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On sequence space representations of Hörmander–Beurling spaces

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ABSTRACT

It is shown that $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega)$ is isomorphic to $(B^c_{p,k}(\Omega))'_b$ (Ω open set in $\mathbb{R}^n$, $1 \leqslant p < \infty$, k Beurling–Björck weight) extending a Hörmander's result (the proof we give is valid in the vector-valued case, too). As a consequence, and using Vogt's representation theorems and weighted L_p-spaces of entire analytic functions, a number of results on sequence space representations of Hörmander–Beurling are given.

1. Introduction and notation

In [13, Chapter XV] Hörmander studies the behaviour of the Fourier–Laplace transform in the space $B^c_{2,k}(\Omega) = \operatorname{ind}_{\substack{\rightarrow \\ K \Subset \Omega}} [B_{2,k} \cap \mathcal{E}'(K)]$ when Ω is an open convex set in $\mathbb{R}^n$ and k is a temperate weight function on $\mathbb{R}^n$, and then proves a theorem on the representation of solutions of the equation $P(D)u = 0$ by integrals of exponential solutions ($P(D)$ is a constant coefficient partial differential operator). For this he obtains an appropriate collection of seminorms defining the inductive limit topology of $B^c_{2,k}(\Omega)$, proves the isomorphism $(B^c_{2,k}(\Omega))'_b \simeq B^{\mathrm{loc}}_{2,1/\tilde{k}}(\Omega)$ and shows that every continuous seminorm in $B^c_{2,k}(\Omega)$ is bounded by a seminorm of the form $u \to (\int |\hat{u}(\zeta)|^2 e^{-2\phi(\zeta)}\, d\lambda(\zeta))^{1/2}$ where $\hat{u}$ is the Fourier–Laplace transform of u and ϕ is plurisubharmonic (see [13, Section 15.2]). In this paper we extend the former isomorphism to Beurling–Björck weights [1] and as a consequence (and using Vogt's representation theorems [33] and weighted L_p-spaces of entire analytic functions [25,30]) a number of results on sequence space representations of Hörmander spaces in the sense of Beurling and Björck [1] (= Hörmander–Beurling spaces) are given. This research pursues the study on Hörmander–Beurling spaces carried out in [1,6,12,13,29,33] and [24,25,27,28,32] (see also [14]).

The organization of the paper is as follows. Section 2 contains some basic facts about scalar and vector-valued Beurling ultradistributions and the definitions of the spaces which are considered in the paper. In Section 3 we show that $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ is isomorphic to $(B^c_{p,k}(\Omega, E))'_b$ when $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leqslant p < \infty$ and E is a Banach space whose dual E' possesses the Radon–Nikodým property (see Theorem 3.2), and we propose the following question: Are the spaces $BV^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ and $(B^c_{p,k}(\Omega, E))'_b$ isomorphic (E is any Banach space) (Problem 3.4)? In Section 4, by using the previous isomorphism, some representation theorems of Vogt [33, Theorems 5.2, 6.2] and the solution to Problem 4.11 in [24] given by Cembranos and Mendoza in [3], we partially answer Problem 4.10 in [24] (see Theorem 4.4). We also show that, in general, $B^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ is not isomorphic to either $B^{\mathrm{loc}}_{\infty,k}(\Omega)\, \widehat{\otimes}_\varepsilon\, E$ or $B^{\mathrm{loc}}_{\infty,k}(\Omega)\, \widehat{\otimes}_\pi\, E$. Next it is shown that $B^c_{p,k}(\Omega, l_q)$ (resp. $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$) is isomorphic to $\bigoplus_{j=0}^{\infty} G_j$ (resp. $\prod_{j=0}^{\infty} H_j$) where G_0 (resp. H_0) is isomorphic to $l_p(l_q)$ and G_j (resp. H_j) is

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isomorphic to a complemented subspace of $l_p(l_q)$ for $j = 1, 2, \ldots$. Then we describe the structure of the complemented normed subspaces of $B^{\mathrm{loc}}_{p,k}(\Omega)$, $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ and $\prod_{j=1}^{m} B^{\mathrm{loc}}_{p_j,k_j}(\Omega_j, l_p)$. We also give a new proof (based on our representation theorem $B^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$) of a well-known result on linear partial differential operators.

Notation. The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $L_b(E, F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. The dual of E is denoted by E' and is given the strong topology so that $E' = L_b(E, \mathbb{C})$. $E \widehat{\otimes}_\varepsilon F$ (resp. $E \widehat{\otimes}_\pi F$) is the completion of the injective (resp. projective) tensor product of E and F. If E and F are (topologically) isomorphic we put $E \simeq F$. If E is isomorphic to a complemented subspace of F we write $E < F$. We put $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous (we replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F). If $(E_n)_{n=1}^{\infty}$ is a sequence of locally convex spaces, $\prod_{n=1}^{\infty} E_n$ ($E^{\mathbb{N}}$ if $E_n = E$ for all n) is the topological product of the spaces E_n; $\bigoplus_{n=1}^{\infty} E_n$ ($E^{(\mathbb{N})}$ if $E_n = E$ for all n) is the locally convex direct sum of the spaces E_n.

Let $1 \leqslant p \leqslant \infty$, $k : \mathbb{R}^n \to (0, \infty)$ a Lebesgue measurable function, and E a Fréchet space. Then $L_p(E)$ is the set of all (equivalence classes of) Bochner measurable functions $f : \mathbb{R}^n \to E$ for which $\|f\|_p = (\int_{\mathbb{R}^n} \|f(x)\|^p\,dx)^{1/p}$ is finite (with the usual modification when $p = \infty$) for all $\|\cdot\| \in \mathrm{cs}(E)$ (see, e.g. [8]). $L_{p,k}(E)$ denotes the set of all Bochner measurable functions $f : \mathbb{R}^n \to E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|kf\|_p$ for all $f \in L_{p,k}(E)$ and for all $\|\cdot\| \in \mathrm{cs}(E)$, $L_{p,k}(E)$ becomes a Fréchet space isomorphic to $L_p(E)$. When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transform of f, $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\,dx$. If f is a function on $\mathbb{R}^n$ then $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$. The letter C will always denote a positive constant, not necessarily the same at each occurrence.

Finally we recall the definition of A_p^* functions. A positive, locally integrable function ω on $\mathbb{R}^n$ is in A_p^* provided, for $1 < p < \infty$,

$$\sup_R \left(\frac{1}{|R|} \int_R \omega\,dx \right) \left(\frac{1}{|R|} \int_R \omega^{-p'/p}\,dx \right)^{p/p'} < \infty,$$

where R runs over all bounded n-dimensional intervals. The basic properties of these functions can be found in [7, Chapter IV].

2. Spaces of vector-valued (Beurling) ultradistributions

In this section we collect some basic facts about vector-valued (Beurling) ultradistributions and we recall the definitions of the vector-valued Hörmander–Beurling spaces. Comprehensive treatments of the theory of (scalar or vector-valued) ultradistributions can be found in [1,10,15–17]. Our notations are based on [1] and [30, pp. 14–19].

Let $\mathcal{M}$ (or $\mathcal{M}_n$) be the set of all functions ω on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty[$ with the following properties:

(i) $\sigma(0) = 0$,
(ii) $\int_0^\infty \frac{\sigma(t)}{1+t^2}\,dt < \infty$ (Beurling's condition),
(iii) there exist a real number a and a positive number b such that

$$\sigma(t) \geqslant a + b\log(1+t) \quad \text{for all } t \geqslant 0.$$

The assumption (ii) is essentially the Denjoy–Carleman non-quasianalyticity condition (see [1, Section 1.5]). The two most prominent examples of functions $\omega \in \mathcal{M}$ are given by $\omega(x) = \log(1 + |x|)^d$, $d > 0$, and $\omega(x) = |x|^\beta$, $0 < \beta < 1$.

If $\omega \in \mathcal{M}$ and E is a Fréchet space, we denote by $D_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support, such that $\|f\|_\lambda = \int_{\mathbb{R}^n} \|\hat{f}(\xi)\| e^{\lambda\omega(\xi)}\,d\xi < \infty$ for all $\lambda > 0$ and for all $\|\cdot\| \in \mathrm{cs}(E)$. For each compact subset K of $\mathbb{R}^n$, $D_\omega(K, E) = \{f \in D_\omega(E)\colon\ \operatorname{supp} f \subset K\}$, equipped with the topology induced by the family of seminorms $\{\|\cdot\|_\lambda\colon\ \|\cdot\| \in \mathrm{cs}(E),\ \lambda > 0\}$, is a Fréchet space and $D_\omega(E) = \operatorname{ind}_{\underset{K}{\to}} D_\omega(K, E)$ becomes a strict (LF)-space. If Ω is any open set in $\mathbb{R}^n$, $D_\omega(\Omega, E)$ is the subspace of $D_\omega(E)$ consisting of all functions f with $\operatorname{supp} f \subset \Omega$. $D_\omega(\Omega, E)$ is endowed with the corresponding inductive limit topology: $D_\omega(\Omega, E) = \operatorname{ind}_{\underset{K \subset \Omega}{\to}} D_\omega(K, E)$. Let $S_\omega(E)$ be the set of all functions $f \in L_1(E)$ such that both f and $\hat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with $\sup_{x \in \mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha f(x)\| < \infty$ and $\sup_{x \in \mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha \hat{f}(x)\| < \infty$ for all multi-indices α, all positive numbers λ and all $\|\cdot\| \in \mathrm{cs}(E)$. $S_\omega(E)$ with the topology induced by the above family of seminorms is a Fréchet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $S_\omega(E)$. If $E = \mathbb{C}$ then $D_\omega(E)$ and $S_\omega(E)$ coincide with the spaces D_ω and S_ω (see [1]). Let us recall that, by Beurling's condition, the space D_ω is non-trivial and the usual procedure of the resolution of unity can be established with D_ω-functions (see [1, Theorem 1.3.7]). Furthermore, $D_\omega \overset{d}{\hookrightarrow} D$ (see [1, Theorem 1.3.18]) and D_ω is nuclear [33, Corollary 7.5]. On the other hand, $D_\omega = D \cap S_\omega$, $D_\omega \overset{d}{\hookrightarrow} S_\omega \overset{d}{\hookrightarrow} S$ (see [1, Proposition 1.8.6, Theorem 1.8.7]) and S_ω is nuclear also (see [10, p. 320]). If $\mathcal{E}_\omega$ is the set of multipliers on D_ω, i.e., the set of all functions $f:\mathbb{R}^n\to\mathbb{C}$ such that $\varphi f\in D_\omega$ for all $\varphi\in D_\omega$, then $\mathcal{E}_\omega$ with the topology generated by the seminorms $\{f\to\|\varphi f\|_\lambda=\int_{\mathbb{R}^n}|\widehat{\varphi f}(\xi)|e^{\lambda\omega(\xi)}\,d\xi:\ \lambda>0,\ \varphi\in D_\omega\}$ becomes a nuclear Fréchet space (see [33, Corollary 7.5]) and $D_\omega\overset{d}{\hookrightarrow}\mathcal{E}_\omega$. Using the above results and [17, Theorem 1.12] we can identify $S_\omega(E)$ with $S_\omega\widehat{\otimes}_\varepsilon E$. However, though $D_\omega\otimes E$ is dense in $D_\omega(E)$, in general $D_\omega(E)$ is not isomorphic to $D_\omega\widehat{\otimes}_\varepsilon E$ (cf., e.g. [9, Chapter II, p. 83]). A continuous linear operator from D_ω into E is said to be a (Beurling) ultradistribution with values in E. We write $D'_\omega(E)$ for the space of all E-valued (Beurling) ultradistributions endowed with the bounded convergence topology, thus $D'_\omega(E)=L_b(D_\omega,E)$. $D'_\omega(\Omega,E)=L_b(D_\omega(\Omega),E)$ is the space of all (Beurling) ultradistributions on Ω with values in E. A continuous linear operator from S_ω into E is said to be an E-valued tempered ultradistribution. $S'_\omega(E)$ is the space of all E-valued tempered ultradistributions equipped with the bounded convergence topology, i.e., $S'_\omega(E)=L_b(S_\omega,E)$. The Fourier transformation $\mathcal{F}$ is an automorphism of $S'_\omega(E)$.

If $\omega\in\mathcal{M}$, then $\mathcal{K}_\omega$ is the set of all positive functions k on $\mathbb{R}^n$ for which there exists a positive constant N such that $k(x+y)\leqslant e^{N\omega(x)}k(y)$ for all x and y in $\mathbb{R}^n$ [1, Definition 2.1.1] (when $\omega(x)=\log(1+|x|)$ the functions k of the corresponding class $\mathcal{K}_\omega$ are called temperate weight functions, see [13, Definition 10.1.1]). If k, k_1, $k_2\in\mathcal{K}_\omega$ and s is a real number then $\log k$ is uniformly continuous, $k^s\in\mathcal{K}_\omega$, $k_1k_2\in\mathcal{K}_\omega$ and $M_k(x)=\sup_{y\in\mathbb{R}^n}\frac{k(x+y)}{k(y)}\in\mathcal{K}_\omega$ (see [1, Theorem 2.1.3]). If $u\in L_1^{\mathrm{loc}}$ and $\int_{\mathbb{R}^n}\varphi(x)u(x)\,dx=0$ for all $\varphi\in D_\omega$, then $u=0$ a.e. (see [1]). This result, the Hahn–Banach theorem and [5, Chapter II, Corollary 7] prove that if $k\in\mathcal{K}_\omega$, $p\in[1,\infty]$ and E is a Fréchet space, we can identify $f\in L_{p,k}(E)$ with the E-valued tempered ultradistribution $\varphi\to\langle\varphi,f\rangle=\int_{\mathbb{R}^n}\varphi(x)f(x)\,dx$, $\varphi\in S_\omega$, and $L_{p,k}(E)\hookrightarrow S'_\omega(E)$. If $\omega\in\mathcal{M}$, $k\in\mathcal{K}_\omega$, $p\in[1,\infty]$ and E is a Fréchet space, we denote by $B_{p,k}(E)$ the set of all E-valued tempered ultradistributions T for which there exists a function $f\in L_{p,k}(E)$ such that $\langle\varphi,\hat{T}\rangle=\int_{\mathbb{R}^n}\varphi(x)f(x)\,dx$, $\varphi\in S_\omega$. $B_{p,k}(E)$ with the seminorms $\{\|T\|_{p,k}=((2\pi)^{-n}\int_{\mathbb{R}^n}\|k(x)\hat{T}(x)\|^p\,dx)^{1/p}:\ \|\cdot\|\in\mathrm{cs}(E)\}$ (usual modification if $p=\infty$), becomes a Fréchet space isomorphic to $L_{p,k}(E)$. Spaces $B_{p,k}(E)$ are called Hörmander–Beurling spaces with values in E (see [12,13,33] for the scalar case and [25,27,32] for the vector-valued case). We denote by $B_{p,k}^{\mathrm{loc}}(\Omega,E)$ (see [12,13,24,25,27,33]) the space of all E-valued ultradistributions $T\in D'_\omega(\Omega,E)$ such that, for every $\varphi\in D_\omega(\Omega)$, the map $\varphi T:S_\omega\to E$ defined by $\langle u,\varphi T\rangle=\langle u\varphi,T\rangle$, $u\in S_\omega$, belongs to $B_{p,k}(E)$. The space $B_{p,k}^{\mathrm{loc}}(\Omega,E)$ is a Fréchet space with the topology generated by the seminorms $\{\|\cdot\|_{p,k,\varphi}:\ \varphi\in D_\omega(\Omega),\ \|\cdot\|\in\mathrm{cs}(E)\}$, where $\|T\|_{p,k,\varphi}=\|\varphi T\|_{p,k}$ for $T\in B_{p,k}^{\mathrm{loc}}(\Omega,E)$, and $B_{p,k}^{\mathrm{loc}}(\Omega,E)\hookrightarrow D'_\omega(\Omega,E)$. We shall also use the spaces $B_{p,k}^c(\Omega,E)$ which generalize the scalar spaces $B_{p,k}^c(\Omega)$ considered by Hörmander in [13], by Vogt in [33] and by Björck in [1]. If ω, k, p, Ω and E are as above, then $B_{p,k}^c(\Omega,E)=\bigcup_{j=1}^\infty[B_{p,k}(E)\cap\mathcal{E}'_\omega(K_j,E)]$ (here (K_j) is any fundamental sequence of compact subsets of Ω and $\mathcal{E}'_\omega(K_j,E)$ denotes the set of all $T\in D'_\omega(E)$ such that $\operatorname{supp}T\subset K_j$). Since for every compact $K\subset\Omega$, $B_{p,k}(E)\cap\mathcal{E}'_\omega(K,E)$ is a Fréchet space with the topology induced by $B_{p,k}(E)$, it follows that $B_{p,k}^c(\Omega,E)$ becomes a strict (LF)-space (strict (LB)-space if E is a Banach space): $B_{p,k}^c(\Omega,E)=\operatorname{ind}_{\underset{j}{\to}}[B_{p,k}(E)\cap\mathcal{E}'_\omega(K_j,E)]$. These spaces are studied in [24,25,27].

3. The dual of $B_{p,k}^c(\Omega,E)$

In [13, Chapter XV] Hörmander studies the behaviour of the Fourier–Laplace transform in the space $B_{2,k}^c(\Omega)=\operatorname{ind}_{\underset{K}{\to}}[B_{2,k}\cap\mathcal{E}'(K)]$ when Ω is an open convex set in $\mathbb{R}^n$ and k is a temperate weight function on $\mathbb{R}^n$. For this he discusses the inductive limit topology in $B_{2,k}^c(\Omega)$, proves the isomorphism $(B_{2,k}^c(\Omega))'_b\simeq B_{2,1/\tilde{k}}^{\mathrm{loc}}(\Omega)$ [13, Section 15.2] and shows that every continuous seminorm in $B_{2,k}^c(\Omega)$ is bounded by a seminorm of the form

$$u\to\left(\int\left|\hat{u}(\zeta)\right|^2e^{-2\phi(\zeta)}\,d\lambda(\zeta)\right)^{1/2},$$

where $\hat{u}$ is the Fourier–Laplace transform of u and ϕ is plurisubharmonic. In this section we extend the former isomorphism to Hörmander spaces in the sense of Beurling and Björck [1] and prove that $(B_{p,k}^c(\Omega,E))'_b\simeq B_{p',1/\tilde{k}}^{\mathrm{loc}}(\Omega,E')$ when $\omega\in\mathcal{M}$, $k\in\mathcal{K}_\omega$, $1\leqslant p<\infty$ and E is a Banach space. A number of applications of this duality will be given in the next section.

Let us recall that a $D_\omega(\Omega)$-partition of unity in Ω (= open set in $\mathbb{R}^n$) is a sequence (θ_j) in $D_\omega(\Omega)$ such that: (i) $\theta_j\geqslant0$ for $j=1,2,\dots$, (ii) $\sum_j\theta_j\equiv1$ in Ω, (iii) for every compact set $K\subset\Omega$ there exist a positive integer m and a bounded open set W such that $K\subset W\subset\bar{W}\subset\Omega$ and $\sum_{j=1}^m\theta_j\equiv1$ in W.

Lemma 3.1. *Let Ω be an open set in $\mathbb{R}^n$, $\omega\in\mathcal{M}$, $k\in\mathcal{K}_\omega$, $1\leqslant p\leqslant\infty$, and E a Banach space. Let (θ_j) be a $D_\omega(\Omega)$-partition of unity in Ω. Then the inductive limit topology on $B_{p,k}^c(\Omega,E)$ is generated by the seminorms*

$$\|T\|_{(C_j)}=\sum_{j=1}^\infty C_j\|\theta_jT\|_{p,k},\quad T\in B_{p,k}^c(\Omega,E),$$

varying (C_j) in $\mathbb{R}_+^{\mathbb{N}}$.

Proof. See Proposition 3.10 of [27]. □

In the next result we will need the spaces $l_1(C_j, E)$ and $l_\infty(C_j, E)$: If (C_j) is a sequence in $\mathbb{R}_+^{\mathbb{N}}$ and E is a Banach space then $l_1(C_j, E)$ (resp. $l_\infty(C_j, E)$) denotes the set of all sequences $(x_j) \in E^{\mathbb{N}}$ such that $\|(x_j)\|_1 = \sum_{j=1}^{\infty} C_j \|x_j\|_E < \infty$ (resp. $\|(x_j)\|_\infty = \sup_j C_j \|x_j\|_E < \infty$). With the norm $\|\cdot\|_1$ (resp. $\|\cdot\|_\infty$) $l_1(C_j, E)$ (resp. $l_\infty(C_j, E)$) becomes a Banach space.

Theorem 3.2. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leqslant p < \infty$, and let E be a Banach space whose dual E' possesses the Radon–Nikodým property. Then $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ is isomorphic to $(B^c_{p,k}(\Omega, E))'_b$.*

Proof. Choose a fixed $D_\omega(\Omega)$-partition of unity (θ_j) in Ω and let L be an element in $(B^c_{p,k}(\Omega, E))'$. By Lemma 3.1 we can find a sequence (C_j) in $\mathbb{R}_+^{\mathbb{N}}$ such that

$$\left|L(T)\right| \leqslant \sum_{j=1}^{\infty} C_j \|\theta_j T\|_{p,k}, \quad T \in B^c_{p,k}(\Omega, E).$$

Then the linear mapping

$$\begin{aligned} Z : B^c_{p,k}(\Omega, E) &\to l_1\big(C_j, B_{p,k}(E)\big) \\ T &\to (\theta_j T) \end{aligned}$$

is continuous. Furthermore, since each T can be written in the form $T = \sum_{j=1}^{m} \theta_j T$ (m varying with T), we conclude that Z is injective. Now we consider the linear form $L \circ Z^{-1}$. Since $|L \circ Z^{-1}((\theta_j T))| \leqslant \|(\theta_j T)\|_1$, the Hahn–Banach theorem shows that there exists a linear form $(L \circ Z^{-1})^{-} \in (l_1(C_j, B_{p,k}(E)))'$ of norm at most 1 which extends $L \circ Z^{-1}$. Then, by the isometric isomorphism

$$A : l_\infty\left(\frac{1}{C_j}, B_{p',1/k}(E')\right) \to \big(l_1\big(C_j, B_{p,k}(E)\big)\big)'$$

defined by $\langle (T_j), A((S_j))\rangle = (2\pi)^{-n} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \langle \hat{T}_j(x), \hat{S}_j(x)\rangle\, dx$, we can find $(S_j) \in l_\infty(\frac{1}{C_j}, B_{p',1/k}(E'))$ such that $A((S_j)) = (L \circ Z^{-1})^{-}$, and so

$$L \circ Z^{-1}\big((\theta_j T)\big) = L(T) = (2\pi)^{-n} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \big\langle \widehat{\theta_j T}(x), \hat{S}_j(x)\big\rangle\, dx$$

for each $T \in B^c_{p,k}(\Omega, E)$. Next we shall prove that the linear mapping

$$\begin{aligned} \Phi : \big(B^c_{p,k}(\Omega, E)\big)'_b &\to B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E') \\ L &\to \textstyle\sum_{j=1}^{\infty} \theta_j \tilde{S}_j \end{aligned}$$

(the series $\sum_{j=1}^{\infty} \theta_j \tilde{S}_j$ converges in $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ since this space is a Fréchet space and $\sum_{j=1}^{\infty} \|\theta_j \tilde{S}_j\|_{p',1/\tilde{k},\varphi} = \sum_{j=1}^{\infty} \|(\theta_j \varphi)\tilde{S}_j\|_{p',1/\tilde{k}} < \infty$ for each $\varphi \in D_\omega(\Omega)$ in virtue of the properties of the sequence (θ_j)) is an isomorphism. Let us see that Φ is well defined. Let $(L \circ Z^{-1})^{=}$ another extension of $L \circ Z^{-1}$ to $l_1(C_j, B_{p,k}(E))$ and let $(S_j^1) \in l_\infty(\frac{1}{C_j}, B_{p',1/k}(E'))$ the sequence which represents this extension. Let us check that $\sum_{j=1}^{\infty} \theta_j \tilde{S}_j = \sum_{j=1}^{\infty} \theta_j \tilde{S}_j^1$. By Fourier's inversion formula, the properties of the Bochner integral and the embedding $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E') \hookrightarrow D'_\omega(\Omega, E')$ (see Section 2) we have for all $\varphi \in D_\omega(\Omega)$ and all $e \in E$,

$$\left\langle \varphi, \sum_{j=1}^{\infty} \theta_j \tilde{S}_j \right\rangle = \sum_{j=1}^{\infty} \langle \varphi, \theta_j \tilde{S}_j\rangle = \sum_{j=1}^{\infty} \langle \varphi\theta_j, \tilde{S}_j\rangle = (2\pi)^{-n} \sum_{j=1}^{\infty} \langle \widehat{\varphi\theta_j}, \hat{S}_j\rangle$$

and

$$\begin{aligned} (2\pi)^{-n} \left\langle e, \sum_{j=1}^{\infty} \langle \widehat{\varphi\theta_j}, \hat{S}_j\rangle \right\rangle &= (2\pi)^{-n} \sum_{j=1}^{\infty} \big\langle e, \langle \widehat{\varphi\theta_j}, \hat{S}_j\rangle\big\rangle = (2\pi)^{-n} \sum_{j=1}^{\infty} \left\langle e, \int_{\mathbb{R}^n} \widehat{\theta_j\varphi}(x) \hat{S}_j(x)\, dx \right\rangle \\ &= (2\pi)^{-n} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \big\langle \big(\theta_j(\varphi \otimes e)\big)^{\wedge}(x), \hat{S}_j(x)\big\rangle\, dx = L(\varphi \otimes e). \end{aligned}$$

Repeating the argument with $\sum_{j=1}^{\infty} \theta_j \tilde{S}_j^1$ we conclude that $\sum_{j=1}^{\infty} \theta_j \tilde{S}_j = \sum_{j=1}^{\infty} \theta_j \tilde{S}_j^1$. Now let $(C'_j) \in \mathbb{R}_+^{\mathbb{N}}$ another sequence such that $|L(T)| \leqslant \sum_{j=1}^{\infty} C'_j \|\theta_j T\|_{p,k}$ for $T \in B^c_{p,k}(\Omega, E)$. Let Z' be the corresponding operator, let $(L \circ Z'^{-1})^{-}$ be an extension of $L \circ Z'^{-1}$ to $l_1(C'_j, B_{p,k}(E))$ and let $(S'_j) \in l_\infty(\frac{1}{C'_j}, B_{p',1/k}(E'))$ the sequence which represents this extension, then $L(T) = (2\pi)^{-n} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \langle \widehat{\theta_j T}(x), \hat{S}'_j(x)\rangle\, dx$, $T \in B^c_{p,k}(\Omega, E)$, and also $\langle e, \langle \varphi, \sum_{j=1}^{\infty} \theta_j \tilde{S}'_j\rangle\rangle = L(\varphi \otimes e)$ for $\varphi \in D_\omega(\Omega)$ and $e \in E$. Then Φ is well defined. If $\Phi(L) = 0$ then $\langle e, \langle \varphi, \Phi(L)\rangle\rangle = 0 = L(\varphi \otimes e)$ for all $\varphi \in D_\omega(\Omega)$ and all $e \in E$, thus $L = 0$ on $D_\omega(\Omega) \otimes E$. Since this space is dense in $D_\omega(\Omega, E)$ (see Section 2) and $D_\omega(\Omega, E) \overset{d}{\hookrightarrow} B^c_{p,k}(\Omega, E)$ (see Proposition 3.6 of [27]), it follows that $L = 0$. Consequently, Φ is one-to-one. Furthermore, Φ is surjective: Let (χ_j) a sequence in $D_\omega(\Omega)$ such that $\chi_j = 1$ in a compact neighborhood of $\operatorname{supp}\theta_j$. Let S be an element of $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$. Then we have (convergence in $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$) $S = \sum_{j=1}^{\infty} \theta_j S = \sum_{j=1}^{\infty} (\theta_j \chi_j) S = \sum_{j=1}^{\infty} \theta_j(\chi_j S) = \sum_{j=1}^{\infty} \theta_j \tilde{X}_j$ where $X_j = \widetilde{\chi_j S}$. Now we define the functional

$$L(T) = (2\pi)^{-n} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \big\langle \widehat{\theta_j T}(x), \hat{X}_j(x)\big\rangle\, dx, \quad T \in B^c_{p,k}(\Omega, E).$$

Since

$$\big|L(T)\big| \leqslant (2\pi)^{-n} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \big\|\widehat{\theta_j T}(x)\big\|_E k(x) \big\|\hat{X}_j(x)\big\|_{E'} \frac{1}{k(x)}\, dx \leqslant \sum_{j=1}^{\infty} \|\theta_j T\|_{p,k} \|X_j\|_{p',1/k}$$

for all $T \in B^c_{p,k}(\Omega, E)$, it follows that $L \in (B^c_{p,k}(\Omega, E))'$. Then $\Phi(L) = S$ and Φ is surjective.

Now we prove that Φ^{-1} is continuous: Let A be a bounded set in $B^c_{p,k}(\Omega, E)$. Since this space is a strict (LB)-space, there is a compact set M in Ω such that A is contained and bounded in the step $B_{p,k}(E) \cap \mathcal{E}'_\omega(M, E)$ (see [18, (4), p. 223]). Take a sequence (χ_j) in $D_\omega(\Omega)$ such that $\chi_j = 1$ in a compact neighborhood of $\operatorname{supp}\theta_j$, $j = 1, 2, \ldots,$ and let m be such that $\theta_j = 0$ in M for all $j > m$. Then, taking into account Proposition 3.4 of [27] and that every $S \in B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ can be written in the form $S = \sum_{j=1}^{\infty} \theta_j \tilde{X}_j$ with $X_j = \widetilde{\chi_j S}$, we get

$$\begin{aligned}\sup_{T \in A} \big|\Phi^{-1}(S)(T)\big| &= \sup_{T \in A} \left| (2\pi)^{-n} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \big\langle \widehat{\theta_j T}(x), \hat{X}_j(x)\big\rangle\, dx \right| \leqslant \sup_{T \in A} \sum_{j=1}^{m} \|\theta_j T\|_{p,k} \|X_j\|_{p',1/k} \\ &\leqslant \sup_{T \in A} \sum_{j=1}^{m} \|\theta_j\|_{1,M_k} \|T\|_{p,k} \|S\|_{p',1/\tilde{k},\chi_j} \leqslant C \sum_{j=1}^{m} \|\theta_j\|_{1,M_k} \|S\|_{p',1/\tilde{k},\chi_j}\end{aligned}$$

for all $S \in B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ (C is a constant > 0). Hence it follows the continuity of Φ^{-1}. Then Φ becomes an isomorphism since $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ and $(B^c_{p,k}(\Omega, E))'_b$ are Fréchet spaces ($B^c_{p,k}(\Omega, E)$ is a (DF)-space by [18, (4), p. 402] and so its strong dual is a Fréchet space (see [18, (1), p. 397])). The proof is complete. □

Remark 3.3. When $k(x)$ is a temperate weight function, $p = 2$ and $E = \mathbb{C}$, our theorem yields the isomorphism which appears in [13, p. 279].

In [32] the spaces $BV_{p,k}(E)$ are introduced (by using the natural embedding of the space $V_p(k^p\, dx, E)$ of the finitely additive E-valued measures of bounded p-variation into the space $S'_\omega(E)$) and the isometric isomorphism $BV_{p',1/k}(E') \simeq (B_{p,k}(E))'$ is shown (E is any Banach space and $1 \leqslant p < \infty$). In view of this result and our Theorem 3.2 we can define the space

$$BV^{\mathrm{loc}}_{p,k}(\Omega, E) = \big\{T \in D'_\omega(\Omega, E)\colon\ \varphi T \in BV_{p,k}(E) \text{ for all } \varphi \in D_\omega(\Omega)\big\}$$

(equipped with the topology generated by the family of seminorms $\{T \to \|(2\pi)^{-n/p} \widehat{\varphi T}\|_{V_p(k^p\, dx, E)}\colon\ \varphi \in D_\omega(\Omega)\}$ when $p < \infty$ (resp. $\{T \to \|\widehat{\varphi T}\|_{V_\infty(\frac{1}{k}\, dx, E)}\colon\ \varphi \in D_\omega(\Omega)\}$ if $p = \infty$)) and propose the following question.

Problem 3.4. Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leqslant p < \infty$ and let E be a Banach space. Are the spaces $BV^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ and $(B^c_{p,k}(\Omega, E))'_b$ isomorphic?

4. On sequence space representations of spaces of ultradistributions

In this section we give a number of results on sequence space representations of spaces of distributions and ultradistributions. Based on these and using the solution to Problem 4.11 in [24] given by Cembranos and Mendoza in [3], we partially answer Problem 4.10 in [24]. We also give a new proof of a well-known result: The short sequence

$$0 \to N\big(P(D)\big) \to B^{\mathrm{loc}}_{p,k}(\Omega) \xrightarrow{P(D)} B^{\mathrm{loc}}_{p,k/P'}(\Omega) \to 0$$

does not split ($P(D)$ is an elliptic operator with constant coefficients and $P'(\xi) = (\sum_\alpha |\partial^\alpha P(\xi)|^2)^{1/2}$). (The proof we give is based on the isomorphism $B^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$.)

We shall omit the proof of the following simple result.

Lemma 4.1. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leqslant p \leqslant \infty$, and let $(E_j)_{j=1}^\infty$ be a sequence of Banach spaces. Then the space $B^{\mathrm{loc}}_{p,k}(\Omega, \prod_{j=1}^\infty E_j)$ is isomorphic to $\prod_{j=1}^\infty B^{\mathrm{loc}}_{p,k}(\Omega, E_j)$.*

Theorem 4.2. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, and let E be a Banach space. Then*

(1) *$B^{c}_{1,k}(\Omega, E)$ is isomorphic to $(l_1(E))^{(\mathbb{N})}$,*
(2) *$B^{\mathrm{loc}}_{1,k}(\Omega, E)$ is isomorphic to $(l_1(E))^{\mathbb{N}}$,*
(3) *if E is a dual space and has the Radon–Nikodým property then $B^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ is isomorphic to $(l_\infty(E))^{\mathbb{N}}$.*

Proof. (1) and (2). The proof given in [33] is also valid in the vector-valued case and for weights $k \in \mathcal{K}_\omega$. (3) Suppose $E \simeq F'$ and recall that if $(E_j)_{j=1}^\infty$ is a sequence of Banach spaces then the space $(\bigoplus_{j=1}^\infty E_j)'_b$ is isomorphic to $\prod_{j=1}^\infty E'_j$ (see [18, p. 287]). Then, taking into account Theorem 3.2 and (1), we get

$$B^{\mathrm{loc}}_{\infty,k}(\Omega, E) \simeq \big(B^{c}_{1,1/\tilde{k}}(\Omega, F)\big)'_b \simeq \big(\big(l_1(F)\big)^{(\mathbb{N})}\big)'_b \simeq \big(l_\infty(E)\big)^{\mathbb{N}}. \qquad \square$$

Theorem 4.3. *$l_\infty(l_1)$ and $l_1(l_\infty)$ are not isomorphic.*

Proof. See [3, Theorem 1]. $\square$

Next we answer Problem 4.10 in [24] when $q = \infty$.

Theorem 4.4. *If Ω_1 is an open set in $\mathbb{R}^{n_1}$, $\omega_1 \in \mathcal{M}_{n_1}$ and $k_1 \in \mathcal{K}_{\omega_1}$ (resp. Ω_2 open set in $\mathbb{R}^{n_2}$, $\omega_2 \in \mathcal{M}_{n_2}$, $k_2 \in \mathcal{K}_{\omega_2}$), then the spaces $B^{\mathrm{loc}}_{1,k_1}(\Omega_1, B^{\mathrm{loc}}_{\infty,k_2}(\Omega_2))$ and $B^{\mathrm{loc}}_{\infty,k_2}(\Omega_2, B^{\mathrm{loc}}_{1,k_1}(\Omega_1))$ are not isomorphic.*

Proof. By using the previous results we have the isomorphisms

$$B^{\mathrm{loc}}_{1,k_1}\big(\Omega_1, B^{\mathrm{loc}}_{\infty,k_2}(\Omega_2)\big) \simeq B^{\mathrm{loc}}_{1,k_1}\big(\Omega_1, l_\infty^{\mathbb{N}}\big) \simeq \big(B^{\mathrm{loc}}_{1,k_1}(\Omega_1, l_\infty)\big)^{\mathbb{N}} \simeq \big(\big(l_1(l_\infty)\big)^{\mathbb{N}}\big)^{\mathbb{N}} \simeq \big(l_1(l_\infty)\big)^{\mathbb{N}}$$

and

$$B^{\mathrm{loc}}_{\infty,k_2}\big(\Omega_2, B^{\mathrm{loc}}_{1,k_1}(\Omega_1)\big) \simeq B^{\mathrm{loc}}_{\infty,k_2}\big(\Omega_2, l_1^{\mathbb{N}}\big) \simeq \big(B^{\mathrm{loc}}_{\infty,k_2}(\Omega_2, l_1)\big)^{\mathbb{N}} \simeq \big(\big(l_\infty(l_1)\big)^{\mathbb{N}}\big)^{\mathbb{N}} \simeq \big(l_\infty(l_1)\big)^{\mathbb{N}}.$$

Suppose now that our iterated spaces are isomorphic. Then $(l_1(l_\infty))^{\mathbb{N}}$ and $(l_\infty(l_1))^{\mathbb{N}}$ are also isomorphic. Hence it follows (by [4]) that there exist positive integers α, β such that $l_1(l_\infty) < (l_\infty(l_1))^\alpha \simeq l_\infty(l_1)$ and $l_\infty(l_1) < (l_1(l_\infty))^\beta \simeq l_1(l_\infty)$. Then, using Pelczynski's decomposition method, we conclude that $l_1(l_\infty) \simeq l_\infty(l_1)$. This contradicts Theorem 4.3. In consequence, $B^{\mathrm{loc}}_{1,k_1}(\Omega_1, B^{\mathrm{loc}}_{\infty,k_2}(\Omega_2))$ and $B^{\mathrm{loc}}_{\infty,k_2}(\Omega_2, B^{\mathrm{loc}}_{1,k_1}(\Omega_1))$ are not isomorphic. $\square$

Remark 4.5. 1. We must point out that the space $B^{\mathrm{loc}}_{\infty,k_2}(\Omega_2, B^{\mathrm{loc}}_{1,k_1}(\Omega_1))$ even contains no complemented subspace isomorphic to $B^{\mathrm{loc}}_{1,k_1}(\Omega_1, B^{\mathrm{loc}}_{\infty,k_2}(\Omega_2))$ (see the proof of Theorem 4.4 and use the final remarks of [3]).

2. Note also that, in general, $B^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ is not isomorphic to either $B^{\mathrm{loc}}_{\infty,k}(\Omega)\widehat{\otimes}_\varepsilon E$ or $B^{\mathrm{loc}}_{\infty,k} \widehat{\otimes}_\pi E$: In fact, let $1 \leqslant p < \infty$ and assume that $B^{\mathrm{loc}}_{\infty,k}(\Omega, l_p)$ is isomorphic to $B^{\mathrm{loc}}_{\infty,k}(\Omega)\widehat{\otimes}_\varepsilon l_p$. Then, by virtue of [19, (5), p. 282], [19, (2), p. 287], Theorem 4.2 and a result of Cembranos and Freniche [2, Theorem 3.2.1], we get

$$\big(l_\infty(l_p)\big)^{\mathbb{N}} \simeq l_\infty^{\mathbb{N}} \widehat{\otimes}_\varepsilon l_p \simeq (l_\infty \widehat{\otimes}_\varepsilon l_p)^{\mathbb{N}} \simeq \big(C(\beta\mathbb{N}) \widehat{\otimes}_\varepsilon l_p\big)^{\mathbb{N}} \simeq \big(C(\beta\mathbb{N}, l_p)\big)^{\mathbb{N}} > c_0^{\mathbb{N}}.$$

Hence it follows, arguing as in Theorem 4.4, that $l_\infty(l_p)$ contains a complemented copy of c_0. Then, by a result of Leung and Räbiger [2, Theorem 5.1.1], l_p also contains a complemented copy of c_0. This contradiction shows that $B^{\mathrm{loc}}_{\infty,k}(\Omega, l_p)$ and $B^{\mathrm{loc}}_{\infty,k} \widehat{\otimes}_\varepsilon l_p$ are not isomorphic. On the other hand, since by Theorem 4.2 and [19, (5), p. 194] we have

$$B^{\mathrm{loc}}_{\infty,k}(\Omega, l_1) \simeq \big(l_\infty(l_1)\big)^{\mathbb{N}}, \qquad B^{\mathrm{loc}}_{\infty,k}(\Omega) \widehat{\otimes}_\pi l_1 \simeq l_\infty^{\mathbb{N}} \widehat{\otimes}_\pi l_1 \simeq (l_\infty \widehat{\otimes}_\pi l_1)^{\mathbb{N}} \simeq \big(l_1(l_\infty)\big)^{\mathbb{N}},$$

it follows that the spaces $B^{\mathrm{loc}}_{\infty,k}(\Omega, l_1)$ and $B^{\mathrm{loc}}_{\infty,k}(\Omega) \widehat{\otimes}_\pi l_1$ are not isomorphic.

In the next theorem the following elementary fact will be used: "Let $F = \operatorname{ind}_{\underset{j}{\to}} F_j$ be the strict inductive limit of a properly increasing sequence $F_1 \subset F_2 \subset \cdots$ of Banach spaces. Assume that every F_j is a complemented subspace of F_{j+1} and that G_j is a topological complement of F_j in F_{j+1}. Then, the mapping $F_1 \oplus G_1 \oplus G_2 \oplus \cdots \to F : (f_1, g_1, g_2, \ldots) \to f_1 + g_1 + g_2 + \cdots$ is an isomorphism". We will also need the weighted L_p-spaces of vector-valued entire analytic functions $L^K_{p,k}(E)$ and the operators $S_K(f) = \mathcal{F}^{-1}(\chi_K \hat{f})$ (see [25]).

Theorem 4.6. *Let Ω be an open set in $\mathbb{R}^n$. Assume $1 < p, q < \infty$ and let k be a temperate weight function on $\mathbb{R}^n$ with $k^p \in A^*_p$. Then the space $B^c_{p,k}(\Omega, l_q)$ (resp. $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$) is isomorphic to $\bigoplus_{j=0}^{\infty} G_j$ (resp. $\prod_{j=0}^{\infty} H_j$) where G_0 (resp. H_0) is isomorphic to $l_p(l_q)$ and G_j (resp. H_j) is isomorphic to a complemented subspace of $l_p(l_q)$ for $j = 1, 2, \ldots$.*

Proof. Let (K_j) be a covering of Ω consisting of compact sets such that $K_j \subset \mathring{K}_{j+1}$, $K_j = \overline{\mathring{K}_j}$ and $\mathring{K}_j$ has the segment property (we may also assume, without loss of generality, that each K_j is a finite union of n-dimensional compact intervals). Then $B^c_{p,k}(\Omega, l_q) = \operatorname{ind}_{\underset{j}{\to}} [B_{p,k}(l_q) \cap \mathcal{E}'(K_j, l_q)]$. In this inductive limit, the step $B_{p,k}(l_q) \cap \mathcal{E}'(K_j, l_q)$ is isomorphic (via the Fourier transform) to $L^{-K_j}_{p,k}(l_q)$ and this space is isomorphic, by Corollaries 4.2 and 5.1 of [25], to $l_p(l_q)$. Furthermore, $L^{-K_j}_{p,k}(l_q)$ is a complemented subspace of $L^{-K_{j+1}}_{p,k}(l_q)$: $L^{-K_j}_{p,k}(l_q) \oplus [\ker S_{-K_j} \cap L^{-K_{j+1}}_{p,k}(l_q)] = L^{-K_{j+1}}_{p,k}(l_q)$. Thus, the space $G_j = \ker S_{-K_j} \cap L^{-K_{j+1}}_{p,k}(l_q)$ is isomorphic to an infinite-dimensional complemented subspace of $l_p(l_q)$. Then, by using the former result, we obtain

$$B^c_{p,k}(\Omega, l_q) \simeq L^{-K_1}_{p,k}(l_q) \oplus G_1 \oplus G_2 \oplus \cdots \simeq l_p(l_q) \oplus G_1 \oplus G_2 \oplus \cdots.$$

Next, since $1/\tilde{k}$ is a temperate weight function on $\mathbb{R}^n$ such that $1/\tilde{k}^{p'} \in A^*_{p'}$, we see that $B^c_{p',1/\tilde{k}}(\Omega, l_{q'}) \simeq \bigoplus_{j=0}^{\infty} B_j$ where $B_0 \simeq l_{p'}(l_{q'})$ and $B_j < l_{p'}(l_{q'})$ for $j = 1, 2, \ldots$. Therefore, by Theorem 3.2, we get

$$B^{\mathrm{loc}}_{p,k}(\Omega, l_q) \simeq \left(B^c_{p',1/\tilde{k}}(\Omega, l_{q'})\right)'_b \simeq \left(\bigoplus_{j=0}^{\infty} B_j\right)'_b \simeq \prod_{j=0}^{\infty} B'_j = \prod_{j=0}^{\infty} H_j$$

(here $H_j = B'_j$) where $H_0 \simeq l_p(l_q)$ and $H_j < l_p(l_q)$ for $j = 1, 2, \ldots$, and the proof is complete. □

Remark 4.7. (1) Let Ω, p and k as in Theorem 4.6. In [25, Corollary 5.3] the space $B^c_{p,k}(\Omega, E)$ is showed to be isomorphic to $l_p^{(\mathbb{N})}$ if $\dim E < \infty$ or $E = l_p$, and to $(l_p(l_2))^{(\mathbb{N})}$ if $E = l_2$. By duality (Theorem 3.2) it follows that $B^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$, $B^{\mathrm{loc}}_{p,k}(\Omega, l_p) \simeq l_p^{\mathbb{N}}$ and $B^{\mathrm{loc}}_{p,k}(\Omega, l_2) \simeq (l_p(l_2))^{\mathbb{N}}$.

(2) Note that, in general, $B^{\mathrm{loc}}_{p,k}(\Omega, E)$ is not isomorphic to either $B^{\mathrm{loc}}_{p,k}(\Omega) \widehat{\otimes}_\varepsilon E$ or $B^{\mathrm{loc}}_{p,k}(\Omega) \widehat{\otimes}_\pi E$: In fact, let Ω, p, q and k as in Theorem 4.6 and assume that $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ is isomorphic to $B^{\mathrm{loc}}_{p,k}(\Omega) \widehat{\otimes}_\varepsilon l_q$ (resp. $B^{\mathrm{loc}}_{p,k}(\Omega) \widehat{\otimes}_\pi l_q$). Then, by Theorem 4.6, the previous note, [19, (5), p. 282] and [19, (5), p. 194], we get

$$\prod_{j=0}^{\infty} H_j \simeq l_p^{\mathbb{N}} \widehat{\otimes}_\varepsilon l_q \simeq (l_p \widehat{\otimes}_\varepsilon l_q)^{\mathbb{N}} \quad \left(\text{resp. } \prod_{j=0}^{\infty} H_j \simeq (l_p \widehat{\otimes}_\pi l_q)^{\mathbb{N}}\right),$$

where $H_0 \simeq l_p(l_q)$ and $H_j < l_p(l_q)$ for $j = 1, 2, \ldots$. Hence it follows, reasoning as in Theorem 4.4, that $l_p(l_q) \simeq l_p \widehat{\otimes}_\varepsilon l_q$ (resp. $l_p \widehat{\otimes}_\pi l_q$) but this is false when $p' \leqslant q$ (resp. $p \leqslant q'$) by a result of Holub [11, Proposition 3.7] (resp. [11, Proposition 3.6]). In consequence, the spaces $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ and $B^{\mathrm{loc}}_{p,k}(\Omega) \widehat{\otimes}_\varepsilon l_q$ (resp. $B^{\mathrm{loc}}_{p,k}(\Omega) \widehat{\otimes}_\pi l_q$) are not isomorphic when $p' \leqslant q$ (resp. $p \leqslant q'$).

(3) By using the previous results we can describe the structure of the complemented (normed) subspaces of $B^{\mathrm{loc}}_{p,k}(\Omega)$, $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ and $\prod_{i=1}^{m} B^{\mathrm{loc}}_{p_i,k_i}(\Omega_i, l_{p_i})$: (i) Let X be an infinite-dimensional complemented (normed) subspace of $B^{\mathrm{loc}}_{p,k}(\Omega)$ (Ω open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $p \in \{1, \infty\}$ or k temperate weight function on $\mathbb{R}^n$ such that $k^p \in A^*_p$ and $p \in (1, \infty)$). Then $B^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$ and thus X becomes a complemented subspace of l_p. This implies, since l_p is prime [20, Theorems 2.a.3, 2.a.7], that $X \simeq l_p$. (ii) Let X be an infinite-dimensional complemented (normed) subspace of $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ (Ω open set in $\mathbb{R}^n$, $p, q \in (1, \infty)$ and k temperate weight function on $\mathbb{R}^n$ with $k^p \in A^*_p$). Then, since $B^{\mathrm{loc}}_{p,k}(\Omega, l_q) < (l_p(l_q))^{\mathbb{N}}$ in virtue of Theorem 4.6, X becomes a complemented subspace of $l_p(l_q)$. This implies, in the case $q = 2$, that X is isomorphic to either l_2, l_p, $l_2 \oplus l_p$ or $l_p(l_2)$ by a result of Odell [26]. (iii) Let X be an infinite-dimensional complemented (normed) subspace of $\prod_{i=1}^{m} B^{\mathrm{loc}}_{p_i,k_k}(\Omega_i, l_{p_i})$ (Ω_i open set in $\mathbb{R}^n$, $1 < p_1 < \cdots < p_m < \infty$, k_i temperate weight function on $\mathbb{R}^n$ with $k_i^{p_i} \in A^*_{p_i}$, $i = 1, \ldots, m$). Then, since

$$\prod_{i=1}^{m} B^{\mathrm{loc}}_{p_i,k_i}(\Omega_i, l_{p_i}) \simeq \prod_{i=1}^{m} l_{p_i}^{\mathbb{N}} \simeq (l_{p_1} \oplus \cdots \oplus l_{p_m})^{\mathbb{N}},$$

we have that $X < l_{p_1} \oplus \cdots \oplus l_{p_m}$ and so there exist $1 \leqslant i_1 < \cdots < i_k \leqslant m$ such that $X \simeq l_{p_{i_1}} \oplus \cdots \oplus l_{p_{i_k}}$ in virtue of [20, Theorem 2.c.14].

(4) We omit the proof of the following result:

$$B^{\mathrm{loc}}_{p_1,k_1}(\Omega_1, l_{q_1}) \simeq B^{\mathrm{loc}}_{p_2,k_2}(\Omega_2, l_{q_2}) \iff p_1 = p_2 \text{ and } q_1 = q_2$$

(Ω_i open set in $\mathbb{R}^n$, $p_i, q_i \in (1,\infty)$, k_i temperate weight function on $\mathbb{R}^n$ with $k_i^{p_i} \in A^*_{p_i}$, $i = 1, 2$).

We conclude this section by showing a result on linear partial differential operators (the result is well known, see e.g. [21,22,31,34]). The proof we give is based on our representation theorem $B^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$.

Theorem 4.8. *Let Ω be an open set in $\mathbb{R}^n$ ($n \geqslant 2$), $1 < p < \infty$, k a temperate weight function on $\mathbb{R}^n$ such that $k^p \in A^*_p$ and $P(D)$ an elliptic operator with constant coefficients. Then the short sequence*

$$0 \to N\big(P(D)\big) \to B^{\mathrm{loc}}_{p,k}(\Omega) \xrightarrow{P(D)} B^{\mathrm{loc}}_{p,k/P'}(\Omega) \to 0$$

is exact and does not split, i.e., the operator $P(D)$ has no continuous linear right inverse (here $N(P(D))$ is the kernel of $P(D)$).

Proof. $P(D)$ is well defined by [13, Theorem 10.1.11] and the short sequence is exact in virtue of [13, Corollary 10.8.2] and [13, Theorem 10.6.7]. The closed subspace $N(P(D))$ of $B^{\mathrm{loc}}_{p,k}(\Omega)$ coincides, algebraic and topologically, with the subspace $N(\Omega) = \{f \in \mathcal{E}(\Omega)\colon\ P(D)f = 0\}$ of $\mathcal{E}(\Omega)$ (by [12, Theorem 1.11.10], [12, Theorem 1.11.11] and the closed graph theorem) and thus it is a nuclear Fréchet space. Note also that, for every connected component O of Ω, the space $N(O)$ equipped with the topology induced by $\mathcal{E}(O)$, is a nuclear Fréchet space with continuous norms (since all $f \in N(O)$ is real analytic in O, see e.g. [1, Corollary 4.1.4]) isomorphic to a complemented subspace of $N(P(D))$. Now assume that the short sequence splits. Then $N(P(D))$ is a complemented subspace of $B^{\mathrm{loc}}_{p,k}(\Omega)$. Since this space is isomorphic to $l_p^{\mathbb{N}}$ by Remark 4.7(1), it follows that, for any connected component O of Ω, the space $N(O)$ becomes isomorphic to an infinite-dimensional ($n \geqslant 2$) complemented subspace of $l_p^{\mathbb{N}}$. This implies, taking into account a result of Metafune and Moscatelli [23, Theorem 1.2], that $N(O)$ is isomorphic to either l_p, $l_p \times \omega$, ω or $l_p^{\mathbb{N}}$. This contradiction completes the proof. $\square$

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