Linear interpolation problems for matrix classes
and a transformational characterization of
\(M\)-matrices

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Abstract

The linear interpolation problem (LIP) for a class of matrices \(\mathcal{C}\) asks for which pairs of vectors \(x, y\) there exists a matrix \(A \in \mathcal{C}\) such that \(Ax = y\). The LIP is solved for \(M\)-matrices, \(P\)-matrices, \(H\)-matrices, and \(H^+\)-matrices. In addition, a transformational characterization is given for \(M\)-matrices that refines the known one for \(P\)-matrices. There is no such characterization for \(H\)- or \(H^+\)-matrices. © 2001 Elsevier Science Inc. All rights reserved.

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All matrices throughout are real and square (unless otherwise indicated). A \(P\)-matrix is one whose principal minors are positive, a \(Z\)-matrix is one whose off-diagonal entries are nonpositive, and an (invertible) \(M\)-matrix is one that is both a \(P\)-matrix and an \(Z\)-matrix. The comparison matrix \(C(A)\) of the \(n \times n\) matrix \(A = (a_{ij})\) is \(C(A) = (c_{ij})\), in which \(c_{ii} = |a_{ii}|, i = 1, \ldots, n\), and \(c_{ij} = -|a_{ij}|, i \neq j\); \(A\) is called an \(H\)-matrix if \(C(A)\) is an \(M\)-matrix and, further, called an \(H^+\)-matrix if, in addition,

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A has positive diagonal entries. An $H^+$-matrix is a $P$-matrix. The Hadamard product of two vectors (or matrices of the same dimensions) is the entry-wise product and is denoted with $\circ$. We will make use of the fact that principal submatrices of $P$-, $M$-, $H$-, and $H^+$-matrices, are themselves $P$-, $M$-, $H$-, and $H^+$-matrices, respectively.

We call two nonzero vectors $x, y \in \mathbb{R}^n$ sign-related if $x \circ y \not\leq 0$ (entry-wise). It is known [2, Theorem 2.4] that $A$ is a $P$-matrix if and only if for every vector $x \neq 0$, $x$ and $Ax$ are sign-related. For $M$-matrices many characterizations are known [1–3], but we add another that is reminiscent of the above “transformational characterization” of $P$-matrices. In addition, we solve the “linear interpolation problem” for each of $M$-, $P$-, $H$-, and $H^+$-matrices. By the linear interpolation problem (LIP) for a class of matrices $\mathcal{C}$, we mean the identification of all pairs $x, y \in \mathbb{R}^n$, $0 \neq x$, such that there is an $A \in \mathcal{C}$ for which $Ax = y$. Clearly, a transformational characterization of a class $\mathcal{C}$ is related to a solution of the LIP for $\mathcal{C}$. However, we note that, while the LIP always has a solution for a class $\mathcal{C}$, there may be no transformational characterization of $\mathcal{C}$. If, for example, there is a class $\mathcal{C'}$ that properly contains $\mathcal{C}$, but for which the solution to the LIP is the same as that for $\mathcal{C}$, then there can be no transformational characterization of $\mathcal{C}$. This happens to be the case for both $\mathcal{C}$ the $H$-matrices and the $H^+$-matrices.

We first give a transformational characterization of $M$-matrices. For this we need a refinement of the sign-related condition. Suppose $0 \neq x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Let $P$ be a permutation matrix chosen so that

$$Px = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

in which $X_1 > 0$, $X_2 < 0$, and $X_3 = 0$ (entry-wise) and suppose that

$$Py = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

is partitioned conformally with $x$. (Since $M$-matrices are closed under permutation similarity, this partitioning applies to any pair of vectors $0 \neq x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. That is, we may assume any pair of vectors $0 \neq x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ have the partitioned form of $Px$ and $Py$.) Except for $X_1$ and $X_2$, any one or two of $X_1$ (and then $Y_1$), $X_2$ (and then $Y_2$), and $X_3$ (and then $Y_3$) may be empty. We say that $x$ and $y$ are doubly sign-related if

(P1) \[ X_1 \circ Y_1 \not\leq 0 \quad \text{and} \quad X_2 \circ Y_2 \not\leq 0 \]

and

(P2) if $X_1$ is empty and $X_3$ is not, then $Y_3 \geq 0$ and if $X_2$ is empty and $X_3$ is not, then $Y_3 \leq 0$. (Note that (P1) implies $x$ and $y$ are sign-related.)

We then have the following theorem.
Theorem 1. If \( A \in M_n(\mathbb{R}) \), then \( A \) is an \( M \)-matrix if and only if for every \( 0 \neq x \in \mathbb{R}^n \), \( x \) and \( Ax \) are doubly sign-related.

Proof. Let \( A \in M_n(\mathbb{R}) \). If \( n = 1 \), the result is clear. Assume hereafter that \( n \geq 2 \). For sufficiency, suppose that for any nonzero vector \( x \), the pair \( x, y = Ax \) satisfies (P1) and (P2). Now (P1) implies that \( A \) is a \( P \)-matrix. So we just need to show \( A \in Z \). Assume the contrary, say \( a_{ij} \geq 0 \), some \( i \neq j \). If \( x = -e_j \), then \( X_1 \) is empty and \( X_3 \) is not. But, \( Y_3 \neq 0 \), which contradicts (P2) and completes the proof of sufficiency.

Conversely, suppose that \( A \) is an \( M \)-matrix, \( 0 \neq x \in \mathbb{R}^n \), and \( y = Ax \). Writing \( x \) in partitioned form and partitioning \( y \) and \( A \) conformally with \( x \), we have

\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix}
\]

in which \( X_1 > 0 \), \( X_2 < 0 \), and \( X_3 = 0 \). Thus, assuming \( X_1 \) is nonempty, \( A_{11}X_1 + A_{12}X_2 = Y_1 \) or, solving for \( X_1 \), we have \( X_1 = (A_{11})^{-1}(Y_1 - A_{12}X_2) \). Now, if \( Y_1 \leq 0 \), then, since \( (A_{11})^{-1} \) and \( A_{12}X_2 \) are nonnegative, it follows that \( X_1 \leq 0 \), a contradiction. Hence, \( Y_1 \not\leq 0 \) so that \( Y_1 \circ Y_1 \not< 0 \). Similarly, if \( X_2 \) is nonempty, it follows that \( X_2 \circ Y_2 \not< 0 \). So (P1) holds.

Moreover, if \( X_1 \) is empty and \( X_3 \) is not, then, partitioning \( y \) and \( A \) conformally with \( x \), we have

\[
y = \begin{bmatrix}
Y_2 \\
Y_3
\end{bmatrix} = \begin{bmatrix}
A_{22} & A_{23} \\
A_{32} & A_{33}
\end{bmatrix} \begin{bmatrix}
X_2 \\
X_3
\end{bmatrix} = Ax
\]

(since \( x \neq 0 \), \( X_2 \) is nonempty). Thus, \( Y_3 = A_{32}X_2 + A_{33}X_3 = A_{32}X_2 \geq 0 \). The case in which \( X_2 \) is empty and \( X_3 \) is nonempty implies \( Y_3 \leq 0 \) is similar. So, (P2) holds which completes the proof. \( \square \)

We next turn to linear interpolation problems. In the case of \( M \)-matrices, we know from Theorem 1 a necessary condition for the LIP. Interestingly, it is also sufficient.

Lemma 1. Let \( 0 < x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \). If \( x \) and \( y \) are sign-related, then there is an \( M \)-matrix \( A \in M_n(\mathbb{R}) \) such that \( Ax = y \).

Proof. Let \( 0 < x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \), and suppose that \( x, y \) are sign-related. Then \( y \) has at least one positive entry. Since \( M \)-matrices are closed under positive diagonal equivalence, we may assume that the entries of \( x \) are all \( 1 \) and \( y \) are all \(-1\), \( 0 \), or \( 1 \). And, since \( M \)-matrices are closed under permutation similarity, we may assume that \( \{ i : x_i = 1, y_i = 1 \} = \{ 1, \ldots, p \} \), \( \{ i : x_i = 1, y_i = 0 \} = \{ p + 1, \ldots, q \} \), and \( \{ i : x_i = 1, y_i = -1 \} = \{ q + 1, \ldots, n \} \) in which \( 0 \leq p \leq q \leq n \). Let

\[
A = \begin{bmatrix}
I_q & 0 & 0 \\
A_{21} & I_{q-p} & 0 \\
A_{31} & 0 & I_{n-q}
\end{bmatrix}
\]
in which $A_{21} = [-e(q - p) \ 0]$ and $A_{31} = [-2e(n - q) \ 0]$ ($e(r)$ denoting the $r$-vector consisting of all ones). Then $A$ is an $M$-matrix satisfying $Ax = y$. □

Note that $Ax = y \iff A(-x) = -y$, so that the conclusion of the lemma also holds if $x < 0$. Also, notice that, in either of these cases, sign-related is equivalent to doubly sign-related.

**Theorem 2.** If $0 \neq x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, then there is an $M$-matrix $A \in M_n(\mathbb{R})$ such that $Ax = y$ if and only if $x$ and $y$ are doubly sign-related.

**Proof.** Let $0 \neq x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. As mentioned prior to the lemma, we need only to show sufficiency of the doubly sign-related condition. So suppose that $x, y$ satisfy (P1) and (P2). Then $y \neq 0$ also. As before, we may assume that the entries of $x$ and $y$ are all $-1, 0, or 1$, and, further, that

$$x = \begin{bmatrix} e(p) \\ -e(q) \\ 0 \end{bmatrix}.$$  

Partition $y$ conformally with $x$ as

$$y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix},$$

and, by permutation similarity, we may assume that

$$Y_3 = \begin{bmatrix} e(r) \\ -e(s) \\ 0 \end{bmatrix}.$$  

By the lemma there is a $p \times p M$-matrix $A_{11}$ satisfying $A_{11}e(p) = Y_1$ and a $q \times q M$-matrix $A_{22}$ satisfying $A_{22}(-e(p)) = Y_2$. Let the first columns of $A_{31}$ and $A_{32}$ be

$$\begin{bmatrix} 0 \\ -e(s) \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -e(r) \\ 0 \\ 0 \end{bmatrix},$$

respectively, partitioned conformally with $Y_3$, and let the other entries of $A_{31}$ and $A_{32}$ be 0. Then, if

$$A = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & I_{n-p-q} \end{bmatrix},$$

$A$ is an $M$-matrix such that $Ax = y$, which completes the proof. □

Notice that a dual result holds for inverse $M$-matrices.

In the case of $P$-matrices, the transformational characterization (sign-related) mentioned earlier also gives a necessary condition for the $P$-matrix LIP. Interestingly, the
fact that the condition is also necessary seems not to have been noticed. A natural proof solves the LIP for $H^+$-matrices at the same time.

**Theorem 3.** For the pair $0 \neq x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the following statements are equivalent:

(i) there is an $H^+$-matrix $A$ such that $Ax = y$;

(ii) there is a $P$-matrix $A$ such that $Ax = y$; and

(iii) $x$ and $y$ are sign-related.

**Proof.** Certainly (i) implies (ii) since $H^+$-matrices are $P$-matrices. That (ii) implies (iii) follows from the transformational characterization of $P$-matrices [2]. So it remains to show that (iii) implies (i). Thus, $x$ and $y$ agree in sign in some position. So, by permutation similarity, we can assume $x_1y_1 > 0$. It is straightforward to check that

$$
A = \begin{bmatrix}
\frac{y_1}{x_1} & 0 & \ldots & 0 \\
\frac{y_2-x_2}{x_1} & 1 & \ddots & \\
\frac{y_3-x_3}{x_1} & 0 & \ddots & \\
\vdots & \vdots & \ddots & 0 \\
\frac{y_n-x_n}{x_1} & 0 & \ldots & 0 \\
\end{bmatrix}
$$

is an $H^+$-matrix satisfying $Ax = y$. □

Since the $H^+$-matrices are properly contained in the $P$-matrices, yet, according to Theorem 3, the solution to the LIP is the same for both, there can be no transformational characterization of $H^+$-matrices. Any constraint in the relation between $x$ and $Ax$ for $H^+$-matrices must admit general $P$-matrices as well.

We also note the solution to the LIP for $H$-matrices. Again, we shall see that it does not lead to a transformational characterization of $H$-matrices.

**Theorem 4.** If $0 \neq x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, then there is an $H$-matrix $A$ such that $Ax = y$ if and only if $x \circ y \neq 0$.

**Proof.** Let $0 \neq x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ and let $A$ be an $H$-matrix such that $Ax = y$. Then $y \neq 0$ also. Suppose $x \circ y = 0$. Then, by permutation similarity, we may assume $x$, $y$, and $A$ are partitioned conformally as

$$
x = \begin{bmatrix} X_1 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ Y_2 \end{bmatrix},
$$

and

$$
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
$$

in which $X_1 \neq 0$. Then
Ax = \begin{bmatrix} A_{11}X_1 \\ A_{21}X_1 \end{bmatrix} = \begin{bmatrix} 0 \\ Y_2 \end{bmatrix},

which implies \( A_{11}X_1 = 0 \), contradicting the fact that \( A_{11} \), a principal submatrix of an \( H \)-matrix, is nonsingular. Thus, \( x \circ y \neq 0 \).

Since \( H \)-matrices are invariant under invertible diagonal multiplication, the converse follows from Theorem 3 ((iii) \( \rightarrow \) (i)) and noting that if \( x \circ y \neq 0 \), there is an invertible diagonal matrix \( D \) such that \( x \) and \( Dy \) are sign-related. \( \square \)

There is, however, a larger class for which the LIP has the same solution set as for \( H \)-matrices. Call a matrix \( A \in M_n(\mathbb{R}) \) \emph{principally nonsingular} (PN) if every principal submatrix is nonsingular; \( H \)-matrices are PN because a principal submatrix of an \( H \)-matrix is an \( H \)-matrix; the containment is proper. The same proof shows that, for \( 0 \neq x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \), there is a PN-matrix \( A \) such that \( Ax = y \) if and only if \( x \circ y \neq 0 \). Thus, there is no transformational characterization of \( H \)-matrices. It is also straightforward to observe that \( A \) is a PN-matrix if and only if for each \( x \neq 0 \), \( x \circ Ax \neq 0 \).

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**References**

