/\delta^2. \]  

(2.9)
EXAMPLES OF BOUNDARY VALUE TECHNIQUES

Substituting (2.6)–(2.9) into (2.1), we obtain

\[ \frac{\varepsilon}{\delta^2} Y''_L(X) - X^2 \delta Y'_L(X) - Y_L(X) = 0. \]  

(2.10)

By a dominant balance argument we have

\[ \frac{\varepsilon}{\delta^2} = 1 \]  

(2.11)

\[ \delta = \sqrt{\varepsilon} \]  

(2.12)

Introducing (2.12) into (2.10) yields

\[ Y''_L(X) - X^2 \sqrt{\varepsilon} Y'_L(X) - Y_L(X) = 0. \]  

(2.13)

As \( \varepsilon \to 0 \), (2.13) reduces to

\[ Y''_L(X) - Y_L(X) = 0 \]  

(2.14)

whose solution is given by

\[ Y_L(X) = a_1 e^x + a_2 e^{-x}. \]  

(2.15)

As \( X \) becomes larger, \( Y_L(X) \) becomes unbounded. To compensate for this we require

\[ a_1 = 0. \]  

(2.16)

Equation (2.15) is then written as

\[ Y_L(X) = a_2 e^{-x}. \]  

(2.17)

By the initial condition in (2.2), we have at \( X = 0 \),

\[ Y_L(0) = a_2 = 1. \]  

(2.18)

The left-hand boundary layer solution is then expressed as

\[ Y_L(X) = e^{-x} = e^{-x/\sqrt{\varepsilon}}. \]  

(2.19)

For the right-hand boundary layer independent variable we define

\[ \tilde{x} = \frac{1 - x}{\delta(\varepsilon)}, \]  

(2.20)

where \( \delta(\varepsilon) \) is to be determined.
We define $Y_R(\bar{X})$, the right-hand boundary layer solution and its derivatives as

\begin{align*}
y(x) &= Y_R(\bar{X}) \\
y'(x) &= -\frac{Y_R'(\bar{X})}{\delta} \\
y''(x) &= \frac{Y_R''(\bar{X})}{\delta^2}.
\end{align*}

Substituting (2.20)–(2.23) into (2.1), we obtain

\begin{equation}
\frac{\varepsilon}{\delta^2} \frac{\partial^2}{\partial X^2} Y_R''(\bar{X}) - \left(1 - \delta \bar{X}\right)^2 \frac{\partial}{\partial \bar{X}} \left(\frac{-Y_R'(\bar{X})}{\delta}\right) - Y_R(\bar{X}) = 0.
\end{equation}

By dominant balance

\begin{equation}
\frac{\varepsilon}{\delta^2} = \frac{1}{\delta} \\
\varepsilon = \delta.
\end{equation}

Introducing (2.26) into (2.24), we have

\begin{equation}
Y_R''(\bar{X}) + \left(1 - \varepsilon \bar{X}\right)^2 \frac{\partial}{\partial \bar{X}} \left(\frac{-Y_R'(\bar{X})}{\delta}\right) - \varepsilon Y_R(\bar{X}) = 0.
\end{equation}

As $\varepsilon \to 0$, (2.27) reduces to

\begin{equation}
Y_R''(\bar{X}) + Y_R'(\bar{X}) = 0,
\end{equation}

where the solution is

\begin{equation}
Y_R(\bar{X}) = b_0 + b_1 e^{-\bar{x}},
\end{equation}

where $b_0$, $b_1$ are constants to be determined.

By (2.20) and (2.26),

\begin{equation}
\bar{X} = \frac{1 - x}{\varepsilon}.
\end{equation}

At $x = 1$ (or $\bar{X} = 0$) by the boundary conditions (2.2),

\begin{equation}
Y_R(0) = b_0 + b_1 = 1.
\end{equation}

As $\varepsilon \to 0$, $\bar{X} \to \infty$,

\begin{equation}
Y_R(\bar{X}) \to y_0(x) = 0.
\end{equation}
(The right boundary layer solution approximates the outer solution which is identically zero.) and by (2.29)

\[ Y_R(X) = b_0 = 0. \] (2.33)

From (2.31) and (2.33), we have

\[ b_1 = 1. \] (2.34)

Therefore, it follows that (2.29) can be written as

\[ Y_R(\bar{X}) = e^{-X} = e^{-(1-x)/\epsilon}. \] (2.35)

This problem is unusual since no outer solution exists. It is not necessary to specify \( x_f^{(1)} \) or \( x_f^{(R)} \) for the problem may be solved directly without the thickness of the boundary layers entering into the analysis as in other problems we have discussed.

4. Example 3 (Bender and Orszag [1, p. 449])

For singular perturbation problems, a simple but useful way to gain insight into the nature of the solution is to conduct a parameter or continuation study of the original differential equation. In particular, we may generate the solution of the differential equation for values of \( \epsilon \) which are not necessarily small and for which numerical integration of the original system is possible. Such a study can delineate: (1) the shape of the solution, (2) the location of boundary layers, (3) the nature of the boundary layers, (4) the deformation of the solution as a function of \( \epsilon \), and (5) the determination of the value of \( \epsilon \) where numerical integration of the original system fails.

Armed with this knowledge the investigator can attack the singular perturbation problem with some assurance of what to expect. As our first approach to solving the fourth-order equation

\[ \epsilon^2 y'''(x) - (1 + x) y''(x) = 1, \quad 0 \leq x \leq 1, \] (4.1)

with the boundary conditions

\[ y(0) = y'(0) = 1 \] (4.2)

\[ y(1) = y'(1) = 1, \] (4.3)

we integrate (4.1)-(4.3) by the Bulirsch–Stoer method described by Gear [2].
A parameter study for $\varepsilon = 0.10, 0.05, 0.04$ was carried out and plotted in Fig. 3. We could not solve (4.1)–(4.3) for $\varepsilon < 0.04$ without overflow. The curves rise sharply at $x = 0$, reach a maximum value in the range of $0.3 \leq x \leq 0.4$, decline sharply to a minimum in the range of $0.9 \leq x \leq 0.96$ and rise sharply near $x = 1.0$. From these plots we can expect as $\varepsilon \to 0$, both left-hand and right-hand boundary layers, with the maximum shifting toward $x = 0.4$ and the minimum becoming less pronounced and shifting toward $x = 1.0$. The slope at $x = 1.0$ is positive and becomes steeper as $\varepsilon$ becomes smaller.

As before, we will develop the equations for the left- and right-hand boundary layers as well as for the outer solution. To obtain the differential equation of the outer solution, we set $\varepsilon = 0$ in (4.1) and write

$$-(1 + x) y''_0(x) = 1.$$  \hspace{1cm} (4.4)

The outer solution is then given by

$$y_0(x) = -(1 + x) \ln(1 + x) + a_0 x + b_0,$$  \hspace{1cm} (4.5)

where $a_0$ and $b_0$ are constants to be determined.

From our parameter study we have concluded that both left- and right-
hand boundary layers exist, so we proceed on this basis. As before, we rescale for the left-hand boundary layer as

\[ X = x/\delta(\varepsilon) \]  

and we write

\[ y(x) = Y_L(X) \]  
\[ y'(x) = Y'_L(X)/\delta \]  
\[ y''(x) = Y''_L(X)/\delta^2 \]  
\[ y'''(x) - Y'''_L(X)/\delta^3 \]  
\[ y''''(x) = Y''''_L(X)/\delta^4. \]

Substituting (4.6), (4.9), (4.11), into (4.1) yields the left-hand boundary layer solution equations

\[ \frac{\varepsilon^2}{\delta^4} Y''''_L(X) - \frac{(1 + X\delta)}{\delta^2} Y'_L(X) = 1. \]  

By a dominant balance argument, we write

\[ \frac{\varepsilon^2}{\delta^4} = \frac{1}{\delta^2} \]

and conclude

\[ \delta = \varepsilon. \]

Substituting (4.14) into (4.6) and (4.12) gives

\[ Y''''_L(X) - (1 + X\varepsilon) Y'_L(X) = \varepsilon^2. \]

As \( \varepsilon \to 0 \), (4.15) is approximated by

\[ Y''''_L(X) - Y'_L(X) = 0. \]

The solution of (4.16) and its derivative are given by

\[ Y_L(X) = A_0(1 + X) + B_0e^X + C_0e^{-X} \]  
\[ Y'_L(X) = A_0 + B_0e^X - C_0e^{-X}. \]

From (4.2), (4.6), (4.8), (4.14), the initial conditions are

\[ Y_L(0) = 1 \]  
\[ Y'_L(0) = \varepsilon. \]
For bounded solutions of (4.17), the coefficient of $e^X$ must vanish, since $e^X \to \infty$ as $X \to \infty$. Therefore,

$$B_0 = 0.$$  \hspace{1cm} (4.21)

At $X = 0$, (4.17) and its derivative appear as

$$Y_L(0) = A_0 + C_0 = 1$$  \hspace{1cm} (4.22)
$$Y'_L(0) = A_0 - C_0 = \varepsilon,$$  \hspace{1cm} (4.23)

from which it follows that

$$A_0 = \frac{1 + \varepsilon}{2}$$  \hspace{1cm} (4.24)
$$C_0 = \frac{1 - \varepsilon}{2}.$$  \hspace{1cm} (4.25)

The left-hand inner solution and its derivative are expressed as

$$Y_L(X) = \left(\frac{1 + \varepsilon}{2}\right) (1 + X) + \left(\frac{1 - \varepsilon}{2}\right) e^{-X}$$  \hspace{1cm} (4.26)
$$Y'_L(X) = \left(\frac{1 + \varepsilon}{2}\right) - \left(\frac{1 - \varepsilon}{2}\right) e^{-X}.$$  \hspace{1cm} (4.27)

Up to this point we have established the outer solution equation, but have not evaluated its constants $a_0$ and $b_0$. We have also generated the left-hand boundary layer solution. To proceed further, we need to form the right-hand boundary layer solution.

For the right-hand boundary layer, consider the rescaling

$$\tilde{X} = \frac{1 - x}{\delta(\varepsilon)}.$$  \hspace{1cm} (4.28)

As before, we have

$$y(x) = Y_R(\tilde{X})$$  \hspace{1cm} (4.29)
$$y'(x) = - \frac{Y'_R(\tilde{X})}{\delta}$$  \hspace{1cm} (4.30)
$$y''(x) = \frac{Y''_R(\tilde{X})}{\delta^2}$$  \hspace{1cm} (4.31)
$$y'''(x) = \frac{-Y'''_R(\tilde{X})}{\delta^3}$$  \hspace{1cm} (4.32)
$$y''''(x) = \frac{Y''''_R(\tilde{X})}{\delta^4}.$$  \hspace{1cm} (4.33)
Substituting (4.28), (4.31), and (4.33) into (4.1), we have
\[
\frac{\varepsilon^2 Y''''(\bar{X})}{\delta^4} - \frac{(2 - \bar{X}\delta) Y''(\bar{X})}{\delta^2} = 1. \tag{4.34}
\]

By dominant balance arguments
\[
\frac{\varepsilon^2}{\delta^4} = \frac{1}{\delta^2} \tag{4.35}
\]
\[
\delta(\varepsilon) = \varepsilon. \tag{4.36}
\]

Introducing (4.36) into (4.34) we have
\[
Y''''(\bar{X}) - (2 - \bar{X}\varepsilon) Y''(\bar{X}) = \varepsilon^2. \tag{4.37}
\]

As \( \varepsilon \to 0 \), (4.37) is approximated by
\[
Y''''(\bar{X}) - 2 Y''(\bar{X}) = 0. \tag{4.38}
\]

The solution to (4.38) is written as
\[
Y_R(\bar{X}) = \bar{A}_0(1 + \bar{X}) + \bar{B}_0 e^{\sqrt{2}\bar{X}} + \bar{C}_0 e^{-\sqrt{2}\bar{X}}. \tag{4.39}
\]

The derivative is
\[
Y'_R(\bar{X}) = \bar{A}_0 + \sqrt{2} \bar{B}_0 e^{\sqrt{2}\bar{X}} - \sqrt{2} \bar{C}_0 e^{-\sqrt{2}\bar{X}}. \tag{4.40}
\]

In order for \( Y_R(\bar{X}) \) to be bounded, it is necessary that
\[
\bar{B}_0 = 0. \tag{4.41}
\]

The constants in (4.39) may be determined from the terminal boundary conditions in (4.3). At \( x = 1, \bar{X} = 0 \),
\[
y(1) = Y_R(0) = 1 \tag{4.42}
\]
\[
y'(1) = -\frac{Y'_R(0)}{\varepsilon} = 1. \tag{4.43}
\]

Using (4.42) and (4.43), we evaluate (4.39) and (4.40) at \( \bar{X} = 0 \),
\[
Y_R(0) = \bar{A}_0 + \bar{C}_0 = 1 \tag{4.44}
\]
\[
Y'_R(0) = \bar{A}_0 - \sqrt{2} \bar{C}_0 = -\varepsilon. \tag{4.45}
\]
Solving (4.44) and (4.45) for $\bar{A}_0$ and $\bar{C}_0$ we have

\begin{align*}
\bar{A}_0 &= \frac{\sqrt{2} - \varepsilon}{1 + \sqrt{2}} \\
\bar{C}_0 &= \frac{1 + \varepsilon}{1 + \sqrt{2}}.
\end{align*}

(4.46)  
(4.47)

The right-hand boundary layer solution and its derivative are given by

\begin{align*}
Y_R(\bar{x}) &= \left(\frac{\sqrt{2} - \varepsilon}{1 + \sqrt{2}}\right)(1 + \bar{x}) + \left(\frac{1 + \varepsilon}{1 + \sqrt{2}}\right)e^{-\sqrt{2} \bar{x}} \\
Y'_R(\bar{x}) &= \frac{\sqrt{2} - \varepsilon}{1 + \sqrt{2}} \left(\frac{\sqrt{2}(1 + \varepsilon)}{1 + \sqrt{2}}\right)e^{-\sqrt{2} \bar{x}}.
\end{align*}

(4.48)  
(4.49)

Let us now summarize where we are and what is remaining to be done. At this point we have

1. the analytical solution to the outer solution

\[ y_0(x) = -(1 + x) \ln(1 + x) + a_0 x + b_0; \quad (4.50) \]

2. the left-hand boundary layer solution

\[ Y_L(X) = A_0(1 + X) + C_0 e^{-X}, \quad (4.51) \]

where $X = x/\varepsilon$, $A_0 = (1 + \varepsilon)/2$, $C_0 = (1 - \varepsilon)/2$;

3. the right-hand boundary layer solution

\[ Y_R(\bar{x}) = \bar{A}_0(1 + \bar{x}) + \bar{C}_0 e^{-\sqrt{2} \bar{x}}, \quad (4.52) \]

where $\bar{x} = (1 - x)/\varepsilon$, $\bar{A}_0 = (\sqrt{2} - \varepsilon)/(1 + \sqrt{2})$, $\bar{C}_0 = (1 + \varepsilon)/(1 + \sqrt{2})$.

What we do not have are values for the constants $a_0$ and $b_0$ in the outer solution. We now proceed to determine $a_0$ and $b_0$.

To meld the outer solution with the left- and right-hand boundary layers we need to specify the points where the solutions join. In particular, we must specify $x_{fL}$ and $x_{fR}$, where $x_{fL}$ is the terminal point (or the thickness) of the left-hand boundary layer and where $x_{fR}$ is the initial point of the right-hand boundary layer. The width or thickness of the right-hand boundary layer is $(1 - x_{fR})$.

On specifying $x_{fL}$ and $x_{fR}$, we can solve (4.50) for $a_0$ and $b_0$ by employ-
EXAMPLES OF BOUNDARY VALUE TECHNIQUES

From (4.51) at \( x = x_{fL} \) and (4.52) at \( x = x_{fR} \). At \( x_{fL} \) we equate \( y_0(x_{fL}) \) and \( Y_L(X_{fL}) \), where \( X_{fL} = x_{fL}/\varepsilon \),

\[
y_0(x_{fL}) = -(1 + x_{fL}) \ln(1 + x_{fL}) + a_0 x_{fL} + b_0 = Y_L(X_{fL}) \tag{4.53}
\]

\[
= A_0(1 + X_{fL}) + C_0 e^{-X_{fL}}.
\]

At \( x_{fR} \) we equate \( y_0(x_{fR}) \) and \( Y_R(X_{fR}) \), where \( X_{fR} = (1 - x_{fR})/\varepsilon \),

\[
y_0(x_{fR}) = -(1 + x_{fR}) \ln(1 + x_{fR}) + a_0 x_{fR} + b_0 = Y_R(X_{fR}) \tag{4.54}
\]

\[
= \bar{A}_0(1 + X_{fR}) + \bar{C}_0 e^{-\sqrt{2}X_{fR}}.
\]

We may simplify (4.53) and (4.54) by rearranging

\[
a_0 x_{fL} + b_0 = S_1, \tag{4.55}
\]

\[
a_0 x_{fR} + b_0 = S_2, \tag{4.56}
\]

where \( S_1 \), and \( S_2 \) are known quantities defined as

\[
S_1 = A_0(1 + X_{fL}) + C_0 e^{-X_{fL}} + (1 + x_{fL}) \ln(1 + x_{fL}) \tag{4.57}
\]

\[
S_2 = \bar{A}_0(1 + X_{fR}) + \bar{C}_0 e^{-\sqrt{2}X_{fR}} + (1 + x_{fR}) \ln(1 + x_{fR}). \tag{4.58}
\]

Solving (4.55)-(4.56) for \( a_0 \) and \( b_0 \), we have

\[
a_0 = \frac{S_1 - S_2}{x_{fL} - x_{fR}} \tag{4.59}
\]

\[
b_0 = \frac{x_{fL} S_2 - x_{fR} S_1}{x_{fL} - x_{fR}}. \tag{4.60}
\]

It is interesting to note that the left- and right-hand boundary layer problems are initial value problems whose solutions depend only on \( \varepsilon \) and \( x \). The outer solution, on the other hand, is dependent on \( a_0 \) and \( b_0 \) which in turn are dependent on \( \varepsilon \), \( x_{fL} \), and \( x_{fR} \). The outer solution serves as the bridge between the two inner solutions.

The proper choice of \( x_{fL} \) and \( x_{fR} \) is determined by the smallest norm on the derivatives at \( x_{fL} \) and \( x_{fR} \). That is, \( \| y'(x_{fL}) - Y'_L(X_{fL})/\varepsilon \| + \| y'(x_{fR}) - \{ - Y'_R(X_{fR})/\varepsilon \} \| \) is to be minimized.

To review the process:

1. Specify \( \varepsilon \).
2. Calculate \( A_0 = (1 + \varepsilon)/2 \), \( C_0 = (1 - \varepsilon)/2 \).
3. Calculate \( \bar{A}_0 = (\sqrt{2} - \varepsilon)/(1 + \sqrt{2}) \), \( \bar{C}_0 = (1 + \varepsilon)/(1 + \sqrt{2}) \).
4. Specify \( x_{fL} \) and \( x_{fR} \) or \( (X_{fL} = x_{fL}/\varepsilon, \bar{X}_{fR} = (1 - x_{fR})/\varepsilon) \).
5. Calculate \( a_0, b_0 \),

\[
a_0 = \frac{S_1 - S_2}{x_{fL} - x_{fR}}, \quad b_0 = \frac{x_{fL}S_2 - x_{fR}S_1}{x_{fL} - x_{fR}},
\]

where \( S_1, S_2 \) are defined in (4.57)–(4.58).

6. Generate the left-hand boundary layer solution and its derivative \( Y_L(X) \) and \( Y'_L(X) \), (4.26) and (4.27), over \([0, x_{fL}]\).

7. Generate the outer solution (4.5) over the interval \([x_{fL}, x_{fR}]\).

8. Generate the right-hand boundary layer solution and its derivative, \( Y_R(X) \) and \( Y'_R(X) \), (4.39)–(4.40), over the interval \([x_{fR}, 1]\).

9. Form the derivative norms

\[
\|y'_0(x_{fL}) - \frac{Y'_L(x_{fL})}{\varepsilon}\| \quad \text{and} \quad \|y'_0(x_{fR}) - \left\{ -\frac{Y'_R(x_{fR})}{\varepsilon} \right\}\|
\]

10. If the norms are not sufficiently small, adjust \( x_{fL} \) and \( x_{fR} \) and return to item 4.

11. Otherwise, terminate.

### 5. Numerical Experience: Example 3,

\[
e^2y'''(x) - (1 + x)y''(x) = 1.
\]

Initially, we did not know what values to choose for \( x_{fL} \) and \( x_{fR} \), so guided by Fig. 3 we made some exploratory guesses and examined the norms in the errors in the derivatives at \( x_{fL} \) and \( x_{fR} \). We also knew from Fig. 3 the maximum value of \( y_0(x) \approx 1.08 \) with the maximum occurring in the vicinity of \( x = 0.4 \). In fact the maximum occurs at \( x_{max} = e^{(\alpha_0 - 1)} - 1 \).

#### Table IV

| Code BEN1: Example 3: \( e^2y'''(x) - (1 + x)y''(x) = 1 \)—Error Norms for \( \varepsilon = 0.01 \) |
|---|---|---|---|
| Run No. | B41 | B42 | B43 |
| \( \varepsilon \) | 0.01 | 0.01 | 0.01 |
| \( x_{fL} \) | 0.01 | 0.001 | 0.0001 |
| \( x_{fR} \) | 0.99 | 0.999 | 0.9999 |
| \( \|y'_0(x_{fL}) - \frac{Y'_L(x_{fL})}{\varepsilon}\| \) | 31.83 | 5.33 | 1.10 |
| \( \|y'_0(x_{fR}) - \left\{ -\frac{Y'_R(x_{fR})}{\varepsilon} \right\}\| \) | 43.55 | 6.49 | 0.48 |
| \( a_0 \) | 1.4614649 | 1.3859715 | 1.3861192 |
| \( b_0 \) | 1.1875355 | 1.0030091 | 1.0000861 |
| \( x_{max} \) | 0.5864 | 0.4710 | 0.4712 |
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TABLE V

Code BEN1: Example 3: \( \varepsilon^2 y'''(x) - (1 + x) y''(x) = 1 \) — Error Norms for \( \varepsilon = 0.001 \)

<table>
<thead>
<tr>
<th>Run No.</th>
<th>B44</th>
<th>B45</th>
<th>B46</th>
<th>B47</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon )</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>( x_{fL} )</td>
<td>0.001</td>
<td>0.0001</td>
<td>0.00001</td>
<td>0.000001</td>
</tr>
<tr>
<td>( y'_{fR} )</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.99999</td>
<td>0.999999</td>
</tr>
<tr>
<td>( | y_0(x_{fL}) - y_1(x_{fL})/\varepsilon | )</td>
<td>316.26</td>
<td>48.14</td>
<td>5.58</td>
<td>1.11</td>
</tr>
<tr>
<td>( | y_0(x_{fR}) - { y'_1(X_fR) } - \varepsilon | )</td>
<td>442.59</td>
<td>76.01</td>
<td>6.92</td>
<td>0.48</td>
</tr>
<tr>
<td>( a_0 )</td>
<td>1.4733393</td>
<td>1.3876438</td>
<td>1.3862915</td>
<td>1.3862926</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>1.1842829</td>
<td>1.0024775</td>
<td>1.0000310</td>
<td>1.000009</td>
</tr>
<tr>
<td>( x_{\text{max}} )</td>
<td>0.6053</td>
<td>0.4735</td>
<td>0.4715</td>
<td>0.4715</td>
</tr>
</tbody>
</table>

To illustrate some of the results of our probing we list in Tables IV and V for \( \varepsilon = 0.01 \) and 0.001, respectively, the norms of the errors in the derivatives at \( x_{fL} \) and \( x_{fR} \). The last column in Tables IV and V gives the best results of our survey. The profiles associated with these runs are listed in Tables VI and VII. Tables IV and V also include the values for \( a_0 \) and \( b_0 \), the constants in the outer solution, and the location \( x_{\text{max}} \) of the maximum value of \( y_0(x) \). Our computed values for \( a_0 \) and \( b_0 \) converge to 1.3862 and 1.0 as the norms become smaller. As an independent check on \( a_0 \) and \( b_0 \) we note that Bender and Orszag [1, p. 451] determine from asymptotic considerations

\[
a_0 = 2 \ln 2 = 1.3862944
\]

\[
b_0 = 1.
\]

In Tables VI and VII for \( \varepsilon = 0.01 \) and 0.001, respectively, we list the profiles as a function of \( x \), the outer solution independent variable. The profiles include the left-hand boundary layer solution, the outer solution, and the right-hand boundary layer solution. For Table VI, the data are \( \varepsilon = 0.01 \), \( x_{fL} = 10^{-4} \), \( x_{fR} = 0.9999 \) and for Table VII the data are \( \varepsilon = 0.001 \), \( x_{fL} = 10^{-6} \), \( x_{fR} = 0.999999 \). The second and third columns in Tables VI and VII list the solution and the first derivative as calculated by the boundary value method. The fourth column is the numerical evaluation of the solution as developed by Bender and Orszag. The boundary value method and the asymptotic method of Bender and Orszag give comparable results. Note in Tables VI and VII for the boundary value method that the boundary condition specifications (4.2) and (4.3) are satisfied.

While the error norm or (the matching of the derivatives) are not as small as we would like, we possibly could reduce the norms by adjustment.
TABLE VI

Example 3: \( x^2 y''''(x) - (1 + x) y''(x) = 1 \),

Left- and Right-Hand Boundary Layer and Outer Solutions—\( \varepsilon = 0.01 \), \( x_{fL} = 0.0001 \), \( x_{fR} = 0.9999 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>Boundary value ( y(x)^a )</th>
<th>Boundary value ( y'(x)^a )</th>
<th>Bender and Orszag ( y(x)^b )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Left boundary layer</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td>1.0061371</td>
</tr>
<tr>
<td>1.0(10^{-5})</td>
<td>1.00000102</td>
<td>1.049475</td>
<td>1.0061409</td>
</tr>
<tr>
<td>2.0(10^{-5})</td>
<td>1.0000210</td>
<td>1.0989011</td>
<td>1.0061448</td>
</tr>
<tr>
<td>4.0(10^{-5})</td>
<td>1.0000440</td>
<td>1.1976045</td>
<td>1.0061525</td>
</tr>
<tr>
<td>6.0(10^{-5})</td>
<td>1.0000689</td>
<td>1.2961108</td>
<td>1.0061602</td>
</tr>
<tr>
<td>8.0(10^{-5})</td>
<td>1.0000958</td>
<td>1.3944202</td>
<td>1.0061680</td>
</tr>
<tr>
<td>1.0(10^{-4})</td>
<td>1.0001247</td>
<td>1.4925332</td>
<td>1.0061757</td>
</tr>
<tr>
<td><strong>Outer solution</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0(10^{-4})</td>
<td>1.0001247</td>
<td>3.8601924(10^1)</td>
<td>1.0061714</td>
</tr>
<tr>
<td>2.0(10^{-2})</td>
<td>1.0076098</td>
<td>3.6631661(10^{-1})</td>
<td>1.0128022</td>
</tr>
<tr>
<td>1.0(10^{-1})</td>
<td>1.0338568</td>
<td>2.9080906(10^{-1})</td>
<td>1.0356149</td>
</tr>
<tr>
<td>2.0(10^{-1})</td>
<td>1.0585240</td>
<td>2.0379768(10^{-1})</td>
<td>1.0559893</td>
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<tr>
<td>3.0(10^{-1})</td>
<td>1.0748483</td>
<td>1.2375947(10^{-1})</td>
<td>1.0680207</td>
</tr>
<tr>
<td>4.0(10^{-1})</td>
<td>1.0837476</td>
<td>4.9647002(10^2)</td>
<td>1.0723522</td>
</tr>
<tr>
<td>5.0(10^{-1})</td>
<td>1.0849480</td>
<td>-1.9345869(10^{-2})</td>
<td>1.0695347</td>
</tr>
<tr>
<td>6.0(10^{-1})</td>
<td>1.0797518</td>
<td>-8.3884390(10^{-2})</td>
<td>1.0600456</td>
</tr>
<tr>
<td>7.0(10^{-1})</td>
<td>1.0683015</td>
<td>-1.4450901(10^{-1})</td>
<td>1.0443024</td>
</tr>
<tr>
<td>8.0(10^{-1})</td>
<td>1.059655</td>
<td>-2.0166743(10^{-1})</td>
<td>1.0226735</td>
</tr>
<tr>
<td>9.0(10^{-1})</td>
<td>1.0280710</td>
<td>-2.5573465(10^{-1})</td>
<td>0.9954861</td>
</tr>
<tr>
<td>9.8(10^{-1})</td>
<td>1.0059512</td>
<td>-2.9697761(10^{-1})</td>
<td>0.9699320</td>
</tr>
<tr>
<td>9.999(10^{-1})</td>
<td>0.9999416</td>
<td>-3.0697794(10^{-1})</td>
<td>0.9630682</td>
</tr>
<tr>
<td><strong>Right boundary layer</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.99990</td>
<td>0.9999416</td>
<td>1.6917725(10^{-1})</td>
<td>0.9907898</td>
</tr>
<tr>
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<td>0.9999466</td>
<td>3.3440317(10^{-1})</td>
<td>0.9907837</td>
</tr>
<tr>
<td>0.99994</td>
<td>0.9999550</td>
<td>5.0009707(10^{-1})</td>
<td>0.9907775</td>
</tr>
<tr>
<td>0.99996</td>
<td>0.9999666</td>
<td>6.6626029(10^{-1})</td>
<td>0.9907714</td>
</tr>
<tr>
<td>0.99998</td>
<td>0.9999816</td>
<td>8.3289941(10^{-1})</td>
<td>0.9907652</td>
</tr>
<tr>
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<td>1.0000000</td>
<td>1.0000000</td>
<td>0.9907591</td>
</tr>
</tbody>
</table>

\( a \) Roberts (Run B43, Code BEN1)

\( b \) Bender and Orszag (Run BBB7, Code BEN3)

in \( x_{fL} \) and \( x_{fR} \). However, the solutions provided by the boundary value method are sufficiently close to asymptotically derived solution of Bender and Orszag that it is doubtful whether the solution would be changed much by any perturbation in \( x_{fL} \) or \( x_{fR} \).

As a matter of interest we list here the Bender and Orszag solution. The left-hand boundary layer is given by

\[
Y_1^B(X) = 1 + x + (2 \ln 2 - 2)(x - \varepsilon),
\]

(5.1)
TABLE VII

Example 3: \(e^{y''(x) - (1 + x)} y''(x) = 1\),
Left- and Right-Hand Boundary Layer and Outer Solutions—\(\varepsilon = 0.001\), \(x_{FL} = 0.000001\), \(x_{FR} = 0.999999\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>Boundary value method</th>
<th>Boundary value method</th>
<th>Bender and Orszag</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(y(x)^a)</td>
<td>(y'(x)^a)</td>
<td>(y(x)^b)</td>
</tr>
<tr>
<td>Left boundary layer</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.00000000</td>
<td>1.00000000</td>
<td>1.0006137</td>
</tr>
<tr>
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<td>1.00000001</td>
<td>1.0498475</td>
<td>1.0006137</td>
</tr>
<tr>
<td>2.0(10^{-7})</td>
<td>1.00000002</td>
<td>1.0998900</td>
<td>1.0006138</td>
</tr>
<tr>
<td>4.0(10^{-7})</td>
<td>1.00000004</td>
<td>1.1997600</td>
<td>1.0006139</td>
</tr>
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<td>6.0(10^{-7})</td>
<td>1.00000007</td>
<td>1.2996101</td>
<td>1.0006139</td>
</tr>
<tr>
<td>8.0(10^{-7})</td>
<td>1.00000010</td>
<td>1.3994402</td>
<td>1.0006140</td>
</tr>
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<td>1.00000012</td>
<td>1.4992503</td>
<td>1.0006141</td>
</tr>
<tr>
<td>Outer solution</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>1.00000012</td>
<td>3.8629161(10^{-1})</td>
<td>1.0006141</td>
</tr>
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<td>1.0080547</td>
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<td>1.0339709</td>
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<td>2.0397105(10^{-1})</td>
<td>1.0582246</td>
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<tr>
<td>3.0(10^{-1})</td>
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<td>1.2392834(10^{-1})</td>
<td>1.0741354</td>
</tr>
<tr>
<td>4.0(10^{-1})</td>
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<td>4.9820369(10^{-2})</td>
<td>1.0823462</td>
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<td>1.0834080</td>
</tr>
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</tr>
<tr>
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<td>1.0248769</td>
</tr>
<tr>
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<td>0.9999994</td>
<td>3.0685408(10^{-1})</td>
<td>0.9963036</td>
</tr>
<tr>
<td>Right boundary layer</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.9999990</td>
<td>0.9999994</td>
<td>1.7130354(10^{-1})</td>
<td>0.9990762</td>
</tr>
<tr>
<td>0.9999992</td>
<td>0.9999995</td>
<td>3.3697070(10^{-1})</td>
<td>0.9990761</td>
</tr>
<tr>
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<td>0.9999996</td>
<td>5.0265770(10^{-1})</td>
<td>0.9990761</td>
</tr>
<tr>
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<td>0.9999997</td>
<td>6.6839158(10^{-1})</td>
<td>0.9990760</td>
</tr>
<tr>
<td>0.9999998</td>
<td>0.9999998</td>
<td>8.3417234(10^{-1})</td>
<td>0.9990759</td>
</tr>
<tr>
<td>1.0000000</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td>0.9990759</td>
</tr>
</tbody>
</table>

\(^a\) Roberts (Run B47, Code BEN1)
\(^b\) Bender and Orszag (Run BBBB, Code BEN3)

where the superscript \(B\) indicates the Bender and Orszag solution. The outer solution is given by

\[
y^B_0(x) = - (1 + x) \ln(1 + x) + (2 \ln 2) x + 1 \\
+ \varepsilon \left( \frac{\ln 2}{\sqrt{2} - (\sqrt{2} + 2 \ln 2)} \right) x + (2 - 2 \ln 2) \right)
\]

(5.2)
and the right-hand boundary layer solution is

\[ Y^B_R(x) = x + \left( \frac{2 - \ln 2}{\sqrt{2}} \right) \left[ \sqrt{2(1 - x)} - \varepsilon \right]. \]  

(5.3)

Bender and Orszag point out that as \( \varepsilon \to 0 \), \( y^B_0(x) \) satisfies the boundary conditions

\[ y^B_0(0) = 1 \]  

(5.4)

\[ y^B_0(1) = 1. \]  

(5.5)

However, the derivative \( y^B_0'(x) \) does not satisfy the boundary conditions. In fact,

\[ y^B_0'(0) = 2 \ln 2 - 1 \neq 1 \]  

(5.6)

\[ y^B_0'(1) = \ln 2 - 1 \neq 1 \]  

(5.7)

Bender and Orszag claim that boundary layers appear, therefore, to adjust the slope so the derivative boundary conditions are satisfied.

REFERENCES