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A proportional view: The mathematics of James Glenie (1750–1817)

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Abstract

The mathematical work of James Glenie (1750–1817) was published at irregular intervals during a turbulent life. His ideas, mostly deriving from his time as an Assistant in Mathematics at St Andrews University in Scotland, were developed intermittently over a period of thirty-seven years. His mathematical achievements, underestimated by previous historians, were deeply rooted in Euclidean geometry and his own generalized theory of proportion. Among them are many new geometrical constructions and proofs, a novel demonstration of the binomial theorem, and an alternative approach to the differential calculus.

Résumé

La vie de James Glenie (1750–1817) était turbulente, et ses ouvrages mathématiques furent publiés irrégulièrement. Ses idées, prenant leur origine, par la plupart, dans la periode où il était Assistant en mathématique à l'Université de St Andrews en Écosse, furent developpées pendant trente-sept ans environ. Ses succès mathématiques, sousestimés auparavant, furent enracinés dans la géometrie Euclidienne et dans sa théorie originale des proportions géneralisées. Parmi eux il y a plusieurs constructions et preuves géometriques nouvelles, une démonstration originale du théorème binomial, et une approche alternative au calcul différentiel.

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1. Introduction

The life and works of James Glenie (1750–1817) have been considered by just a few historians. After study at St Andrews University in Scotland, and brief employment there as an Assistant in Mathematics, his turbulent life encompassed service in Canada as a military engineer and then radical politician, controversies in England over planned sea defences, and involvement in various legal proceedings. In his own day, Glenie was a quite well-known and even notorious figure. An account of his life is given in the next section, where references to fuller historical studies are given. The remaining sections focus on his mathematical writings. Short summaries of his publications on gunnery, fortification, and mathematics have earlier appeared in Goodwin [1889]; Roberts [1970]; Johnson [1997, 1998, 2004]. Johnson deserves credit for bringing the work of Glenie to the attention of modern readers: his papers contain much information, but also a few inaccuracies, and his discussion of Glenie's mathematics is not profound.

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During his eventful and unquiet life, it is surprising that Glenie retained his love of mathematics, and intermittently published works of considerable interest. Though it seems that he made no great attempt to "keep up with the subject," he continued to deploy the skills acquired during his days as a student and assistant at St Andrews University in the 1770s. These centered mainly on Euclidean geometry and the theory of proportion, which he applied and extended in unexpected ways. His geometrical constructions and proofs established many new results and proved old ones left unproven by Matthew Stewart. His *Doctrine of Universal Comparison* [1789] and his *Antecedental Calculus* [1793], though unwieldy, gave a workable alternative approach to the differential calculus that was more firmly founded than modern writers have allowed. And his Euclidean demonstration of the binomial theorem [Glenie, 1799], for arbitrary real powers, deserves recognition as perhaps the earliest attempt that *almost* stands up to modern scrutiny.[1]

The best-known among these little-remembered publications is Glenie's *Antecedental Calculus*. Charles Hutton's *A Mathematical and Philosophical Dictionary* of 1796 describes this as "a branch of general geometrical proportion, or universal comparison ... invented by Mr. James Glenie and published by him in 1793" [Hutton, 1796, 1: 121].[2] In 1919, Florian Cajori discussed Glenie's work in a chapter devoted to "abortive attempts at arithmetisation" of the calculus, all of which he deemed "either a complete failure or so complicated as to be prohibitive." Implying that Glenie's work belongs to the latter category, he rightly observed that it "plays no part in the later history of fluxions" and that "the style of exposition is poor" [Cajori, 1919, 237–238]. Glenie's calculus is also briefly discussed by Guicciardini [1989, 104], who comments on the "obvious weakness of Glenie's mathematics" in simply deleting higher-order powers of small increments in order to obtain his equivalent of Newton's fluxions.

It is the contention of this paper that the mathematical achievements of James Glenie have been underestimated. No doubt, these played little part in the subsequent development of mathematics; and his mathematical language, based on Euclidean geometry and the theory of proportion, was already falling from fashion in his own lifetime. The unfavorable judgment of Guicciardini is based, I believe, on incomplete readings of Glenie's publications; but Cajori's criticism of the complexity and near-unreadability of some of Glenie's works is well founded. Nevertheless, these works repay close study, for he employed his traditional tools with insight and originality to establish many new results.

Many of Glenie's publications were privately printed pamphlets, with only a few in established journals. Just two papers were published in each of the *Philosophical Transactions of the Royal Society of London* and the *Transactions of the Royal Society of Edinburgh*, and a few pieces appeared in Maseres' eccentric *Scriptores Logarithmici* and in the *Ladies' Diary*. Glenie's main papers on his Universal Comparison and Antecedental Calculus and his several papers on fortification all appeared as pamphlets, and some work remained unpublished in manuscripts that survive in the archives of the Royal Society of London. It may be significant that no papers by Glenie were published in journals of the Royal Society of London after he quarrelled spectacularly with its President in 1784. But he certainly remained involved, for he presented copies of his pamphlets to the Society, and some of his works in manuscript seem to have been read at Society meetings, though not published.

Glenie's first three scientific publications appeared in 1775–1777: two papers in *Philosophical Transactions of the Royal Society of London*, and his 163-page *A History of Gunnery*.[3] It has been suggested that Glenie's interest in gunnery may have been stimulated by Charles Hutton at Woolwich. However, Glenie's work was published in Edinburgh in 1776 and, though dedicated to Viscount Townshend, Master Gunner at the Ordnance, it was surely written while he was working in St Andrews as assistant to the Professor of Mathematics, Nicolas Vilant.

The recurrent themes of Glenie's mathematical publications are new theorems in Euclidean geometry and his "Doctrine of Universal Comparison or General Proportion." The latter involved his own algebra of ratios, by which *ratios of ratios* were manipulated according to various rules. He had begun this work on proportion and also his related "Antecedental Calculus" by 1774, while working as Vilant's assistant. In 1793, the latter calculus was published as a short 16-page tract in continuation of his work on general proportion. Having just been elected a fellow of the Royal Society of Edinburgh, in 1794 Glenie prepared a further paper on the principles of his Antecedental Calculus for that society's *Transactions*. This Antecedental Calculus was Glenie's bold attempt to supplant Newton's fluxionary calculus by one

[1] The next such proof of which the present author is aware dates from 1809, and is due to another Scot, William Spence, who employed more modern analytical methods [Spence, 1809, v–viii].

[2] Hutton and Glenie were friends and associates, with common interests in ballistics and mathematics. Also, during 1773–1818, Hutton edited the *Ladies' Diary* to which Glenie was a contributor.

[3] A full list of Glenie's publications is given in the References at the end of this paper.

based purely on the analysis of his Doctrine of General Proportion, which was free from any extraneous considerations of motion. Several other mathematical papers, mainly geometrical but one giving his novel demonstration of the binomial theorem, were published between 1784 and 1812.

Glenie was born in 1750, just fourteen years after J.L. Lagrange and one year after P.S. Laplace. Even the elderly Leonhard Euler (d. 1783) was still active for seven more years after Glenie published his first papers. But Glenie's mathematical scope did not embrace continental analysis. As a result, his work has long been forgotten, for expertise in Euclidean geometry is no longer a much-valued skill. Furthermore, the complexity of some of his writings, and his overblown claims, would have alienated many readers, even in his own day; and his geometrical and proportional techniques would surely have been regarded as retrograde by those keen to embrace the "new analysis."

It is easy to dismiss Glenie as a reactionary, like Francis Maseres and William Frend who objected to the introduction of modern methods and even to the use of negative quantities in arithmetic. But this seems a misjudgment; for, within his rather obvious limitations, Glenie was seeking to innovate. His work remains interesting on two main counts. First, for his contributions to pure geometry, particularly his proofs of Matthew Stewart's many challenging propositions and his extensions thereof; and, second, for his demonstrations that Euclid's theory of proportion, and Glenie's own extension of it, were capable of providing a basis for the differential calculus and a proof of the binomial theorem.

2. Life of James Glenie

James Glenie (1750–1817)[4] was born in Leslie, Fife, the son of an Army officer. He first matriculated at St Andrews in 1766 with a Foundation Bursary and later won Chancellor's prizes in the second Mathematics class (1769) and the Natural Philosophy class (1770). After graduating M.A. in 1770, he studied at St Mary's College during 1770–1773, supported by a Foundation Bursary in 1770 and 1772, and he won a class prize in 1771 [Smart, 2004, 328]. At St Mary's he earned a reputation as "a keen polemic and theologian," and was expelled from the Theological Society "on account of some conceited speeches of his meant to reflect contempt upon the Club."[5] Quite soon thereafter (the exact dates are unknown), he became Vilant's assistant, probably for two sessions.

It seems quite likely that Glenie met the Astronomer Royal Nevil Maskelyne when the latter was in Scotland during 1774–1775 to conduct his famous experiment to measure the gravitational attraction of the mountain Schehallion. Though Maskelyne does not mention Glenie among his visitors (who included several professors from Glasgow and Edinburgh) [Maskelyne, 1776], it was Maskelyne who in 1775 communicated Glenie's first mathematical paper to the *Philosophical Transactions of the Royal Society*. Its title identifies Glenie as "A.M. of the University of Edinburgh"—a mistake for St Andrews—and it was published with additional notes by Samuel Horsley.[6] Later, in 1779, Maskelyne and Horsley were the first two signatories on the certificate of recommendation of Glenie as a fellow of the Royal Society of London.[7]

Military ambition replaced clerical and, with the support of the Earl of Kinnoull, Glenie was "nominated" an artillery cadet at the Military Academy at Woolwich [Goodwin, 1889]. However, Glenie was by then much older and better educated than the typical cadet, and it seems unlikely that he underwent the usual training. Probably he was fast-tracked to a commission, perhaps following examination of his mathematical, surveying, and artillery knowledge by Charles Hutton, Woolwich's professor of mathematics. Certainly, Glenie's name does not appear in the Army list as a cadet [Johnson, 1998, 204], but he was commissioned as a second lieutenant in the Artillery in November 1776. He was immediately sent to Canada, the American war having begun the year before.[8]

Glenie soon established a reputation as "a promising officer and skillful engineer," building roads, bridges, and garrison buildings. Following involvement in constructing a garrison at the entry of the St Lawrence River into Lake Ontario, he was transferred in February 1779 to the Corps of Engineers as a Practitioner Engineer. This transfer

4 His name is spelt "Glennie" or "Glenny" in some documents.

5 [Chambers, 1870]; [Roberts, 1970, 23] quoting St. Mary's College Theological Soc. Minutes, March 1771.

6 Tantalizingly, a "Ja^{s}. Glennie" [sic] enrolled for just one year at Edinburgh, in the 1774–1775 class of anatomy and surgery of Alexander Monro.

7 A facsimile is in the Royal Society Electronic Archive at http://royalsociety.org.

8 Fuller information about Glenie's activities in Canada is given in Godfrey [2000]; Stanley [1942] and more briefly by Chambers [1870]; Goodwin [1889]; Johnson [1998, 2004].

was at Glenie's request, for he had fallen foul of his commanding officer, a Captain Aubrey, against whom he had organized a petition and lodged a formal complaint. These unwise actions led to Glenie's court-martial in August 1780, when it was recommended that he be discharged from the Army. However, the Commander-in-Chief of Canada, Sir Frederick Haldimand, furnished Glenie with a letter to the authorities in London, recommending clemency in view of his talents. Back in London, Glenie received further support from Marquis Townshend, Master-General of Artillery (to whom Glenie had dedicated his book *A History of Gunnery*). King Geoge III duly remitted Glenie's sentence in February 1781 and restored him to his former rank, while admonishing him for his conduct and urging him to show due deference to his superiors.

In 1780 Glenie married Mary Anne Locke of Plymouth, and they had three children [Johnson, 2004]. Glenie was elected a fellow of the Royal Society of London in 1779, signing the membership roll two years later when back in London. In 1784, he took part in an acrimonious debate following the dismissal by the Royal Society's President, Sir Joseph Banks, of the Society's Foreign Secretary, Charles Hutton, ostensibly for failing to perform his duties. Glenie was one of several fellows who objected to Banks' unilateral action and his prejudice against mathematicians.[9] Through Marquis Townshend, Glenie became involved with the Duke of Richmond's costly proposals for fortifying southern England against a possible French invasion. Glenie was typically outspoken, describing the scheme as "impractical and absurd" [Glenie, 1785]. This pamphlet elicited a strong reply from the Duke, but Glenie's intervention was instrumental in persuading the House of Commons to reject the proposal. Later papers by Glenie on this matter, and on fortification generally, appeared in 1805 and 1807.

Having been reinstated, Glenie was back in New Brunswick working as an engineer during September–October 1785. He was promoted to first lieutenant in 1787 but retired soon afterwards, doubtless aware that, with such a powerful enemy as the Duke, his future in the military was bleak.

In the same year, he and his family left to make a new life in Canada. Settling in Sunbury County, New Brunswick, he soon became involved in radical politics, criticizing the New Brunswick administration and winning a by-election to its Assembly in 1791. When war broke out between Britain and France, Glenie sent an ill-received memorandum to London, criticizing a plan to raise a provincial defence corps as both militarily useless and harmful to the local community. The provincial legislature was several times dissolved, but Glenie was each time reelected. In February 1795 he argued "furiously for religious toleration and colonial autonomy—features highly reminiscent of the American Revolution itself" [Johnson, 1998, 220]. Though his motion was passed by the House of Assembly, it was opposed by the Governor and the House was again dissolved. Now at the height of his popularity, Glenie was awarded the freedom of the city of St John. Back in a new Assembly, he unsuccessfully attempted to censure both the Governor and Council. Thereafter, Glenie's political reputation began to decline, and he was increasingly characterized as a tiresome hothead. Disputes continued, and in 1797 Glenie was wounded in the thigh in a duel.

He returned to England for a time, but was soon back in Canada, applying his engineering expertise to the possibility of building a canal connecting the Bay of Fundy to the St Lawrence River. Though the plan was supported by the press, it came to nothing. Glenie was narrowly reelected to the Assembly, serving until 1803. But his financial situation was becoming desperate, with the failure of a business venture to supply ships' timber and masts to the British government. Accordingly, in 1805, at the age of fifty-five, he returned to England to seek a suitable post.

Back in England, he was retained as "engineer-extraordinary" by the Earl of Chatham, and was appointed an instructor in artillery at the East India Company College. Chatham also secured his appointment as an inspecting engineer to some of the West Indian islands, but it is unlikely that he ever went there. However, in 1810 Glenie was dismissed from all these posts following strong criticism of his testimony at the highly politicized trial of a Member of Parliament, G.L. Wardle, that involved the reputation of members of the Royal Family.

About the last period of his life, less is known. A few publications on geometry and fortification appeared, he did editorial work for reissues of a few mathematical books, and he taught some private pupils. A trip to Copenhagen to negotiate the purchase of a plantation for a Member of Parliament resulted in further financial loss when his claim for compensation failed. Financially ruined, he lived for just five more years. He died at Ebury House in Chelsea,

[9] These objections were published as [Horsley et al., 1784] = [Glenie, 1784a]. Glenie was not allowed to finish his speech. The published version (pp. 67–76) notes on p. 71 that "Here, Mr. Glenie was interrupted—the remainder of what he was going to say, is what follows." A more moderate account of the dispute is Kippis [1784], and it is revisited by Heilbron [1993].

London in November 1817, at the age of sixty-six.[10] His body was interred in St Martin-in-the-Fields churchyard, but at a later date many coffins were removed elsewhere and the present location of his grave is unknown. Brief obituary notices appeared in Leybourn's *Mathematical Repository* and the *Gentleman's Magazine*. The latter is rather inaccurate, referring to Glenie's advances in "squaring the circle," but the former is better informed.[11]

3. The mathematical environment

Before describing Glenie's own work, it is pertinent to recall the central place of Euclidean geometry at this time. It was not only the core of mathematical instruction, but for many remained the one rigorously established branch of mathematics, apart from simple arithmetic. Nowhere was this more so than in Scotland, which had a rich geometrical tradition associated with such luminaries as Robert Simson, Colin MacLaurin, Matthew Stewart, and John Playfair. Furthermore, the "Common Sense" philosophers of the Scottish Enlightenment, especially Thomas Reid, Adam Ferguson, Dugald Stewart, and later Sir William Hamilton, all had mathematical interests and emphasized the primacy of geometry in their writings: see, especially, Davie [1961]; Olson [1975]. (Despite this fact, several Scots were in the forefront of the introduction of "continental" calculus and analysis into Britain; but that is another story.)

There was still much debate about the status of algebra, which gave rise to "impossible" or "imaginary" quantities that apparently had no meaning. For instance, John Playfair's first mathematical paper [Playfair, 1778] defended the usefulness of such "impossible" quantities in yielding true results that might not easily be discovered by "a more rigid analysis" based directly on geometry. In his view, the algebraic method involving complex quantities succeeded by establishing an analogy between certain properties of the hyperbola and those of the circle, properties that, once discovered, might then be firmly established by geometrical analysis.[12] Typical, too, was the attitude of John Robison, Edinburgh's Professor of Natural Philosophy during 1775–1805, who favored geometrical exposition in his textbook *Elements of Mechanical Philosophy* [Robison, 1804] as better suited to instruction than were algebraic manipulations: using the latter, a student could "proceed without ideas of any kind and obtain a result without meaning and *without being conscious of any process of reasoning*."[13] Nevertheless, Robison knew the astronomical work of d'Alembert and Lagrange and appreciated the need for algebraic analysis in such advanced topics. Similarly, Playfair was an early advocate of "continental analysis" and the *Mécanique céleste* of Laplace. Yet as late as 1834, John Leslie (successor to both Robison and Playfair as Edinburgh's Professor of Natural Philosophy) objected not only to "impossible quantities" in algebra, but also to negative numbers. He maintained that numbers were defined by the operation of counting, and that positive and negative signs denoted arithmetical operations that were legitimate only when a given number could be added to or taken from another. (Such a retrograde view was shared also by Francis Maseres and the Cambridge-educated William Frend.)[14]

Though history tells us that this antipathy to complex algebra was misplaced, the philosophical objections had still not been satisfactorily resolved: and Scots students were particularly well versed in philosophy because of its privileged place in the Arts curriculum of every university. It would be wrong to suppose that adherence to the geometrical tradition was the main cause of the delay in British acceptance of algebra and analysis. Rather, there was a spectrum of views, ranging from the head-in-the-sand denials of Maseres and Frend to the cautious welcomes by Robison and Playfair. In particular, the strong Scottish philosophical tradition encouraged efforts to explore and reform the foundations of analytical mathematics. Colin MacLaurin had been a master of Newton's calculus of fluxions and fluents, but felt it necessary, in his *A Treatise on Fluxions* [Maclaurin, 1742], to provide a firmer basis employing the "method of the ancients"—essentially the geometrical method of exhaustion of Eudoxos, Euclid, and Archimedes. And Matthew Stewart sought to enlarge the scope of geometry itself. This is most apparent in Stewart's *Some General Theorems of Considerable Use in the Higher Parts of Mathematics* [Stewart, 1746], described by John Playfair as "among the most beautiful, as well as the most general, propositions known in the whole compass of geometry" [Playfair, 1788, 6].

[10] It is suggested in Johnson [1998, 222] that this may have been a refuge for destitute old soldiers, connected with the Royal Hospital for Chelsea Pensioners.

[11] *Leybourn's Mathematical Repository*, **4** (new series), p. 76. London 1819; *The Gentleman's Magazine* **87** (1817), ii, pp. 571–572.

[12] See Olson [1975, 164–167].

[13] Quoted by Olson [1975, 161].

[14] [Leslie, 1834], also quoted in Olson [1975, 190–192].

Matthew Stewart (1717–1785), father of Dugald Stewart, was Professor of Mathematics at Edinburgh during 1747–1775, in succession to Colin MacLaurin. Both MacLaurin and Matthew Stewart had studied with Robert Simson in Glasgow, and were imbued with Simson's love of Euclidean geometry. Unlike MacLaurin, whose work was more wide-ranging, Stewart approached mathematics and astronomy from a strictly geometrical standpoint, even though calculus (or fluxions) and analysis were by then replacing geometry as investigative tools.

Stewart's *General Theorems*... was his main geometrical publication. Written when he was serving as a parish minister of the Church of Scotland, it secured Stewart's appointment to the Edinburgh chair. In it, sixty-four Propositions are stated, all but the first five without proof. In his Preface, Stewart wrote that "to explain, in a proper way, so many theorems, so general, and of so great difficulty as most of these are, would require a greater expence [sic] of time and thought than can be expected soon from one in the author's situation. He therefore thought it was better they should appear in the way they now are, than lie by him till an uncertain hereafter." The proofs may also have been withheld in deference to Robert Simson, whose work on porisms was under preparation, and to which Stewart made several contributions. But Simson's work was not to be published in his lifetime. The recent scholarly edition [Tweddle, 2000] contains more information about the collaboration of Simson and Stewart, as also does [Playfair, 1788].

The novelty of Stewart's theorems, now easily overlooked, is that they extend the scope of Euclidean geometry to powers of quantities greater than the third. Recall that, in Euclid, magnitudes could be only numbers, angles, lines, areas or volumes: areas were defined by the two adjacent sides (or lines) of a rectangle, and volumes by the three adjacent sides of a rectangular parallelepiped. As carefully discussed in Grattan-Guinness [1996], Euclid did *not* regard areas as products of two lines, or volumes as products of three lines: this inference was drawn only later, with the advent of algebra. Even allowing this later interpretation, no geometrical meaning was attached to products of four or more lines, and there are no theorems in Euclid that involve them. However, in algebra, quantities can be raised to any power; and Stewart deduced many geometrical theorems concerning higher integer powers of lines constructed in various ways.

Though many apparently tried to prove Stewart's *General Theorems*, few succeeded. The only two to publish proofs were Robert Small, a Church of Scotland minister in Dundee,[15] and our present subject, James Glenie. All of Small's demonstrations [Small, 1790] concern "sums of squares and 4th powers of lines drawn in a certain manner" *not* involving a circle. Though he promised that "The theorems that respect the cubes and other higher powers, may afford materials for another paper," no such paper was ever published. Small proved Stewart's theorems numbered 6 to 14 and 16 to 18.

Glenie's teacher and employer at St Andrews, Nicolas Vilant, was also well versed in geometry, which of course formed a considerable part of his two mathematical classes. But Vilant was also interested in analysis, and published a rather odd work entitled *The Elements of Mathematical Analysis, abridged. For the Use of Students* [Vilant, 1783/1798], parts of which were for a time used in his courses—or rather those delegated to his later assistants: see Craik and Roberts [2008]. Sections of this work are devoted to algebraic series, both finite and infinite. Despite Glenie's later acquaintance with John Playfair and Dugald Stewart in Scotland, and Charles Hutton and Nevil Maskelyne in England, it seems likely that Vilant was not just the first but also the greatest influence on Glenie's mathematical outlook. From Vilant he would first have heard of the controversies about the foundations of algebra and calculus and the still-unproved *General Theorems* of Stewart, and he would have gained practice in the manipulation of algebraic series. He perhaps also heard something of the then–current philosophical concerns on the nature of mathematics from Robert Watson, St Andrews' Professor of Logic, Rhetoric and Metaphysics.[16]

In 1797, Vilant prepared a "28-page Synopsis of Book V. of Euclid's Elements" that he appended to the enlarged 1798 reissue of his textbook (and which seems no longer to exist separately). This is the first of Euclid's Books devoted to the "theory of proportion," concerning ratios of magnitudes of like sort. Vilant's "Synopsis" is based on Simson's scholarly 1781 edition of Euclid's *Elements*, but employs a much more compact notation. Vilant selected this Book "for the use of Students, who too generally neglect the truly elegant demonstrations of Euclid, in this most useful part of the Elements, and thereby obstruct their improvement in the other branches of Mathematics; and in Natural Philosophy, very considerably" [Vilant, 1783/1798, Preface to "Synopsis"]. Though this work was published

[15] Robert Small (1733–1808) is said to have attended St Andrews University, but no record survives. More information about him is in Norrie [1873, 21–22]; Scott [1915–1928, 5, 316–317].

[16] Robert Watson (1730–1781) held the chair during 1756–1778, and was Principal of the United College during 1778–1781. Though not a noted philosopher, he was a respected historian.

some time after Glenie had left St Andrews, it seems likely that Vilant had impressed Glenie with the notion that Euclid's theory of proportion could provide a basis for improving "other branches of Mathematics": certainly, this is what Glenie tried to do, intermittently, and with some success, throughout his life.

4. Glenie's geometrical works

Glenie's first published paper [Glenie, 1776a], "Propositions Selected from a Paper on the Division of Right Lines, Surfaces, and Solids," gives a first taste of his methods employing the theory of proportions. It had been communicated to the Royal Society of London by the Astronomer Royal, Nevil Maskelyne, and it was published with two clarificatory footnotes by Samuel Horsley, who had objected to some of Glenie's claims.[17]

Proposition I concerns the division of the base of a given triangle into two parts such that the ratio of these parts is equal to the "duplicate proportion" of the other two sides [p. 73].[18] That is, given the triangle ABC, cut the base AB at D, such that $AD : DB :: (AC)^2 : (CB)^2$. Proposition III is "To multiply a square of a given finite right line by any number" [p. 79]. Though the constructions mainly apply to integer multiples, Glenie briefly indicates how these may be extended to fractional multiples "having four for their common denominator and so on." Proposition IV is "To find a right line, the square on which shall be equal to the square on a given right line, divided by any given number." He cursorily observes that this follows immediately from the preceding Proposition, for divisors "2, 4, 8, 16, ..., 1024 &c.; and so on for other numbers, whole, surd, fractional, or mixed" [p. 81]. The final Proposition IX (also called "Theorem IV") is a generalization of Proposition I, concerning two "rhomboids" or parallelograms constructed on the sides AC, BC of a given triangle ABC.

In this paper, Glenie dealt only with ratios of lines and ratios of squares of lines. Later, at the end of his *Antecedental Calculus* [Glenie, 1793], he gave constructions, without proofs, that solve six geometrical problems involving ratios of *cubes* on given lines. Soon after, he posed the related prize-winning problem of the 1794 *Ladies' Diary* [Glenie, 1794/95], that was later enthusiastically (and misguidedly) taken up by Francis Maseres as "Mr Glenie's Problem" in his *Scriptores Logarithmici* [Maseres, 1791–1807, 4: xi–xxii, 333–412; 6: 31–43].

Maseres, a prominent lawyer and a keen amateur mathematician, spent the years 1766–1769 in Canada as Attorney General of Quebec. He then resumed his legal career at the Inner Temple and served as Cursiter Baron of the Court of Exchequer from 1773 until his death in 1824. He inherited a large fortune that he used to finance his many publications. His privately published mathematical compilations are mostly reprints of others' works (including historical pieces by such authors as Napier, Halley, James Gregorie and Raphson as well as more recent articles), accompanied by Maseres' own eccentrically verbose prefaces and additions. He and Glenie would not have met in Canada, but they certainly did so at the Royal Society, for both contributed to the 1784 debate criticizing the president, Sir Joseph Banks.[19]

Maseres' discussion of "Mr Glenie's Problem" does not warrant repetition, and serves only to confirm his inadequacy as a mathematician. But Glenie's prize-winning problem is worth noting, as an example of the popular, and far from easy, mathematical puzzles that were set in the *Ladies' Diary*:

> In the Palace of one of the Persian kings, 'tis said, there was a triangular area, such, that the cubes of two of the sides were, together, equal to thrice the cube of the third side, which was 200 feet in length; and that the area itself contained exactly 10,000 superficial feet. Supposing this to have been really the case, it is required to construct the triangle by common, or plane, Geometry.

[17] The Journal Book of the Society records the occasion, on 18 May 1775, when the paper was read. Glenie had apparently claimed that by his methods the fraction 1/781,250,000,000 of a line may be cut off "in less than half an hour," whereas applying Euclid's method, or "that of Dr. Simpson," "would have taken up all the time that is elapsed since the Creation." But Horsley had objected that "the Author was mistaken in this comparison, & that the advantages were not quite so considerable ... as he seemed to believe" [Journal Book of the Royal Society, XXVIII, 254]. Conveniently, the page is reproduced in Johnson [1998, 210].

[18] For future reference, the following should be noted. The sign ":" means "is proportional to," and "::" is the equality sign for proportions. If $a : b :: c : d$, then d is said to be a "fourth proportional" to a, b, c; and if $a : b :: b : c$ then c is a "third proportional" to a and b. The "compounded" ratio of $a : b$ and $c : d$ is $ac : bd$, and their "decompounded" ratio is $ad : bc$. The duplicate and triplicate ratios of $a : b$ are $a^2 : b^2$ and $a^3 : b^3$, respectively. Though the equivalence in algebra with multiplication and division of fractions a/b, c/d is obvious, the theory of proportion, with its geometric origins, was long regarded as a rather separate branch of mathematics.

[19] See Heilbron [1993].

Glenie's demonstration and three others were published in the *Ladies' Diary* for 1795 and reprinted in Leybourn [1817]. Maseres commended Glenie's construction as "very simple and elegant," which it is.[20]

Between 1778 and 1799, Glenie contributed nine other geometrical problems and their solutions to the *Ladies' Diary* [Leybourn, 1817, Nos. 734, 795, 811, 840, 854, 994, 1007, 1023, 1033] = [Glenie, 1778–1799]. Most are on pure Euclidean geometry, with frequent use of the theory of proportion, but two concern applications to fortification. There is no evidence that Glenie submitted solutions to any other problems set in the *Ladies' Diary*; probably, his itinerant lifestyle meant that he did not receive copies in time to do so.

Furthermore, Glenie never published proofs for all the constructions in his *Antecedental Calculus* involving cubes of lines. The first of these is: "On any given right line as a base to constitute a triangle such, that the cubes on the two other sides shall together be equal to the cube on the said given right line or base." The second and third problems respectively require the sum of the two cubes to be equal to twice and three times the cube on the base line. (Obviously, the third problem is equivalent to that in the *Ladies' Diary*.) The fourth, fifth, and sixth problems require the difference of cubes to be equal, respectively, to the cube, twice the cube, and three times the cube on the base. It is possible that some of the missing proofs had formed part of the 1776 work submitted to the Royal Society, for the resulting paper gave only "selected" propositions. Glenie himself points out that construction of certain surd quantities involving square roots is much facilitated by some propositions in that paper.

Boastfully, Glenie concluded that [Glenie, 1793, 16]:

> In this manner, and with equal facility, could I proceed indefinitely constructing such Problems by plane Geometry, had I only leisure sufficient for this purpose. I have been for years in possession of geometrical principles, by which millions of them can be constructed. Between every two of these also, an indefinite number of others may also be constructed...; and so on without end.
>
> This is a geometrical field, which neither the Ancients nor Moderns seem so much as even to have looked into, unlimited both as to extent and variety; and it furnishes the means of enriching pure Geometry infinitely more than all that has been written by Sir ISAAC NEWTON and other ingenious men on curves and lines of different orders.

Clearly, Glenie believed that he was not just providing a few new geometrical results, but enlarging the field of geometry itself, by applying his proportional methods.

The absence of proofs that Glenie's constructions yield the desired results is frustrating, and some effort is required to verify that they are all correct, which they are. Also, he omits to mention that his solutions are *not unique*, and it is not entirely clear that he appreciated that this is so. To see this, consider a given line AB with endpoints $(0,0)$ and $(0,a)$, and let C at (x,y) be the third point of a triangle such that the cubes on the three sides satisfy the prescribed property. This leads to a relation of the form $f(x,y)=a^3$, which describes a curve in the x–y plane. Clearly, another condition must be imposed for uniqueness. This was done in the *Ladies' Diary* problem by specifying the enclosed area, and this fact suggests that Glenie was aware of its necessity. But he imposes no such subsidiary conditions on his other constructions. So where do his solutions come from? One might have expected that the solutions to his first and fourth problems (to find triangles such that the cube on the base respectively equals the sum and difference of the cubes on the other two sides) would have yielded similar triangles with different bases. However, this is not so: by different constructions, Glenie finds two different members of the general solution. The construction by Euclidean means of other such solutions, and generalizations thereof, were much later explored by another part-time mathematician, Pedro A. Pizà [Pizà, 1945, 1946].

Glenie's most substantial geometrical paper is his 49-page "A Geometrical Investigation of Some Curious and Interesting Properties of the Circle, &c." [Glenie, 1812a], which had been read to the Royal Society of Edinburgh some seven years earlier, in April 1805. This paper was sent by Glenie to Dugald Stewart, the Professor of Moral Philosophy at Edinburgh, with this note: "As the following paper refers in great measure to the *general theorems* published by your father, I now commit it to your care, and that of my friend Mr PLAYFAIR, Professor of Natural Philosophy." He begins by observing that those theorems of Matthew Stewart "as refer to the circle, and to regular figures inscribed in, and circumscribed about it, have not, as far as I can understand, been yet demonstrated. These, with an endless variety of other theorems, are derivable, as corollaries, from the following general though simple geometrical investigation,

[20] The construction is also given on Maseres' p. 336, with the demonstration reworked at greater length by Maseres on pp. 336–338. The square roots of the numbers $\frac{14\pm\sqrt{96}}{3}$ being required, he gives constructions of these on pp. 338–343.

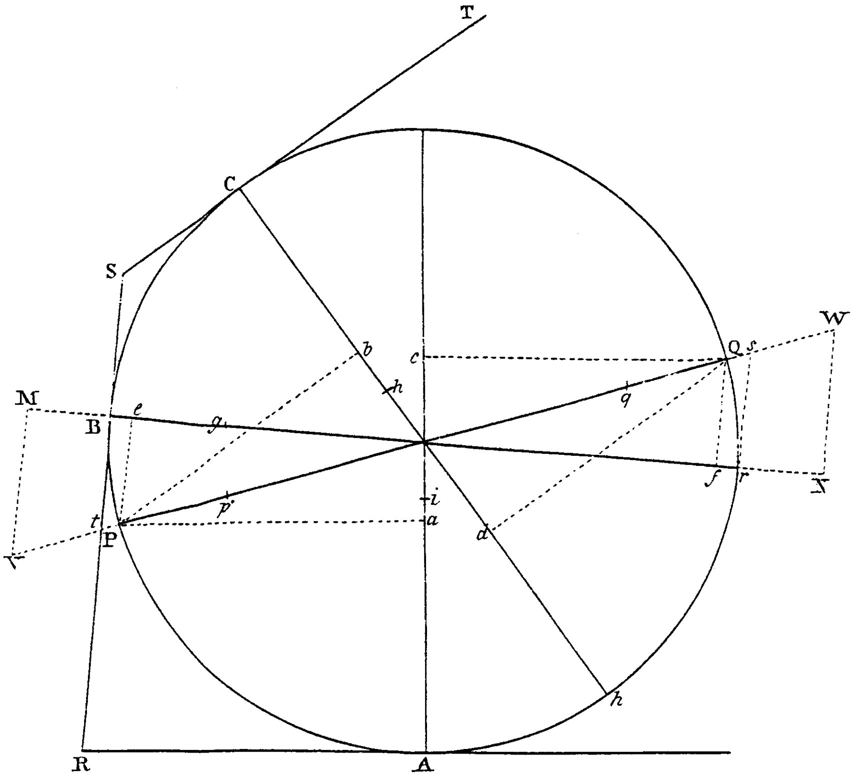


Fig. 1. From Glenie [1805a].

that occurred to me fifteen years ago" [p. 22].[21] Though Glenie would surely have known Small's earlier paper [Small, 1790] on those theorems *not* involving the circle, he does not mention it, perhaps supposing it to be familiar to readers of the journal.

It is inappropriate to attempt here more than a brief summary of Glenie's long and difficult paper: to borrow Glenie's own words from another context, this "would swell this paper beyond its intended size." In its course, Glenie gives proofs of 20 of Stewart's theorems: those numbered 1, 2, 3, 4, 5, 11, 19, 20, 22, 23, 25, 26, 28, 29, 36, 37, 38, 39, 40, and 41, and, incidentally, proofs of several of his own constructions in *Antecedental Calculus*.

For instance, on pp. 59–60, in an example starting with a regular pentagon circumscribing a circle with center G, Glenie constructs lines GD, GF, GV such that $2\overline{GF}^3 - 2\overline{GV}^3 = \overline{GD}^3$. This leads on to two "easy" problems: "Having two equal right lines given, it is required to cut one of them in two parts, and the other in three parts; so that the cubes of the two parts ... shall, together, be equal to the cubes on the three parts, ... taken together"; and "On a given right line, to constitute a triangle, such that twice the difference of the cubes on the other two sides, shall be equal to the cube on the given line" [p. 60]. Though Glenie's great geometrical facility led him to many such new results, his starting with a particular construction (here a regular pentagon) restricted him to particular solutions, rather than yielding the general solutions revealed by analytical geometry and numerical computation.

Fig. 1 shows Glenie's first figure, which provides the basis of his general method. There, A, B, C, etc. are any number n of points in the circumference of the circle; RA, RS, ST, etc. are tangents at A, B, C, etc.; and POQ is any diameter through the center O of the circle. Lines Qc, Qf, Qd, etc. and Pa, Pe, Pb, etc. are constructed perpendicular to the diameters through the points A, B, C, etc. On noting that Aa, Be, Cb, etc. are respectively equal to the perpendiculars from P to the tangents RA, RS, ST, etc., and that Ac, Bf, Cd, etc. are similarly equal to the perpendiculars from Q to the same tangents, a simple argument shows that the sum of all the perpendiculars from P and Q to all the lines touching the circle at A, B, C, etc. is equal to $PQ \times n$. It simply follows that, if p, q are any

[21] Presumably this is 15 years before 1805, when his paper was read.

two points in the diameter PQ that are equally distant from O, the sum of all the perpendiculars from these points to the tangents through A, B, C, etc. is likewise equal to $PQ \times n$ [p. 23].

He next finds the sum of all the squares, and then the sum of all the cubes, of the perpendiculars from P and Q to the tangents through A, B, C, etc. In the special cases when A, B, C, etc. are equally spaced round the circle, these sums reduce to $3nr^2$ and $5nr^3$, respectively, where r is the radius of the circle. Already, his results yield proofs of Stewart's 1st, 2nd, 3rd, 11th, 19th, and 23rd theorems [pp. 24–27].

Next, he considers "third proportionals to the cube of the diameter and the cube of the chords drawn from P and Q to the point A, B, C, &c.," where A, B, C, etc. are n regularly spaced points: these are the quantities $(AP)^6/d^3$, $(AQ)^6/d^3$, $(BP)^6/d^3$, $(BQ)^6/d^3$, etc., where $d(=2r)$ is the diameter of the circle. He thereby shows that "the sum of the sixth power of the chords drawn from either P or Q to the said points, will be equal to twenty times a multiple of the sixth power of the radius, by the number of the sides of the inscribed figure" [pp. 27–28]: in other words, $(AP)^6 + (BP)^6 + (CP)^6 + \text{etc.} = (AQ)^6 + (BQ)^6 + (CQ)^6 + \text{etc.} = 20nr^6$.

Another application of third proportionals, this time to a regular circumscribed polygon, yields a proof of "Dr STEWART's 25th theorem": that, "to speak algebraically, eight times the sum of the fourth powers of perpendiculars from either P or Q to the sides of a regular figure of a greater number of sides than four circumscribed about the circle, and touching it in the points A, B, C, &c. are equal to thirty-five times the multiple of the fourth power of the radius by the number of the sides of the figure" [p. 29]. A similar application of third proportionals to an inscribed regular polygon gives the sums of the eighth powers of the chords from either P or Q to A, B, C, etc. as

$$70nr^8 = n \times \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 2^4 \cdot r^8.$$

He proceeds to consider tenth powers, finding the sums equal to $63n(rd)^5/4$; and then the sums of powers $2m$, where m is any integer, which yields Stewart's 36th and 38th theorems [pp. 29–30].

To establish these results, Glenie first shows that

$$\frac{(AP)^{2m} + (BP)^{2m} + (CP)^{2m} + \&c.}{d^m r^{m-3}} + \frac{(AQ)^{2m} + (BQ)^{2m} + (CQ)^{2m} + \&c.}{d^m r^{m-3}}$$
$$= \frac{(r + Oc)^m + (r - Oc)^m}{r^{m-3}} + \frac{(r + Of)^m + (r - Of)^m}{r^{m-3}} + \frac{(r + Od)^m + (r - Od)^m}{r^{m-3}} + \&c.$$

and that the latter equals

$$2nr^3 + \frac{m}{1} \cdot \frac{m-1}{1} \cdot r \times \big((Oc)^2 + (Of)^2 + (Od)^2 + \&c.\big)$$
$$+ \frac{m}{1} \cdot \frac{m-1}{1} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \times \frac{(Oc)^4 + (Of)^4 + (Od)^4 + \&c.}{r}$$
$$+ \frac{m}{1} \cdot \frac{m-1}{1} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \cdot \frac{m-4}{5} \cdot \frac{m-5}{6} \times \frac{(Oc)^6 + (Of)^6 + (Od)^6 + \&c.}{r^3} + \&c.$$

Here, O is the center of the circle and the points c, f, d are as indicated in Fig. 1. (Glenie carelessly uses d twice over, to denote both this point and also the diameter of the circle. Fortunately, no confusion results, as the point d arises only in combination with O as the line Od.)

He then states a result "more general than any of Dr STEWART's theorems, and will furnish an endless number of new and curious infinite series, with their summations." This result is connected with Glenie's "Theory of Universal Comparison," but is here applied to a regular polygon of n sides, inscribed in a circle. Recalling his derivation of the sum $(AP)^{2m} + (BP)^{2m} + (CP)^{2m} + \text{etc.}$ of the $2m$th powers of the n chords, where m is any positive integer, he asserts, but does not prove, a corresponding result involving the (m/l)th powers of lines where m has to l "any ratio whatsoever." This "more general" result is that shown above, relating powers of $r + Oc$, $r + Of$, etc. to a series of sums of even powers of Oc, Of, etc. But now the integer m is replaced by m/l and so the series in even powers does not usually terminate. Glenie states that this result may "be extended to the chords AP, BP, &c. and expressed in terms of them," but he does not do so.

A connection with the binomial theorem is clear, and Glenie goes on to say that “as to the truth of the binomial and residual theorems, when m has to l the ratio of any two homogeneous magnitudes whatsoever, I must refer the reader to my general demonstration of both in Baron MASERES’S *Scriptores Logarithmici*, vol. 5 and to some of the geometrical formulae in my *Universal Comparison*” [pp. 30–31]. On p. 33, he gives a corresponding general result for circumscribed, rather than inscribed, polygons; and he claims that, together, these results will supply “an unlimited number of other theorems, respecting figures both regular and irregular, circumscribing and inscribed in the circle.”

The above remarks summarize only the first 13 pages of Glenie’s long 49-page paper, but enough has been said to give the flavour of the rest. The reader will surely already be convinced that, despite his boastfulness, Glenie was a very able and original geometer who employed his familiarity with the theory of proportion to good effect. Just as Stewart had believed (rightly) that he was enlarging the scope of geometry to admit higher powers of lines, Glenie also thought that he had taken a large step further, not only giving proofs of Stewart’s theorems that had defied all comers for 60 years, but also generalizing several of them by means of his novel application of proportional techniques. Present-day unfamiliarity with such close geometrical arguments should not be allowed to conceal the extent of his achievements, though this paper cannot be described as easy reading.

Glenie returned to another of Stewart’s Theorems in a late paper sent to the Royal Society of London, published in 1812 as a separate eight-page tract. This is *A Demonstration of Dr. Matthew Stewart’s 42nd Proposition, or 39th Theorem, Which Has Remained without One for Sixty-Five Years* [Glenie, 1812c]. The theorem in question concerns the sum of the sixth powers of lines drawn from an arbitrary point to each corner of a regular polygon inscribed in a circle. Glenie explains that this is the only one of Stewart’s Propositions relative to the circle that he had not proved in his earlier paper. Here, Glenie not only proves Stewart’s result, but generalizes it “alike to regular and irregular figures described in and about the circle.” This is described more fully in Johnson [1998, 214–216].

5. The binomial theorem

In volume V of his *Scriptores Logarithmici*, Maseres wrote that “The third Tract in this Volume is a letter sent to me by James Glenie of the Province of New Brunswick in North America, containing a Demonstration of Sir Isaac Newton’s Binomial and Residual Theorems” where the powers p/q of $(1+x)^{p/q}$ and $(1-x)^{p/q}$ are such that p and q denote “not only whole numbers, but any surd numbers ... or any other quantities whatsoever that are incommensurable to 1” [Maseres, 1791–1807, 5: xxiv]. In other words, Glenie claims to prove the Binomial Theorem for irrational as well as for rational powers. Note that Glenie distinguishes the “binomial theorem” for powers of $1+x$ and the “residual theorem” for powers of $1-x$, x being regarded as positive.

This ten-page tract [Glenie, 1799] was sent to Maseres as a letter dated April 30, 1799. He is characteristically forthright about the importance of his discoveries:

> These theorems, I believe, have never been demonstrated in a general way or when p has to q any ratio whatsoever, by any other person, by means of genuine and common algebräick operations and the doctrine of proportion—At least I can truly aver, that I have never either seen or heard of such demonstrations. [p. 213]

Elsewhere [Glenie, 1812b, 26], he mentions that this demonstration “had lain by me for years before I sent it in 1799 to Francis Maseres Esq.” One must wonder why he did not publish it earlier, and in a more prominent place than Maseres’ volumes: certainly, it is likely to have been noticed by few of his contemporaries.

Before examining Glenie’s demonstration, it should be mentioned that, in most modern textbooks, proofs of the binomial theorem with irrational indices rely either on the differential calculus (where it follows directly from Taylor’s Theorem) or on a limit process that involves an infinite sequence of rational indices that converge to the chosen irrational one. But proofs based on calculus are still considered inadmissible by some, as involving circular argument. In Glenie’s time, British attitudes towards the need for rigorous proof were lax. In his *The Elements of Algebra* of 1740, Nicolas Saunderson stated but did not prove the binomial theorem, even for *rational* powers, saying merely that “I might have demonstrated this theorem ... but I suppose by this time the reader has enough of it, at least till he sees it’s use” [Saunderson, 1740, 1, 616]. As late as 1875, the seventh edition of Isaac Todhunter’s popular *Algebra for the Use of Colleges and Schools* [Todhunter, 1875, 309] claimed that Leonhard Euler’s proof for rational indices establishes the truth of the theorem when the index is *any* positive quantity; and [Kelland, 1871] is similarly unconvincing. Nevertheless, a satisfactory proof for arbitrary real powers had been given in the 1820s by Cauchy, as amended by Abel: see, for example, Edwards [1979, 312]; Jahnke [2003, 172–173, 177–178]. And an almost satisfactory proof was given even earlier in Spence [1809], but lacking discussion of convergence.[22]

Glenie's attempt long pre-dates those of Cauchy and Abel, and almost certainly comes before Lagrange's supposed algebraic proof of Taylor's theorem [Lagrange, 1797] (from which the binomial theorem would follow as a special case, but which is now considered unsatisfactory). In comparison with these authors, Glenie's mathematical approach was very old-fashioned. Yet a case can be made for Glenie as the author of the first *almost* satisfactory proof of the binomial theorem for all real powers; and he was certainly one of the first British authors to assert the need for such a proof for nonrational powers. Here, we concentrate on the key parts of his demonstration, which is entirely couched in the language of ratios. (On notation and terminology, see footnote 18 above.)

First, he observes that, if the ratio $1 + x : 1$ is compounded m times, where m is any positive integer, then the composite ratio is

$$1 + \frac{m}{1}x + \frac{m}{1}\frac{m-1}{2}x^2 + \&\text{c.} : 1. \tag{a}$$

Here, he supposes that x is positive. Next, he considers the ratio

$$1 + \frac{p}{q}x + \frac{p}{q}\frac{p-q}{2q}x^2 + \&\text{c.} : 1, \tag{b}$$

where p and q are any positive numbers. Note that this series is usually infinite, whereas the former terminates after $m + 1$ terms. He then states that the latter ratio compounded n times, where n is any positive integer, is

$$1 + \frac{np}{q}x + \frac{np}{q}\frac{np-q}{2q}x^2 + \&\text{c.} : 1. \tag{c}$$

No proof of this is given, but one may accept it as the result of direct computation of the nth power of the previous polynomial expression. (This result would have been well known to Glenie.)

He then sets out a table of four magnitudes and their multiples:

	The magnitudes	Their multiples
1st	p	np
2nd	q	mq
3rd	ratio (b)	ratio (c)
4th	$1 + x : 1$	ratio (a)

Here, he interprets ratio (c) as the nth *multiple* of ratio (b), and ratio (a) as the mth multiple of the ratio $1 + x : 1$. In his own words,

> Here there are four magnitudes; and any equimultiples whatsoever being taken of the 1st and the 3rd, and also any equimultiples whatsoever being taken of the 2nd to the 4th, it is proved that, if the multiple of the 1st be equal to the multiple of the 2nd, the multiple of the 3rd is also equal to the multiple of the 4th; if greater, greater; and if less, less. Wherefore (5 Def. E.5) the magnitudes themselves are proportional, or the ratio $1 + \frac{p}{q}x + \frac{p}{q}\frac{p-q}{2q}x^2 + \&\text{c.} : 1$ is to the ratio $1 + x : 1$ as $p : q$ or $p/q : 1$.
> But the ratio of $\overline{1+x}|^{p/q}$ to 1 is to the ratio of $1+x$ to 1 as $p/q : 1$. Consequently $\overline{1+x}|^{p/q}$ is $= 1 + \frac{p}{q}x + \frac{p}{q}\frac{p-q}{2q}x^2 + \&\text{c.}$ universally, whatever be the ratio of p to q.

Having dealt with $(1 + x)^{p/q}$, Glenie goes on to discuss $(1 - x)^{p/q}$, $(1 + x)^{-p/q}$, and $(1 - x)^{-p/q}$ in similar fashion but more briefly.

[22] Other less worthy British attempts around this time are Robertson [1795, 1806] and Sewell [1796], restricted to "fractional" (rational) indices.

By "5 Def. E.5", Glenie means the 5th definition of Book V of Euclid's *Elements*. This apparently rather contorted definition establishes when two ratios of magnitudes are, or are not, equal by associating them with inequalities satisfied by any integer "equimultiples": see, for example, Simson [1781]. In his dictionary article on "Proportion," Charles Hutton characterized this definition as "obscure" [Hutton, 1796, v.2, 296] and, in his *Elements of Geometry*, John Playfair devotes a Note of over four pages to it [Playfair, 1795]. Playfair alludes to "the difficulty that most people find of reconciling the idea of proportion which they have already acquired, with the account of it that is given in this definition." But, after considering alternatives, he concludes that "All the attempts, indeed, that have been made to demonstrate the properties of proportionals rigorously, by means of other definitions, only serve to evince the excellence of the method by the Greek geometer, and his singular address in the application of it." And it was Euclid's Book 5 that Glenie's own teacher, Nicolas Vilant, singled out for special treatment in a supplement to his *Elements of Mathematical Analysis* [Vilant, 1783/1798].

The subtlety of the definition lies in the fact that it admits consideration of ratios of quantities p and q that are mutually incommensurable. The inequalities of the definition establish that $(1+x)^{p/q}$ and series (b) (or, as Glenie would have it, their ratios to unity) are equal, where the index p/q may be irrational. This follows by comparing arbitrary "equimultiples" m and n of the chosen expressions, where m/n is always a rational number. In effect, this procedure enables irrationals p/q to be trapped by the rationals m/n which, if one wishes, may be taken indefinitely close to p/q. But Euclid's definition avoids the need to apply such a limit concept, which would only later enter analysis. (In a sense, Euclid's definition can be interpreted as serving as a definition of the real numbers, not unrelated to Dedekind's much later "cuts" of the real line.)

To prove the greater and lesser inequalities for the ratios (c) and (a) when np is greater or less than mq, Glenie simply uses term-by-term comparison. Though the result is obvious for the first $m+1$ terms of the corresponding series (since all are then positive), difficulties arise with later terms that are zero in (a) and alternate in sign in (c). For these higher terms, more careful arguments are required than are given by Glenie. But we may charitably allow him this gap in his proof.

A greater difficulty is his use of Euclid's 5th definition as the key to his proof. This was intended to apply to all "magnitudes," whether numbers, angles, lines, surfaces, or volumes, provided that, when ratios are taken, like is compared with like. Euclid surely did not mean to admit *ratios* as "magnitudes," but this is precisely what Glenie does. However, ratios are not subject to the same laws of addition and multiplication as those that apply to numbers. Rather, they have their own rules for manipulation. In particular, the nth *multiple of a ratio* is that ratio compounded n times with itself: that is to say, the nth multiple of the ratio $a:b$ is the ratio $a^n : b^n$. Accordingly, "multiplying" a ratio $a:b$ by n is algebraically equivalent to raising the fraction a/b to its nth power, $(a/b)^n$. This last operation is clearly not a "multiple" according to Euclid's definition.[23] Ratios may be "compounded," not "multiplied," and confusing the two has linguistic rather than mathematical origins. It may be that Glenie was simply misled by the then current terminology for compound ratios as "duplicate," "triplicate," and "multiplicate." More likely, he believed that, by extending Euclid's definition to admit ratios as magnitudes, he was merely defining their law of equality. But, given the correspondence between ratios and fractions, this is surely illegitimate without further justification, as it brings exponentiation as well as multiplication into the definition.

Robert Simson of Glasgow was aware of such linguistic difficulties. In his Notes on Euclid's Book V, he objected to early demonstrations of Proposition X "for the words *greater*, *the same* or *equal*, *lesser* have a quite different meaning when applied to magnitudes and ratios. ... and it seems that he who has given the demonstration, has been deceived in applying what was manifest when understood of magnitudes, unto ratios" [Simson, 1781, 311–312]. Though such a critical judgment may also apply to Glenie's reasoning, his proof can be rescued by using logarithms as follows. Take the third and fourth magnitudes to be the logarithm of the series in (b) and the logarithm of $(1+x)$ respectively; then choose the third multiple to be the logarithm of the series in (c), which is n times the new third magnitude, and let the fourth multiple be the logarithm of the series in (a), which is m times the new fourth magnitude. Euclid's Definition 5.5 is then directly applicable to quantities, rather than ratios, and the result follows. Glenie might not have been happy with this abandonment of his beloved proportionals, but he would have been familiar with the idea of logarithms as the "measure of ratios." (For instance, Saunderson [1740, 2: 619] wrote that "Logarithms are so called from their being the arithmetical or numerical exponents of ratios.") Logarithms rarely appear in Glenie's writings,

[23] On the connections and distinctions between algebraic operations and the procedures of Euclid's *Elements*, see Grattan-Guinness [1996].

though they do so in connection with numerical calculations in his *Gunnery* [Glenie, 1776b]. It does not seem fanciful to imagine that he or one of his contemporaries *could* have amended the proof of the binomial theorem in this way, but no one did so.

Though using logarithms, even incidentally, renders the proof "analytic" rather than "algebraic" (since logarithms are transcendental functions), it seems to the present writer that Glenie's demonstration, with the above modifications, can be made quite satisfactory, apart from his complete lack of discussion of convergence of series. But he received little or no credit for it, no doubt because of his reliance on Euclidean methods, and also because it was buried as a short letter in Maseres' turgid volumes. Yet he was not alone in looking to the past in his quest for mathematical rigour. He would have been familiar with the attempt of Colin Maclaurin to base Newton's theory of fluxions on an extension of Euclidean and Archimedean geometry [Maclaurin, 1742, vol. 1]. This retained Newton's idea of motion as the basis of fluxions; but it did so rigorously, recovering the standard results of fluxions of powers, exponentials and logarithms, though at the large cost of lengthy geometrical arguments employing the "method of exhaustion." And, as discussed below, Glenie himself attempted to remove motion altogether in his Antecedental Calculus. Only much later was the notion of actual vanishingly small infinitesimals circumvented by the formalization of limits, using the now well-known ε–δ notation: though even this owes much to Greek geometric antecedents, it at last freed analysis from geometry.

6. The doctrine of general proportion

Glenie's first published account of his extended theory of proportion appeared in the *Philosophical Transactions of the Royal Society of London* [Glenie, 1777] in an eight-page paper entitled "The General Mathematical Laws which Regulate and Extend Proportion Universally; or, a Method of Comparing Magnitudes of Any Kind Together, in All the Possible Degrees of Increase and Decrease." He begins by observing that, though Euclid's doctrine of proportion and its application in his *Elements* "form the basis of almost all the geometrical reasoning made use of by mathematicians ... proportional magnitudes have seldom been carried beyond the triplicate *ratio*" that relates the proportion of similar solids to that of their linear dimensions. This, however, is "a very limited portion of universal comparison," which allows an "endless variety of relations" [Glenie, 1777, 450].

As Glenie believed this work to be so important, and as his own accounts of it border on the unreadable, the following summary seems appropriate. His aim is to discuss *ratios of ratios*: that is to say, given any ratio, say $A : B$, another ratio $C : D$ may be chosen such that it has to the given ratio a *ratio* equal to any chosen ratio of magnitudes, say $p : q$. In proportional notation, this is expressed as

$$(C : D) : (A : B) :: p : q,$$

or algebraically as $(C/D)^q = (A/B)^p$. Such ratios may be continually compounded or decompounded, and Glenie's object is to "fix general laws in relation to them" [pp. 451–452]. Here, we use the (then) standard notation for proportion, though Glenie usually preferred to express his often unwieldy relations in words, rather than use proportional symbols.

Glenie begins by explaining that

> ... by such expressions as these $A.\frac{A}{B}$, $A.\frac{A-B}{B}$, $A.\frac{C-D}{D}$, $A.(\frac{A-B}{B})^2$, &c. I mean respectively a third proportional to B and A; a fourth proportional to B, A, and the difference of A and B; a fourth proportional to D, A, and the difference of C and D; a fourth proportional to B, $A.\frac{A-B}{B}$ and $A - B$, &c.

Straightforward geometrical demonstrations follow for the compounding of several ratios $A : B$, $C : D$, $E : F$, $G : H$, etc.: viz. the ratios to B are

> 1. $A + A.\frac{C-D}{D}$, when two *ratios* are compounded.
> 2. $A + A.\frac{C-D}{D} + A.\frac{E-F}{F} + A.\frac{C-D}{D} \cdot \frac{E-F}{F}$, when three are compounded.
> 3. $A + A.\frac{C-D}{D} + A.\frac{E-F}{F} + A.\frac{G-H}{H} + A.\frac{C-D}{D}.\frac{E-F}{F} + A.\frac{C-D}{D}.\frac{G-H}{H} + A.\frac{E-F}{F}.\frac{G-H}{H} + A.\frac{C-D}{D}.\frac{E-F}{F}.\frac{G-H}{H}$, when four *ratios* are compounded, &c. &c.
>
> [pp. 455–456]

Note that these give identities connecting the products of ratios and series involving the differences of quantities: for example, expression 3 divided by B equals $\frac{A}{B} \cdot \frac{C}{D} \cdot \frac{E}{F} \cdot \frac{G}{H}$.

Continuing this process, Glenie finds that the number of terms involving the differences $C - D$, $E - F$, $G - H$, etc. "taken one by one, two by two, three by three, &c. if p denote the number of ratios compounded, is expressed respectively by $\frac{p-1}{1}$, $\frac{p-1}{1} \cdot \frac{p-2}{2}$, $\frac{p-1}{1} \cdot \frac{p-2}{2} \cdot \frac{p-3}{3}$, &c." He then observes that taking all the ratios $A : B$, $C : D$, $E : F$, $G : H$, etc. to be equal immediately yields the result that the pth multiplicate ratio of A to B is the ratio to B of

$$A + \frac{p-1}{1}.A.\frac{A-B}{B} + \frac{p-1}{1} \cdot \frac{p-2}{2}.A.\left(\frac{A-B}{B}\right)^2 + \&\text{c.}A.\left(\frac{A-B}{B}\right)^{p-1} \quad \text{[p. 456].} \tag{d}$$

This, of course, yields the binomial theorem with integer index $p - 1$. Though Glenie infers, rather than demonstrates, his general results for any integer p, the geometrical process is so straightforward that he would surely have considered this unnecessary.

The paper ends at this point, but Glenie mentions in his introduction that in a subsequent paper he will demonstrate how his theorems provide "a method of reasoning with finite magnitudes, geometrically, ... without any consideration of motion or velocity, applicable to every thing to which fluxions have been applied" [p. 452].

He takes up the matter in far greater detail in a 46-page tract [Glenie, 1789] entitled *The Doctrine of Universal Comparison or General Proportion*, which he dedicates to the Prince of Wales (Fig. 2). In a five-page introduction, he explains that the present work extends his 1777 paper, which gave "a cursory view," and also serves as an introduction to several promised subsequent papers "in which, amongst other things, he purposes to deliver the Geometrical Investigation of the Doctrine of Fluxions, Increments, and Measure of Ratios, the Summation of Infinite Series geometrically, &c. &c." [Glenie, 1789, i]. These will also include "a great number of geometrical problems similar to the following one, which must lay open a new and extensive field of solid geometry..." [pp. 2–3]. These are his problems, discussed above, concerning the sums or differences of two cubes equal to any multiple of a given cube.

He begins by recapitulating the gist of his 1777 paper, observing that, since "ratios or relations may admit of all possible degrees of increase and decrease," it is possible to treat irrational, not just rational, powers of quantities, "such as $a^{\sqrt{n}}$, $b^{\sqrt[3]{m}}$, $c^{\frac{1}{p\sqrt{q}}}$," etc. [p. 5]. He observes that series such as (d) above terminate, the ratios being taken as equal; and so too do corresponding series obtained when the same ratios are decompounded instead of compounded.

However, if the ratios of C to D, E to F, G to H, &c. are *not* each equal to the ratio of A to B, but have to it the ratios of i to k, l to m, n to o, &c., and r equals $1 + \frac{i}{k} + \frac{l}{m} + \frac{n}{o} + \&\text{c.}$ then series corresponding to (d) are

$$A + \frac{r-1}{1} \cdot A \cdot \frac{A-B}{B} + \frac{r-1}{1} \cdot \frac{r-2}{2} \cdot A \cdot \left(\frac{A-B}{B}\right)^2 + \frac{r-1}{1} \cdot \frac{r-2}{2} \cdot \frac{r-3}{3} \cdot A \cdot \left(\frac{A-B}{B}\right)^3 + \&\text{c.}$$

or

$$B + \frac{r}{1} \cdot B \cdot \frac{A-B}{B} + \frac{r}{1} \cdot \frac{r-1}{2} \cdot B \cdot \left(\frac{A-B}{B}\right)^2 + \frac{r}{1} \cdot \frac{r-1}{2} \cdot \frac{r-2}{3} \cdot B \cdot \left(\frac{A-B}{B}\right)^3 + \&\text{c.} \quad \text{[p. 9],}$$

which do not terminate unless r is an integer. Equivalent results hold when the ratios are decompounded rather than compounded. As Glenie allows i/k, l/m, etc. to be any magnitudes whatsoever, so too is r. The above series give the binomial expansion with index $r - 1$ and r, respectively, where r is *any* positive number. Though these results are correct, Glenie offers no proof, merely saying that "This might be shown geometrically, but, as the process at full length would swell this paper beyond its intended size, I omit setting it down at present" [p. 8].

Here, Glenie must be suspected of an empty rhetorical flourish to conceal his lack of proof. Even the composition of a pair of ratios, yielding the ratio to B of the finite quantity $A + A.\frac{C-D}{D}$, presents difficulties when the ratios $A : B$ and $C : D$ are incommensurate: for to represent the desired infinite series geometrically would require an infinite sequence of constructions. Certainly, he does not come near to offering a proof like that in his letter to Maseres, as described above.

Glenie continues his tract at great, and unnecessary, length, stating eight Theorems and several Scholia concerning various rearrangements and variations of the above results. These probably did more to repel than illuminate his

THE

DOCTRINE

OF

UNIVERSAL COMPARISON,

OR

GENERAL PROPORTION.

BY JAMES GLENIE, ESQ. F.R.S.

LATE LIEUTENANT IN THE CORPS OF ENGINEERS.

DEDICATED, BY PERMISSION,

TO HIS ROYAL HIGHNESS

THE PRINCE OF WALES.

LONDON:

PRINTED BY J. DAVIS, CHANCERY-LANE,

FOR G. G. J. AND J. ROBINSON, PATERNOSTER-ROW.

M,DCC,LXXXIX.

Fig. 2. Title page of *The Doctrine of Universal Comparison or General Proportion* [Glenie, 1789].

readers. One cannot disagree with Cajori's complaint that "Glenie's calculus involves extremely complicated identities of ratios and examines the antecedents of ratios having given consequents. The style of exposition is poor" [Cajori, 1919, 237]. More accessible are a few arithmetical examples: one such, on p. 18, takes the ratio 7 : 1 and decompounds it with the ratios 3 : 1, 5 : 1, and 9 : 1 to obtain the identity

$$\frac{7}{3\cdot 5\cdot 9}=7\times\left(\frac{1}{1}-\frac{2}{3}-\frac{4}{5}-\frac{8}{9}+\frac{2\cdot 4}{3\cdot 5}+\frac{2\cdot 8}{3\cdot 9}+\frac{4\cdot 8}{5\cdot 9}-\frac{2\cdot 4\cdot 8}{3\cdot 5\cdot 9}\right).$$

Never slow to extol his own achievements, Glenie emphasizes that, unlike Newton's binomial theorem, which is "an arithmetical one only," "the expressions delivered in the preceding theorems are not only geometrical, but universally metrical; that is, they extend equally to geometry, and all the other abstract sciences in general." They are "to the best of my knowledge, in a great measure new; the transitions from them, in their geometrical forms, to their algebraic and numerical ones, are so natural, so scientific, and so beautiful, that they cannot fail to furnish the mind with the highest pleasure, satisfaction, and delight..." [p. 31]. Finally, he notes that "The multiplicity of business in an active profession had induced the Author of this Paper to lay it aside in its present form, when he was going to add to it many general and interesting theorems; and having just been ordered abroad, thought proper to deliver it in its present state, with the assurance of completing it hereafter" [p. 45].

Another concern, in Glenie's geometrical demonstrations of his compounded ratios, is whether it is actually *possible* to construct on lines of magnitudes A and B the required magnitudes C and D with ratio $C:D$ equal to any chosen "multiple" of the ratio $A:B$; that is to say, such that $C/D=(A/B)^r$ where r is any number, rational or irrational, positive or negative. If challenged, Glenie would surely have responded, with characteristic exaggeration, that he had already shown how to do this in a large number of cases, such as in the geometrical propositions of his first published paper [Glenie, 1776a], discussed above.

7. The antecedental calculus

Glenie's 16-page tract [Glenie, 1793] has the lengthy title (Fig. 3): "The Antecedental Calculus, or a Geometrical Method of Reasoning, without any Consideration of Motion or Velocity applicable to every Purpose, to which Fluxions have been or can be applied; with the Geometrical Principles of Increments, &c. and Construction of some Problems as a few selected from an endless and indefinite variety of them respecting Solid Geometry, which he has by him in manuscript." The geometrical constructions on pp. 13–16 have already been discussed above, and have no obvious connection with his Antecedental Calculus.

His title makes clear that he aims to supplant Newton's Method of Fluxions, then still used throughout Britain in preference to the "continental" differential and integral calculus.[24] To do so, he applies his Doctrine of Universal Comparison of ratios. He sets out its advantages:

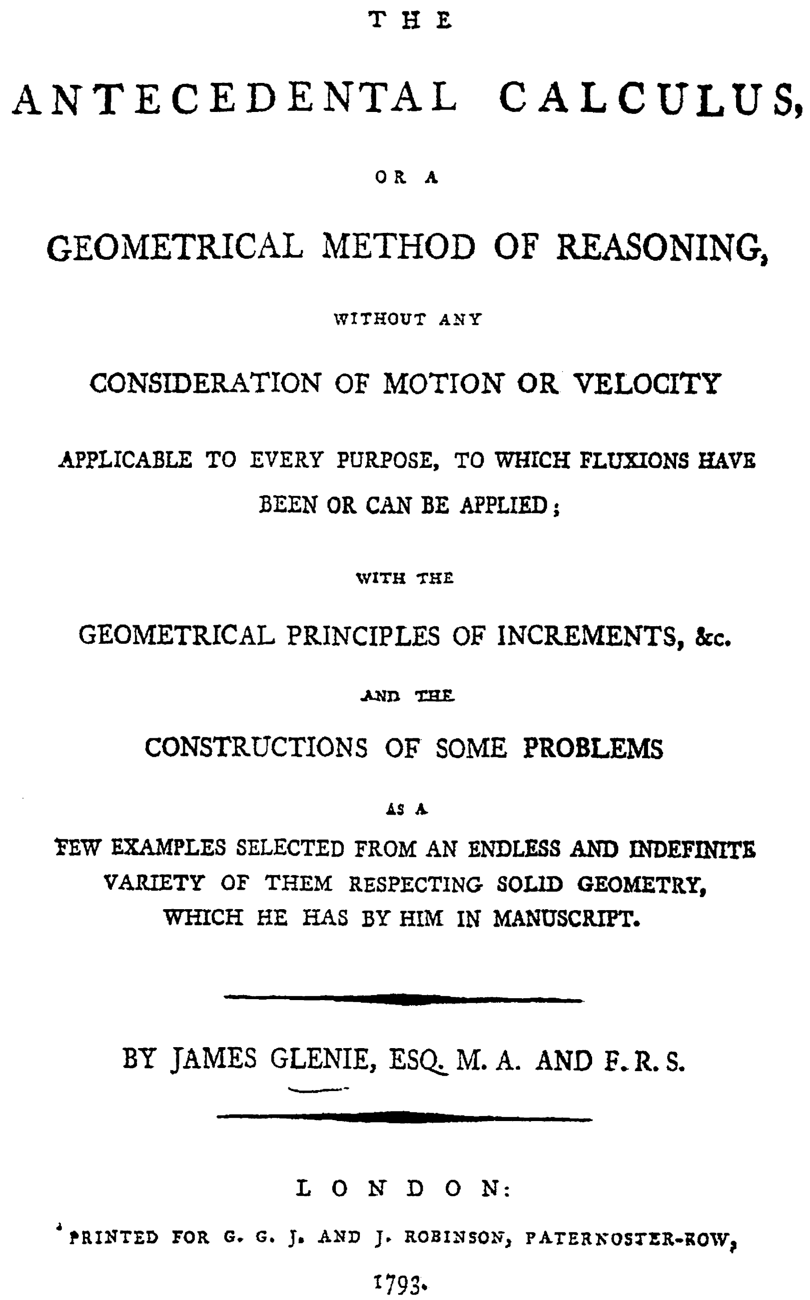
> It first occurred to me in 1774. As it is purely geometrical, and perfectly scientific, I have since that time always made use of it instead of the Fluxionary and Differential Calculi, which are merely arithmetical. Its principles are totally unconnected with the ideas of Motion and Time, which, strictly speaking, are foreign to pure Geometry and abstract Science ...
>
> ... It appears from the writings of that great man, Sir ISAAC NEWTON, that he introduced into Geometry the idea of Velocity, chiefly with a view to avoiding the exceptionable doctrine of Indivisibles ... And in his Doctrine of prime and ultimate Ratios, he had recourse to the idea of time, which, however, there was certainly no necessity for. [p. 10]

He begins by observing that, since the duplicate ratio of A to B, and of $A+N$ to B, are respectively to B as

$$A+A\cdot\frac{A-B}{B},\qquad A+N+(A+N)\cdot\frac{A-B+N}{B},$$

the excess of the latter magnitude over the former is $\frac{2AN+N^2}{B}$. Similarly, for the triplicate ratios, the corresponding excess is $\frac{3A^2N+3AN^2+N^3}{B^2}$. This may be continued to multiplicative ratios of any magnitude, obtaining infinite series if the magnitude is not an integer. This is expressed rather clumsily as follows, where R/Q denotes the "multiple" of the ratios (that is, the respective fractions are $(\frac{A}{B})^{R/Q}$ and $(\frac{A+N}{B})^{R/Q}$):

24 On p. 3, Glenie mentions "a small Performance, written in Latin, and printed the 6th of July, 1776," but this has not been found.



THE

ANTECEDENTAL CALCULUS,

OR A

GEOMETRICAL METHOD OF REASONING,

WITHOUT ANY

CONSIDERATION OF MOTION OR VELOCITY

APPLICABLE TO EVERY PURPOSE, TO WHICH FLUXIONS HAVE
BEEN OR CAN BE APPLIED;

WITH THE

GEOMETRICAL PRINCIPLES OF INCREMENTS, &c.

AND THE

CONSTRUCTIONS OF SOME PROBLEMS

AS A

FEW EXAMPLES SELECTED FROM AN ENDLESS AND INDEFINITE
VARIETY OF THEM RESPECTING SOLID GEOMETRY,
WHICH HE HAS BY HIM IN MANUSCRIPT.

BY JAMES GLENIE, ESQ. M. A. AND F. R. S.

LONDON:

PRINTED FOR G. G. J. AND J. ROBINSON, PATERNOSTER-ROW,

1793.

Fig. 3. Title page of *Antecedental Calculus* [Glenie, 1793].

And, in general, the excess of the magnitude, which has to B a ratio having to the ratio of $A+N$ to B the ratio of R to Q (when R has to Q any given ratio whatever), above the magnitude, which has to B a ratio having to the ratio of A to B the same ratio of R to Q, is geometrically expressed by

$$\frac{\frac{R}{Q}\cdot A^{\frac{R-Q}{Q}}\cdot N+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot A^{\frac{R-2Q}{Q}}\cdot N^2+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot A^{\frac{R-3Q}{Q}}\cdot N^3+\&\text{c.}}{B^{\frac{R-Q}{Q}}}. \tag{e}$$

At this point, not only did the reader have to struggle with opaque language, but he also had to overcome the vagaries of the typesetting, for the powers of A and B (here corrected) were distinguished from multiplicative factors only by the absence of a "dot." The corresponding result is also written down for the excess of any multiplicative ratio of A to B above the same multiplicative ratio of $A - N$ to B. This differs from (e) only by changes in sign of the terms in even powers of N.

Then Glenie introduces something akin to a limit process:

> But if $A + N$ and $A - N$ stand to B in relations nearer to that of equality than by any given or assigned magnitude of the same kind, these general expressions become $\frac{\frac{R}{Q} \cdot A^{\frac{R-Q}{Q}} \cdot N}{B^{\frac{R-Q}{Q}}}$. This I call the antecedental of the magnitude which has to B such a ratio as has to the ratio of A to B the ratio of R to Q.

It was this last move, deleting higher-order powers of the small increment N, that Guicciardini criticized as an "obvious weakness" [Guicciardini, 1989, 104]: however, as argued below, this judgment seems unjustified.

Glenie next denotes N, the antecedental of A, by $\overset{\bullet}{A}$ or $\overset{\mathrm{a}}{A}$, and sets $Q = 1$, $R = 2, 3, 4, 5$, etc., and then $R = 1$, $Q = 2, 3, 4, 5$, etc., to write down the antecedentals of all positive integer and reciprocal powers of A [p. 4]. Glenie's choice of the word "antecedental" corresponds with the customary term "antecedent" for the first term of a ratio: for any ratio $a : b$, a is the "antecedent" and b the "consequent." In effect, his antecedentals are equivalent to infinitesimal increments of antecedents (though he rejected the notion of infinitesimals). In modern terms, the antecedental of $(\frac{A}{B})^{R/Q}$ is its derivative with respect to A, multiplied by NB.

Compounding the ratios $A : B$, $C : D$, $E : F$, etc., and choosing $A \pm M$, $C \pm N$, $E \pm P$ to "stand to A, C, E, &c. respectively, in relations nearer to that of equality than by any given or assigned magnitudes of the same kind," yields the antecedentals of $\frac{A.C}{D}$, $\frac{A.C.E}{D.F}$, etc. (where B, D, F, etc. are fixed). These results are the equivalents of the product rule of differentiation, for two, three, etc. factors. Next, he *decompounds* the ratio $A : B$ with $C : D$ to find the antecedental of $\frac{A.D}{C}$ as $\frac{CD\overset{\bullet}{A} - AD\overset{\bullet}{C}}{C^2}$, B and D being fixed. This, of course, is equivalent to the quotient rule of differentiation if D is set equal to unity.

After gathering his results in a table, he gives geometrical interpretations, including the antecedentals of squares and cubes on a given line, and the equality of the ratio of the antecedentals of "absciss" and "semi-ordinate" to "the ratio of the sub-tangent and the semi-ordinate" [p. 9]. Two pages of general discussion follow, making connections with corresponding results in the standard theory of fluxions, and arguing for the superiority of his antecedental calculus, as both being more general and avoiding ideas of velocity. He ends the section with a brief account of finite increments associated with successively compounded ratios and a discussion "of the measure of ratios" provided by his "Universal Comparison," which he considers superior to Roger Cotes' *Harmonia Mensurarum*, which is "merely arithmetical ... with scarce any regard to the geometrical management of ratios" [pp. 12–13]. Finally, the tract ends with an unrelated account of his geometrical constructions, already described above.

A year later, he returned to his theme in "A Short Paper on the Principles of the Antecedental Calculus" read to the Royal Society of Edinburgh on December 1st, 1794 but not published until four years later [Glenie, 1798]. He explains that he was encouraged to write this paper by several friends, in order to remove misunderstandings caused by the brevity of his previous treatment. Here, he gives a rigorous justification for neglecting terms in N^2 and higher powers, while retaining the term in N, in expressions such as (e) above. He does so by showing that the ratio of any nonzero term to that which precedes it can be made less than any chosen quantity, by taking N sufficiently small. Accordingly, the leading term of series such as (e) differs from the complete sum by an arbitrarily small amount, for small enough N. Specifically, he considers the excess of $(\frac{A+N}{B})^{R/Q}$ over $(\frac{A}{B})^{R/Q}$ (given in (e) above) and the corresponding expression for the excess of $(\frac{A}{B})^{R/Q}$ over $(\frac{A-N}{B})^{R/Q}$. By careful geometrical argument, he proves that each of these excesses

> has to N a ratio nearer to the ratio of $\frac{\frac{R}{Q} \cdot A^{\frac{R-Q}{Q}}}{B^{\frac{R-2Q}{Q}}}$ to B than any given or assigned ratio, or than any given or assigned magnitude, when $A + N$ and $A - N$ have either to A or B ratios nearer to that of equality than any given, or assigned ratio, or than any given or assigned magnitude, and R and Q are two given magnitudes of the same kind. [pp. 73–74]

His proof is given in pp. 73–78, and the paper continues with several Scholia, which provide similar formal justifications for the antecedentals of products and quotients.

In other words, Glenie here gives a formal justification for his previous neglect of all but the lowest power of N when N is sufficiently small, and so provides a firm logical basis for his Antecedental Calculus. Thereby, he answers not only the critics of his own day, but also [Guicciardini, 1989], as mentioned in the Introduction. One cannot disagree with his statement that

> It is manifest then, that in this calculus no indefinitely small or infinitely little magnitudes are supposed, but only magnitudes less than any that may be given or assigned, and ratios nearer to that of equality than any that may be given or assigned, and that it is equally geometrical with the method of exhaustion of the ancients, who never supposed lines, surfaces, or solids, to be resolved into indefinitely small or infinitely little elements....
>
> This geometrical calculus, though it has no connection with the various modifications of motion, is equally convenient in its application with the method of fluxions. [pp. 79–80]

The paper concludes with five examples: these apply his calculus to geometrical problems involving tangents to a circle and a parabola, the surface areas of a sphere and a parabola, and the division of a straight line AB into two parts at C such that the rectangle $AC.CB$ is a maximum. Throughout this paper, Glenie's language is more restrained and less bombastic than usual: one wonders whether he took some advice in its preparation. If so, John Playfair would seem the most likely contact. At this time, Playfair was actively involved with the *Transactions of the Royal Society of Edinburgh*; in fact, the same volume contains papers by Playfair, William Wallace, and James Ivory.

8. Other writings

Glenie's main contributions to mathematics are described above. Here, we summarize his several other writings.

8.1. Summation of series

Related to his earlier papers [Glenie, 1776a, 1777] is an eight-page unpublished manuscript entitled "A short introductory paper to the geometrical summation of an endless number and great variety of infinite series, and to many of which the formulae hitherto delivered are not applicable." Though undated, this is contained in the Royal Society's Letters and Papers for 1805 [Glenie, 1805a].

Here, Glenie considers a triangle ABC inscribed in a semicircle with center O and diameter AB, having sides AC, BC, and AB respectively denoted by a, b, and d. Points E, F, G, H, etc. are constructed on AB such that $AE : BE :: AC : BC$, $AF : BF :: AC^2 : BC^2$, $AG : BG :: AC^3 : BC^3$, etc. It follows that

$$AE = \frac{ad}{a+b}, \qquad OE = \frac{d}{2} \cdot \frac{a-b}{a+b}, \qquad AF = \frac{a^2 d}{a^2+b^2}, \qquad EF = \frac{abd}{a^2+b^2} \cdot \frac{a-b}{a+b}, \qquad \text{etc.}$$

Now, the radius $OB = d/2$ equals the infinite sum of the terms $OE + EF + FG + \cdots$, of which the general term is

$$\frac{a^n b^n d}{a^{n+1}+b^{n+1}} \cdot \frac{a-b}{a^n+b^n}.$$

(Note that d is a factor of this equation.) By assigning particular values to a and b, many numerical series may thereby be summed. For instance, putting $a = 2$ and $b = 1$ gives

$$\frac{1}{2.3} + \frac{2}{3.5} + \frac{4}{5.9} + \frac{8}{9.17} + \frac{16}{17.33} + \frac{32}{33.65} + \cdots = \frac{1}{2}.$$

Glenie's general result may be re-expressed algebraically as

$$\frac{1}{2} = \sum_{n=0}^{\infty} \frac{r^n(1-r)}{(1+r^{n+1})(1+r^n)} \quad (0 < r < 1),$$

where $r = b/a$. Though this series does not appear in the standard modern compendium [Gradshteyn and Ryzhik, 1965], it occurs in Whittaker and Watson [1915, 48, Example 2], where their $z = r - 1$. An analytical summation easily follows from the identity

$$\frac{r^n(1-r)}{(1+r^{n+1})(1+r^n)} = \frac{1}{2} \cdot \left(\frac{1-r^{n+1}}{1+r^{n+1}} - \frac{1-r^n}{1+r^n}\right).$$

Whittaker and Watson point out that this series exhibits nonuniform convergence, with sums $1/2$, 0, or $-1/2$ depending on whether $0 < r < 1$, $r = 1$, or $r > 1$. Though this would have been unknown to Glenie, his construction with the roles of a and b reversed gives a simple geometrical explanation.

8.2. *Quadrature of the circle*

James Glenie's last papers concern aspects of the so-called quadrature of the circle. An unpublished manuscript of 20 large folio pages survives in the Archive Papers of the Royal Society of London [Glenie, 1811]. This has been given the title "Respecting the Quadrature (or the squaring) of the Circle"; but its true title appears on p. 3: this is the same as that of a 30-page pamphlet published in July of the next year. The manuscript is similar, but not identical, to the printed version: *Of the Circle, and the infinite Incommensurability of its Area to the Square of the Diameter, or of the Circumference to the Diameter; together with very useful and rapid Geometrical Approximations for both &c.* [Glenie, 1812b]. Once again, Glenie claims that his work is by no means recent: "The materials, from which I have extracted this paper, have been in my possession for upwards of twelve years" [pp. 14–15 of ms., p. 25 of pamphlet].

In this work, Glenie addresses two main issues. The first is the theoretical one of whether two given magnitudes "incommensurable in length" are also incommensurable in "all superior powers." Obviously, lines of length $\sqrt{2}$ and 2 are incommensurable in length, but they are "commensurable in power" since their squares are in the ratio of 1 to 2. On the other hand, "$2 - \sqrt{2}$ and 2 are not only incommensurable to each other, ... but are ... infinitely incommensurable ... for no multiplicative ratio whatever of $2 - \sqrt{2}$ to 2 will produce a ratio, which has its antecedent to the consequent, as number to number." He gives more examples, and introduces a novel shorthand notation involving a "dot" to denote continued square roots: thus, for instance, $\sqrt{\cdot 2} - \sqrt{\cdot 2} + \sqrt{2}$ means $\sqrt{2 - \sqrt{2 + \sqrt{2}}}$. The advantage of this notation becomes plain when he goes on to consider repeated geometrical constructions of regular polygons inscribed in a circle.

These constructions arise in connection with the second, and main, aim of his paper, which is the calculation of approximations for the ratio $(\pi/4)$ of the area of a circle to that of the square on its diameter, and the ratio (π) of the circumference of a circle to its diameter. Such work was by no means novel, as highly accurate computations for π had been known for over 150 years. It is therefore unsurprising that this paper was not published in the *Philosophical Transactions* of the Royal Society, and only a brief description suffices here.

Glenie gives algorithms for computing estimates of the ratios, based upon inscribed regular polygons, and he gives numerical results for several examples. The published pamphlet contains 5 such examples, for polygons of 5, 12, 24, 48, and 96 sides; and the manuscript has 10 examples, for polygons of 5, 8, 12, 15, 24, 48, 60, 96, 120, and 240 sides. Glenie rightly states that "These simple geometrical approximations are much nearer to the truth than the arithmetical ones given by Archimedes for a circumscribed polygon of 96 sides and an inscribed one" [p. 27], but he ignores the fact that far better approximations had long been known. His estimates give values for π that are correct to four decimal places for a 24-sided polygon, six places for a 96-sided polygon, and eight places for a 240-sided polygon.

8.3. *Gunnery*

This is not the place to give a detailed account of Glenie's contributions to gunnery and fortification, but some brief mention is necessary, since this work is at times intimately connected with his mathematics.

Glenie's 163-page book, *The History of Gunnery* [Glenie, 1776b], is an early work, mostly written while he was an assistant in mathematics in St Andrews. Its two parts are rather disparate. The first, occupying the first 86 pages, outlines practical and theoretical developments in gunnery from earliest times up to his own day. In this, he devotes much space to discussing the work of Isaac Newton and Benjamin Robins on the motion of projectiles subject to air resistance. On pages 59–67, he lists 24 problems "to exercise ... the ingenuity of mathematicians and scientific engineers, who are above tying themselves down to the erroneous practices of their predecessors and companions" [p. 58]. These problems mainly concern the motion of projectiles under various assumed laws of resistance. Though Glenie claims that "Most of them I have already considered, and the rest I shall probably make the subject of my future examination," he seems to have published no such work.

Rather, his Part II is devoted to *A New Method of deriving the Theory of Projectiles in Vacuo, from the properties of the Square and Rhombus*: that is to say, he ignores the influence of air resistance. This part strikes a modern reader as distinctly odd. Though he was well aware that the projectile's path is a parabola, this fact is barely mentioned. Instead, he deduces numerous geometrical theorems and corollaries concerning constructions of squares and rhombuses, from which, eventually, it is possible to find the ranges of projectiles on either horizontal or inclined planes. The geometrical arguments presage his later *Doctrine of General Proportion*, for he makes much use of ratios and proportions of lines and areas. He relates such quantities to the projectile's motion in a given time, with horizontal and vertical velocity components that are respectively constant and uniformly accelerating. Eventually, he arrives at several complicated rules, requiring the use of logarithms, for finding the range of the projectile. Some further details are given in Johnson [1997].

As Glenie gave no tables of results that might have had practical application, it is unsurprising that his book did not prove popular. He was also unlucky in his timing. Charles Hutton at Woolwich was in course of conducting important new experiments on artillery, for which he was awarded the Royal Society's Copley Medal in 1778. And, just a year after Glenie's book was issued, there appeared an English translation by Hugh Brown of Leonhard Euler's commentary on Benjamin Robins' *Gunnery* [Euler, 1777].

8.4. Fortification

In 1807, Glenie published a substantial 121-page pamphlet entitled A *Few Concise Observations on Military Construction with some Rules for it, &c*. [Glenie, 1807b]. Parts of this work had been written much earlier, but apparently not published. An eight-page pamphlet [Glenie, 1784b] dated 1784 and presented to the Royal Society in 1793 consists of pp. 21–30 of [Glenie, 1807b], and an undated fifteen-page manuscript, "An Introductory paper on Fortification ..." [Glenie, [n.d. (1)]], certainly also precede the published work. In these, Glenie provides geometrical rules for constructing fortifications in the form of both regular and irregular polygons, "always making the Flanks either equal to the Perpendiculars to the exterior sides or in any given ratio to them" (quoting from full title to [Glenie, [n.d. (1)]]).

This theoretical paper is in marked contrast with his pamphlets concerning the plans of the Duke of Richmond to fortify the English coastline against the possibility of attack from France. His *A Short Essay on the Modes of Defence best adapted to the Situation and Circumstances of this Island* [Glenie, 1785] dismissed the Duke's plan as "absurd and impractical." An acrimonious exchange ensued, and Glenie's intervention doubtless caused the Duke's plan to be rejected in 1786 by just a single vote in the House of Commons. He returned to the matter in his *Observations on the Duke of Richmond's Extensive Plans of Fortification* [Glenie, 1805b], and *Observations on the Defence of Great Britain and its Principal Dockyards* [Glenie, 1807a]. These works display Glenie's grasp of military strategy and his clear view of the impracticality of constructing and defending a long line of forts. It was far better, in his opinion, to build an effective Navy that could defend the whole coastline against invasion: see also [Johnson, 1997].

In 1805, Glenie also edited and enlarged a work on *Military Memoirs* [Glenie, 1805c], about the time that he was appointed to teach mathematics at the East India Company Royal Military College at Addiscombe.

Glenie's last few years were spent in poverty, the only evidence of mathematical activity, and a small source of income, being found in two manuscript receipts. In 1815, he got three guineas for revising Symond's *Practical Gager*, and in 1816 he received five guineas for revising and arranging the seventeenth edition of Robert Simson's *Euclid* [Glenie, 1815/16]. One can be sure that the latter was a task more to his taste than the former.

9. Conclusion

Though by no means a professional mathematician, Glenie was rather well connected in mathematical and scientific circles. In Scotland, he knew John Playfair and Dugald Stewart, and of course Nicolas Vilant. His early work on gunnery brought him to the notice of Charles Hutton of Woolwich, and he also met Nevil Maskelyne around this time. At the Royal Society of London, his contacts included Samuel Horsley and Francis Maseres, as well as Hutton and

Maskelyne. It is more than likely that he also met James Ivory, a former St Andrews student from just after Glenie's time there, and also William Wallace and Thomas Leybourn, who were Ivory's colleagues at the Royal Military College, Marlow. Glenie must also have known his St Andrews successor as Assistant, John West, whom he may have taught briefly; but West left for Jamaica in 1784 and the two probably never met again. Others who supported Glenie (at least for a time) were Viscount Townshend, Master Gunner at the Ordnance, Sir Frederik Haldimand, commander-in-chief of Canada during 1778–1785, and the Earl of Chatham. Another acquaintance, to whom he wrote about New Brunswick affairs, was the politician William Windham, whom he had met "at the Mathematical Club in Fleet Street, London" [Glenie, [n.d. (2)]].

Glenie's mathematical attitudes and ambitions seem to have remained fixed in the 1770s, acquired when he was a student and assistant at St Andrews University. No doubt his busy and itinerant life afforded him few opportunities to learn new mathematics: he had little enough time to write up for publication his own ideas, many of them conceived and partially worked out years earlier. These intermittent publications mainly concerned his "Doctrine of General Proportion" and its applications. He employed proportional arguments to good effect to prove and extend the theorems of Matthew Stewart concerning squares, cubes, and higher powers of lines. Convinced that his theory of proportion also provided a more satisfactory basis for the differential and integral calculus than that given by Newton, he used it to develop his Antecedental Calculus, which is more viable that previous writers have allowed. And he employed Euclid's far-from-intuitive definition of the equality of ratios of magnitudes to give an *almost* satisfactory proof of the binomial theorem.

There is no doubt that Glenie displayed both originality and skill in allying traditional methods of Euclidean geometry, the theory of ratios, and manipulation of algebraic and infinite series. Thereby, he deduced many new results and proofs and constructed his own Doctrine of General Proportion which he believed to be of great importance. He clearly believed that the theory of ratios gave a uniquely rigorous approach that could (in the words of his teacher Nicolas Vilant) bring "improvement in the other branches of Mathematics; and in Natural Philosophy." But his conviction of its importance as a means of rigorous analysis was not shared by his contemporaries, and even less is shared by mathematicians today. Even in his own day, the traditional theory of proportion was already falling from favour as a topic to be treated separately from algebra. And Glenie's extension of it led to great complexities of notation and language that many must have dismissed out of hand as either incomprehensible or not worth the effort of mastering. Glenie certainly did not help his case by his overblown claims and his sometimes long-winded, contorted and repetitive expositions. Nevertheless, his mathematical publications, interspersed throughout his eventful life, deserve more attention than they received. The present account may serve as a guide and summary for those willing to approach his writings with an open and tolerant mind.

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