On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings

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Abstract


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1. Introduction and preliminaries

Throughout this paper we assume that $E$ is a real Banach space, $F(T)$ and $D(T)$ are the set of fixed points and the domain of $T$, respectively.

Recall that $E$ is said to satisfy Opial condition, if for each sequence $\{x_n\}$ in $E$, the condition that the sequence $x_n \to x$ weakly implies that
\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|
\]
for all $y \in E$ with $y \neq x$.

Definition 1. Let $D$ be a closed subset of $E$ and $T : D \to D$ be a mapping.

1. $T$ is said to be demi-closed at the origin, if for each sequence $\{x_n\}$ in $D$, the condition $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly implies $Tx_0 = 0$.
2. $T$ is said to be semi-compact, if for any bounded sequence $x_n$ in $D$ such that $\|x_n - Tx_n\|\to 0$ ($n \to \infty$), then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in D$.
3. $T$ is said to be asymptotically nonexpansive [3], if there exists a sequence $\{h_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} h_n = 1$ such that
\[
\|T^nx - T^ny\| \leq h_n \|x - y\|, \quad \forall n \geq 1, x, y \in D.
\]

Proposition 1.

(1) Let $K$ be a nonempty subset of $E$, $\{T_i\}_{i=1}^N : K \to K$ be $N$ asymptotically nonexpansive mappings. Then there exists a sequence $\{h_n\} \subset [1, \infty)$ with $h_n \to 1$ such that
\[
\|T^nx - T^ny\| \leq h_n \|x - y\|, \quad \forall n \geq 1, x, y \in K, i = 1, 2, \ldots, N. \quad (1.1)
\]

(2) $\{T_1, T_2, \ldots, T_N\}$ is uniformly Lipschitzian with a Lipschitzian constant $L \geq 1$, i.e., there exists a constant $L \geq 1$ such that
\[
\|T^nx - T^ny\| \leq L \|x - y\|, \quad \forall n \geq 1, x, y \in K, i = 1, 2, \ldots, N. \quad (1.2)
\]

Proof. (1) Since for each $i = 1, 2, \ldots, N$, $T_i : K \to K$ is an asymptotically nonexpansive mapping, there exists a sequence $\{h_n^{(i)}\} \subset [1, \infty)$, with $h_n^{(i)} \to 1$ ($n \to \infty$) such that
\[
\|T^nx - T^ny\| \leq h_n^{(i)} \|x - y\|, \quad \forall n \geq 1. \quad (1.3)
\]
Letting
\[ h_n = \max \{ h_n^{(1)}, h_n^{(2)}, \ldots, k_n^{(N)} \}, \]  
then we have that \( \{ h_n \} \subset [1, \infty) \) with \( h_n \to 1 \) \((n \to \infty)\) and
\[ \| T_i^n x - T_i^n y \| \leq h_n^{(i)} \| x - y \| \leq h_n \| x - y \|, \quad \forall n \geq 1, \]
for all \( x, y \in K \) and for each \( i = 1, 2, \ldots, N \).

(2) Taking \( L = \sup_{n \geq 1} h_n \), then the conclusion (2) can be obtained from the conclusion (1) immediately.

**Definition 2.** Let \( K \) be a nonempty closed convex subset of \( E \) satisfying \( K + K \subset K \), \( x_0 \in K \) be any given point and \( \{ T_1, T_2, \ldots, T_N \} : K \to K \) be \( N \) asymptotically nonexpansive mappings. Let \( \{ \alpha_n \} \) be a sequence in \([0, 1]\) and \( \{ u_n \} \) be a bounded sequence in \( K \).

Then the sequence \( \{ x_n \} \subset K \) defined by
\[
\begin{align*}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1 + u_1, \\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2 + u_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N + u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1} + u_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N} + u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1}) T_1^3 x_{2N+1} + u_{2N+1}, \\
&\vdots 
\end{align*}
\]
is called the implicit iterative sequence with errors for a finite family of asymptotically nonexpansive mappings \( \{ T_1, T_2, \ldots, T_N \} \).

Since for each \( n \geq 1 \), it can be written as \( n = (k - 1)N + i \), where \( i = i(n) \in \{ 1, 2, \ldots, N \} \), \( k = k(n) \geq 1 \) is a positive integer and \( k(n) \to \infty \), as \( n \to \infty \). Hence we can write the above table in the following compact form:
\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^k x_n + u_n, \quad \forall n \geq 1. \tag{1.5}
\]

Especially, if \( T_1, T_2, \ldots, T_N : K \to K \) are \( N \) asymptotically nonexpansive mappings, \( \{ \alpha_n \} \) is a sequence in \([0, 1]\) and \( x_0 \) is a given point in \( K \), then the sequence \( \{ x_n \} \) defined by
\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n, \quad \forall n \geq 1, \tag{1.6}
\]
is called the implicit iterative sequence for a finite family of asymptotically nonexpansive mappings \( \{ T_1, T_2, \ldots, T_N \} \).

Recently concerning the convergence problems of an implicit (or non-implicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces
have been considered by several authors (see, for example, Bauschke [1], Goebel and Kirk [3], Gornicki [4], Halpern [5], Lions [6], Reich [7], Schu [8], Sun [9], Tan and Xu [10], Wittmann [12], Xu and Ori [13], Zhou and Chang [14]).

The purpose of this paper is to study the weak and strong convergence of implicit iteration sequences \( \{x_n\} \) defined by (1.5) and (1.6) to a common fixed point for a finite family of asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces.

The following theorems are the main results of this paper.

**Theorem 1.** Let \( E \) be a real uniformly convex Banach space satisfying Opial condition, \( K \) be a nonempty closed convex subset of \( E \) with \( K + K \subset K \), \( \{T_1, T_2, \ldots, T_N\} : K \to K \) be \( N \) asymptotically nonexpansive mappings with \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \) (the set of common fixed points of \( \{T_1, T_2, \ldots, T_N\} \)). Let \( \{u_n\} \) be a bounded sequence in \( K \), \( \{\alpha_n\} \) be a sequences in \([0,1]\) and \( \{h_n\} \) be the sequence defined by (1.1) and \( L = \sup_{n \geq 1} h_n \geq 1 \) satisfying the following conditions:

(i) \( \sum_{n=1}^{\infty} u_n < \infty \);
(ii) \( \sum_{n=1}^{\infty} (h_n - 1) < \infty \);
(iii) there exist constants \( \tau_1, \tau_2 \in (0, 1) \) such that
\[
\tau_1 \leq (1 - \alpha_n) \leq \tau_2, \quad \forall n \geq 1.
\]

Then the implicit iterative sequence \( \{x_n\} \) defined by (1.5) converges weakly to a common fixed point of \( \{T_1, T_2, \ldots, T_N\} \) in \( K \).

**Theorem 2.** Let \( E \) be a real uniformly convex Banach space satisfying Opial condition, \( K \) be a nonempty closed convex subset of \( E \), \( \{T_1, T_2, \ldots, T_N\} : K \to K \) be \( N \) asymptotically nonexpansive mappings with \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \), and \( \{\alpha_n\} \) be a sequences in \([0,1]\) and \( \{h_n\} \) be the sequence defined by (1.1) and \( L = \sup_{n \geq 1} h_n \geq 1 \) satisfying the following conditions:

(i) \( \sum_{n=1}^{\infty} (h_n - 1) < \infty \);
(ii) there exist constants \( \tau_1, \tau_2 \in (0, 1) \) such that
\[
\tau_1 < 1 - \alpha_n < \tau_2.
\]

Then the implicit iterative sequence \( \{x_n\} \) defined by (1.6) converges weakly to a common fixed point of \( \{T_1, T_2, \ldots, T_N\} \) in \( K \).

**Theorem 3.** Let \( E \) be a real uniformly convex Banach space, \( K \) be a nonempty closed convex subset of \( E \) with \( K + K \subset K \), \( \{T_1, T_2, \ldots, T_N\} : K \to K \) be \( N \) asymptotically nonexpansive mappings with \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \) and at least there exists an \( T_l, 1 \leq l \leq N \), which is semi-compact (without loss of generality, we can assume that \( T_1 \) is semi-compact). Let \( \{u_n\} \) be a bounded sequence in \( K \), \( \{\alpha_n\} \) be a sequences in \([0,1]\) and \( \{h_n\} \) be the sequence defined by (1.1) and \( L = \sup_{n \geq 1} h_n \geq 1 \) satisfying the following conditions:

(i) \( \sum_{n=1}^{\infty} u_n < \infty \);
(ii) \(\sum_{n=1}^{\infty} (h_n - 1) < \infty\);
(iii) there exist constants \(\tau_1, \tau_2 \in (0, 1)\) such that
\[
\tau_1 \leq (1 - \alpha_n) \leq \tau_2, \quad \forall n \geq 1.
\]

Then the implicit iterative sequence \(\{x_n\}\) defined by (1.5) converges strongly to a common fixed point of \(\{T_1, T_2, \ldots, T_N\}\) in \(K\).

**Theorem 4.** Let \(E\) be a real uniformly convex Banach space, \(K\) be a nonempty closed convex subset of \(E\), \(\{T_1, T_2, \ldots, T_N\} : K \to K\) be \(N\) asymptotically nonexpansive mappings with \(F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset\) and at least there exists an \(T_l, 1 \leq l \leq N\) is semi-compact. Let \(\{\alpha_n\}\) be a sequences in \([0, 1]\), \(\{h_n\}\) be the sequence defined by (1.1) and \(L = \sup_{n \geq 1} h_n \geq 1\) satisfying the following conditions:

(i) \(\sum_{n=1}^{\infty} (h_n - 1) < \infty\);
(ii) there exist constants \(\tau_1, \tau_2 \in (0, 1)\) such that
\[
\tau_1 \leq 1 - \alpha_n < \tau_2.
\]

Then the implicit iterative sequence \(\{x_n\}\) defined by (1.6) converges strongly to a common fixed point of \(\{T_1, T_2, \ldots, T_N\}\) in \(K\).

**Theorem 5.** Let \(E\) be a real uniformly convex Banach space, \(K\) be a nonempty closed convex subset of \(E\) with \(K + K \subset K\), \(\{T_1, T_2, \ldots, T_N\} : K \to K\) be \(N\) nonexpansive mappings with \(F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset\). Let \(\{u_n\}\) be a bounded sequence in \(K\), \(\{\alpha_n\}\) be a sequences in \([0, 1]\) satisfying the following conditions:

(i) \(\sum_{n=1}^{\infty} \|u_n\| < \infty\);
(ii) there exist constants \(\tau_1, \tau_2 \in (0, 1)\) such that
\[
\tau_1 \leq (1 - \alpha_n) \leq \tau_2, \quad \forall n \geq 1.
\]

(1) if there exists at least an \(T_l, 1 \leq l \leq N\), which is semi-compact, then the implicit iterative sequence \(\{x_n\}\) defined by (1.5) converges strongly to a common fixed point of \(\{T_1, T_2, \ldots, T_N\}\) in \(K\);
(2) if \(E\) is semi-closed, then the implicit iterative sequence \(\{x_n\}\) defined by (1.5) converges weakly to a common fixed point of \(\{T_1, T_2, \ldots, T_N\}\) in \(K\).

In order to prove the main results of this paper, we need the following lemmas.

**Lemma 1** [2,4]. Let \(E\) be a uniformly convex Banach space, \(K\) be a nonempty closed convex subset of \(E\) and \(T : K \to K\) be an asymptotically nonexpansive mapping. Then \(I - T\) is semi-closed at zero, i.e., for each sequence \(\{x_n\}\) in \(K\), if \(\{x_n\}\) converges weakly to \(q \in K\) and \((I - T)x_n\) converges strongly to 0, then \((I - T)q = 0\).
Lemma 2 [11]. Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be three nonnegative sequences satisfying the following condition:

\[
a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0,
\]

where \( n_0 \) is some nonnegative integer, \( \sum_{n=0}^{\infty} c_n < \infty \) and \( \sum_{n=0}^{\infty} b_n < \infty \).

Then

(1) the limit \( \lim_{n \to \infty} a_n \) exists;
(2) if, in addition, there exists a subsequence \( \{a_{n_i}\} \subset \{a_n\} \) such that \( a_{n_i} \to 0 \) (as \( n \to \infty \)).

Lemma 3 [8]. Let \( E \) be a uniformly convex Banach space, \( b, c \) be two constants with \( 0 < b < c < 1 \). Suppose that \( \{t_n\} \) is a sequence in \( [b,c] \) and \( \{x_n\}, \{y_n\} \) are two sequences in \( E \). Then the conditions:

\[
\begin{align*}
&\lim_{n \to \infty} \|t_nx_n + (1 - t_n)y_n\| = d, \\
&\limsup_{n \to \infty} \|x_n\| \leq d, \\
&\limsup_{n \to \infty} \|y_n\| \leq d
\end{align*}
\]

imply that \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \), where \( d \geq 0 \) is some constant.

2. Proof of Theorems

We are now in a position to prove our main results in this paper.

Proof of Theorem 1. Since \( F = \bigcap_{n=1}^{N} F(T_i) \neq \emptyset \), for any given \( p \in F \), it follows from (1.5) and Proposition 1 that

\[
\|x_n - p\| = \|\alpha_n x_{n-1} + (1 - \alpha_n)T_{i(n)}^{k(n)} x_n + u_n - p\|
\]

\[
\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)\|T_{i(n)}^{k(n)} x_n - p\| + \|u_n\|
\]

\[
\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)h_{k(n)} \|x_n - p\| + \|u_n\|.
\]

Letting \( \mu_n = h_{k(n)} - 1, \forall n \geq 1 \), by condition (i) we have

\[
\sum_{i=1}^{\infty} \mu_n < \infty. \tag{2.1}
\]

Therefore we have

\[
\|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)(1 + \mu_n) \|x_n - p\| + \|u_n\|
\]

\[
\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n + \mu_n) \|x_n - p\| + \|u_n\|.
\]

Simplifying we have

\[
\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{\mu_n}{\alpha_n} \|x_n - p\| + \frac{\|u_n\|}{\alpha_n}. \tag{2.2}
\]
By condition (iii), $1 - \tau_2 \leq \alpha_n$, hence from (2.2) we have
\[
\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{\mu_n}{1 - \tau_2} \|x_n - p\| + \frac{\|u_n\|}{1 - \tau_2}.
\]
Simplifying we have
\[
\|x_n - p\| \leq \frac{1 - \tau_2}{1 - \tau_2 - \mu_n} \|x_{n-1} - p\| + \frac{\|u_n\|}{(1 - \tau_2 - \mu_n)(1 - \tau_2)}.
\]
(2.3)

By virtue of (2.1), $\mu_n \to 0$, therefore there exists a positive integer $n_0$ such that $\mu_n \leq \frac{1 - \tau_2}{2}$, $\forall n \geq n_0$. It follows from (2.3) that
\[
\|x_n - p\| \leq \left(1 + \frac{2\mu_n}{1 - \tau_2}ight) \|x_{n-1} - p\| + \frac{2\|u_n\|}{(1 - \tau_2)(1 - \tau_2)}, \quad \forall n \geq n_0.
\]
(2.4)

Taking $a_{n+1} = \|x_n - p\|, b_n = \frac{2\mu_n}{1 - \tau_2}, c_n = \frac{2\|u_n\|}{(1 - \tau_2)(1 - \tau_2)}$ in Lemma 2 and by using conditions (i) and (2.1), it is easy to see that
\[
\sum_{n=1}^{\infty} b_n < \infty; \quad \sum_{n=1}^{\infty} c_n < \infty.
\]

It follows from Lemma 2 that the $\lim_{n \to \infty} \|x_n - p\|$ exists. Without loss of generality we can assume that
\[
\lim_{n \to \infty} \|x_n - p\| = d,
\]
(2.5)

where $d \geq 0$ is some number. Since \{\|x_n - p\|\} is a convergent sequence and so \{\{x_n\}\} is a bounded sequence in $K$. Again since
\[
\|x_n - p\| = \|\alpha_n[x_{n-1} - p + u_n] + (1 - \alpha_n)[T_{i(n)}^{k(n)} x_n - p + u_n]\|.
\]

By condition (i) and (2.5) we have that
\[
\limsup_{n \to \infty} \|x_{n-1} - p + u_n\| \leq \limsup_{n \to \infty} \|x_{n-1} - p\| + \limsup_{n \to \infty} \|u_n\| \leq d;
\]
(2.6)

and that
\[
\limsup_{n \to \infty} \|T_{i(n)}^{k(n)} x_n - p + u_n\| \leq \limsup_{n \to \infty} h_{k(n)} \|x_n - p\| + \limsup_{n \to \infty} \|u_n\| \leq d.
\]
(2.7)

Therefore from (2.5)–(2.7) and Lemma 3 we have that
\[
\lim_{n \to \infty} \|T_{i(n)}^{k(n)} x_n - x_{n-1}\| = 0.
\]
(2.8)

Moreover, since
\[
\|x_n - x_{n-1}\| = \|(1 - \alpha_n) T_{i(n)}^{k(n)} x_n - (1 - \alpha_n)x_{n-1} + u_n\|
\leq (1 - \alpha_n) \|T_{i(n)}^{k(n)} x_n - x_{n-1}\| + \|u_n\|,
\]

\[
\|x_n - x_{n-1}\| \leq \frac{\|T_{i(n)}^{k(n)} x_n - x_{n-1}\|}{1 - \alpha_n} + \|u_n\|.
\]

(2.9)
it follows from (2.8) and condition (i) that
\[ \lim_{n \to \infty} \| x_n - x_{n-1} \| = 0. \] (2.9)

From (2.8) and (2.9) we have
\[ \lim_{n \to \infty} \| x_n - T_{i(n)}^{k(n)} x_n \| \leq \lim_{n \to \infty} \left\{ \| x_n - x_{n-1} \| + \| x_{n-1} - T_{i(n)}^{k(n)} x_n \| \right\} = 0, \] (2.10)
and
\[ \lim_{n \to \infty} \| x_n - x_{n+j} \| = 0, \quad \forall j = 1, 2, \ldots, N. \] (2.11)

Since for any positive integer \( n > N \), it can be written as \( n = (k(n) - 1)N + i(n) \), \( i(n) \in \{1, 2, \ldots, N\} \). Letting \( \sigma_n = \| T_{i(n)}^{k(n)} x_n - x_{n-1} \| \), then from (2.8), we have \( \sigma_n \to 0 \) and
\[ \| x_{n-1} - T_n x_n \| \leq \| x_{n-1} - T_{i(n)}^{k(n)} x_n \| + \| T_{i(n)}^{k(n)} x_n - T_n x_n \| = \sigma_n + L \| T_{i(n)}^{k(n)-1} x_n - x_n \| \]
\[ \leq \sigma_n + L \left\{ \| T_{i(n)}^{k(n)-1} x_n - T_{n-N}^{k(n)-1} x_{n-N} \| + \| T_{n-N}^{k(n)-1} x_{n-N} - x_{(n-N)} \| + \| x_{(n-N)} - x_n \| \right\}. \] (2.12)

Since for each \( n > N \), \( n = (n-N)(mod N) \), again since \( n = (k(n) - 1)N + i(n) \), hence \( n - N = (k(n) - 1 - 1)N + i(n) = (k(n) - N) - 1)N + i(n-N) \), i.e.,
\[ k(n-N) = k(n) - 1 \quad \text{and} \quad i(n-N) = i(n). \]

Therefore we have
\[ \| T_{i(n)}^{k(n)-1} x_n - T_{n-N}^{k(n)-1} x_{n-N} \| = \| T_{i(n)}^{k(n)-1} x_n - T_{i(n)}^{k(n)-1} x_{n-N} \| \leq L \| x_n - x_{n-N} \| \] (2.13)
and
\[ \| T_{n-N}^{k(n)-1} x_{n-N} - x_{(n-N)-1} \| = \| T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{(n-N)-1} \| = \sigma_{n-N}. \] (2.14)

Substituting (2.13) and (2.14) into (2.12) and simplifying we have
\[ \| x_{n-1} - T_n x_n \| \leq \sigma_n + L^2 \| x_n - x_{n-N} \| + L \sigma_{n-N} + L \| x_{(n-N)} - x_n \|. \]

By (2.8) and (2.11) we know that
\[ \lim_{n \to \infty} \| x_{n-1} - T_n x_n \| = 0. \] (2.15)

It follows from (2.9) and (2.15) that
\[ \lim_{n \to \infty} \| x_n - T_n x_n \| \leq \lim_{n \to \infty} \| x_n - x_{n-1} \| + \| x_{n-1} - T_n x_n \| = 0. \] (2.16)

Consequently, for any \( j = 1, 2, \ldots, N \) from (2.11) and (2.16) we have
\[ \| x_n - T_{n+j} x_n \| \leq \| x_n - x_{n+j} \| + \| x_{n+j} - T_{n+j} x_{n+j} \| + \| T_{n+j} x_{n+j} - T_{n+j} x_n \| \]
\[ \leq (1 + L) \| x_n - x_{n+j} \| + \| x_{n+j} - T_{n+j} x_{n+j} \| \to 0, \]
average
\[ \text{as } n \to \infty. \] (2.17)
This implies that the sequence
\[ \bigcup_{j=1}^{N} \{ \|x_n - T_{n+j}x_n\| \}_{n=1}^{\infty} \to 0 \quad \text{as } n \to \infty. \]

Since for each \( l = 1, 2, \ldots, N \), \( \{ \|x_n - T lx_n\| \}_{n=1}^{\infty} \) is a subsequence of \( \bigcup_{j=1}^{N} \{ \|x_n - T_{n+j}x_n\| \}_{n=1}^{\infty} \), therefore we have
\[ \lim_{n \to \infty} \|x_n - T lx_n\| = 0, \quad \forall l = 1, 2, \ldots, N. \quad (2.18) \]

Since \( E \) is uniformly convex, every bounded subset of \( E \) is weakly compact. Since \( \{x_n\} \) is a bounded sequence in \( K \), there exists a subsequence \( \{x_{n_k}\} \subset \{x_n\} \) such that \( \{x_{n_k}\} \) converges weakly to \( q \in K \). Hence from (2.18) we have
\[ \lim_{n_k \to \infty} \|x_{n_k} - T lx_{n_k}\| = 0, \quad \forall l = 1, 2, \ldots, N. \quad (2.19) \]

By Lemma 1, we have that \((I - T l)q = 0\), i.e., \( q \in F(T l) \). By the arbitrariness of \( l \in \{1, 2, \ldots, N\} \), we know that \( q \in F = \bigcap_{l=1}^{N} F(T l) \).

Next we prove that \( \{x_n\} \) converges weakly to \( q \). Suppose the contrary, then there exists some subsequence \( \{x_{n_j}\} \subset \{x_n\} \) such that \( \{x_{n_j}\} \) converges weakly to \( q_1 \in K \) and \( q \neq q_1 \).

Then by the same method as given above, we can also prove that \( q_1 \in F = \bigcap_{l=1}^{N} F(T l) \).

Taking \( p = q \) and \( p = q_1 \) and by using the same method as given in the proof of (2.5) we can prove that the following two limits exist and
\[ \lim_{n \to \infty} \|x_n - q\| = d_1, \quad \lim_{n \to \infty} \|x_n - q_1\| = d_2, \]
where \( d_1, d_2 \) are two nonnegative numbers. By virtue of the Opial condition of \( E \), we have
\[ d_1 = \lim_{n_k \to \infty} \sup_{n_k \to \infty} \|x_{n_k} - q\| < \lim_{n_k \to \infty} \sup_{n_k \to \infty} \|x_{n_k} - q_1\| = \lim_{n_j \to \infty} \sup_{n_j \to \infty} \|x_{n_j} - q_1\| \]
\[ < \lim_{n_j \to \infty} \sup_{n_j \to \infty} \|x_{n_j} - q\| = d_1. \]

This is a contradiction. Hence \( q = q_1 \). This implies that \( \{x_n\} \) converges weakly to \( q \).

The proof of Theorem 1 is completed. \( \square \)

**Proof of Theorem 2.** Taking \( \gamma_n = 0, \forall n \geq 1 \), and \( \beta_n = 1 - \alpha_n, \forall n \geq 1 \), in Theorem 1, then the conclusion of Theorem 2 can be obtained from Theorem 1 immediately. \( \square \)

**Proof of Theorem 3.** For any given \( p \in F = \bigcap_{i=1}^{N} F(T i) \), by the same method as given in proving (2.5) and (2.18), we can prove that
\[ \lim_{n \to \infty} \|x_n - p\| = d, \quad (2.20) \]
where \( d \geq 0 \) is some nonnegative number, and
\[ \lim_{n \to \infty} \|x_n - T lx_n\| = 0, \quad \forall l = 1, 2, \ldots, N. \quad (2.21) \]

Especially, we have
\[ \lim_{n \to \infty} \|x_n - T_1x_n\| = 0. \quad (2.22) \]
By the assumption of Theorem 3, $T_1$ is semi-compact, therefore it follows from (2.22) that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x_* \in K$. Hence from (2.21) we have that

$$\|x_* - T_1x_*\| = \lim_{n_i \to \infty} \|x_{n_i} - T_1x_{n_i}\| = 0, \quad \forall l = 1, 2, \ldots, N.$$  

This implies that

$$x_* \in F = \bigcap_{i=1}^{N} F(T_i).$$  

By the arbitrariness of $p \in F = \bigcap_{i=1}^{N} F(T_i)$, in (2.20) taking $p = x_*$, similarly we can also prove that

$$\lim_{n \to \infty} \|x_n - x_*\| = d_1,$$

where $d_1 \geq 0$ is some nonnegative number. From $x_{n_i} \to x_*$ we know that $d_1 = 0$, i.e., $x_n \to x_*$.  

This completes the proof of Theorem 3.  

**Proof of Theorem 4.** Taking $\gamma_n = 0$ and $\beta_n = 1 - \alpha_n$, $\forall n \geq 1$ in Theorem 3, the conclusion of Theorem 4 can be obtained from Theorem 3 immediately.  

**Proof of Theorem 5.** Since each nonexpansive mapping from $K$ into $K$ is an asymptotically nonexpansive mapping from $K \to K$ with $h_n = 1$, $\forall n \geq 1$ and $L = 1$. Therefore all conditions in Theorems 1 and 3 are satisfied. The conclusions of Theorem 5 can be obtained from Theorems 1 and 3 immediately.  

This completes the proof of Theorem 5.  

**Remark.** (1) Theorems 2 and 4 give an affirmative answer to the following open question raised by Xu and Ori [13]: “It is unclear what assumptions on the mappings \{T_1, T_2, \ldots, T_N\} and/or the parameters \{\alpha_n\} are sufficient to guarantee the strong convergence of the sequence \{x_n\} defined by (1.6).”  

(2) Theorems 1 and 3 improve and generalize Theorem 3.3 of Sun [9] to the case of implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings and the boundedness condition of the set $K$ in this theorem is deleted.  

(3) Theorems 1 and 3 also generalize and improve the corresponding results in Zhou and Chang [14] and the key condition (v) in [14, Theorem 1]: there exists a constant $L > 0$ such that for any $i, j \in \{T_1, T_2, \ldots, N\}$, $i \neq j$,

$$\|T_i^n x - T_j^n y\| \leq L\|x - y\|, \quad \forall n \geq 1, \forall x, y \in K,$$

is deleted.  

(4) Theorems 1–5 also generalize and improve the corresponding results of [1,3–8, 10,12].
References