Available online at www.sciencedirect.com

ScienceDirect

Theoretical Computer Science 363 (2006) 289–300

Theoretical Computer Science

www.elsevier.com/locate/tcs

Approximation algorithms for facility location problems with a special class of subadditive cost functions

Adriana F. Gabor[1], Jan-Kees C.W. van Ommeren*

Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

Abstract

In this article we focus on approximation algorithms for facility location problems with subadditive costs. As examples of such problems, we present three facility location problems with stochastic demand and exponential servers, respectively inventory. We present a $(1+\varepsilon, 1)$-reduction of the facility location problem with subadditive costs to the soft capacitated facility location problem, which implies the existence of a $2(1+\varepsilon)$-approximation algorithm. For a special subclass of subadditive functions, we obtain a 2-approximation algorithm by reduction to the linear cost facility location problem.

MSC: 68W25; 90B06; 60K30

Keywords: Approximation algorithms; Subadditive functions; Stochastic facility location; Multi server queues; Inventory models

1. Introduction

In the last years, facility location problems have been extensively studied in the literature. In a facility location problem, we are given a set of demand points and a set of locations where facilities may be opened. The goal is to decide at which location to open facilities and how to assign demand points to facilities such that the total cost of opening facilities and connecting demand points to facilities is minimized.

The facility location problem and its variants has proved to be a very useful tool in modelling many network design or location problems, such as location of plants or warehouses [23,8] and placement of caches [11].

In this paper we study a variant of the facility location problem, where the total costs incurred at facilities for opening and purchasing the necessary resources for satisfying the demand are subadditive. The subadditive costs suggest that sharing the resources between demand points is never less profitable then installing resources for each demand point separately. We are interested in the relationship between this problem and known facility location problems, in particular from the perspective of approximation algorithms.

As examples of a facility location problemwith subadditive costs, we present three facility location problems with stochastic demands. At demand points a stochastic number of requests for items is generated. At open facilities, some

* Corresponding author.

E-mail addresses: a.f.gabor@tue.nl (A.F. Gabor), j.c.w.vanommeren@ewi.utwente.nl (J.-K.C.W. van Ommeren).

[1] Present address: EURANDOM and Faculty of Mathematics and Computer Science, TUE, Eindhoven, The Netherlands.

0304-3975/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2006.04.013

resources have to be purchased in order to satisfy the demand. Usual examples of such resources are inventory and servers.

We will call a polynomial time algorithm a ρ-approximation algorithm if it always finds a feasible solution with *objective function value* within ρ times the optimum. The value ρ is called the *performance* (*approximation*) *guarantee* of the algorithm.

The simplest version of a facility location problem, *the metric uncapacitated facility location problem* (UFLP), that is the facility location problem with no restrictions on the facilities or the assignment of demand points and with the transportation costs being a metric, is known to be NP-hard. If the transportation costs are unrestricted, approximating the UFLP is as hard as approximating set cover, and therefore cannot be approximated better than within a factor of $O(\log n)$ the optimum, unless $\mathbf{NP} \subseteq \tilde{\mathbf{P}}$. In this article, we assume, for all the facility locations mentioned, that the transportation costs form a metric. There are several approximation algorithms for the UFLP known in the literature [1,7,10,14–16,19,23]. The currently known best performance guarantee for the UFLP is 1.52, due to Mahdian et al. [19]. Guha and Khuller [10] and Sviridenko [24] have proved that a better factor than 1.463 for the UFLP is not possible unless $\mathbf{NP} \subseteq \tilde{\mathbf{P}}$.

The problem in which each facility has a certain capacity, but more facilities may be opened at a location if the demand exceeds the capacity of one facility, is known as the *soft capacitated* facility location problem. The approximation algorithms for the soft capacitated facility location problems are usually based on reductions to the uncapacitated version of the problem [15,14,19,20]. The best approximation algorithm for this problem has an approximation ratio of 2 and was proposed by Mahdian et al. [20]. In [13], the authors show that the variant in which the cost of facilities are concave functions of the number of demand points served can be easily reduced to the UFLP, and propose a 1.861 approximation algorithm based on the technique of dual fitting and factor revealing LPs introduced in [14]. For the *hard capacitated* facility location problem with splittable demands, where each facility has a certain capacity, only one facility may be open at a location and a demand point may be served by several locations, the best approximation algorithm is due to Zhang et al. [27], and achieves an approximation ratio between $3 + 2\sqrt{2} - \varepsilon$ and $3 + 2\sqrt{2} + \varepsilon$, for any given constant $\varepsilon > 0$.

Subadditive cost functions appear very often in several variants of stochastic facility location problems (problems where the demand is stochastic or/and the service offered by facilities is of stochastic nature). In the OR literature, several heuristics for these problems are known (see e.g. [2–4,21,25]). However, only recently approximation algorithms for stochastic facility location problems started to be developed. To the best of our knowledge, the first approximation algorithm for a stochastic facility location problem was proposed by Ravi and Sinha in [22] and was improved by Mahdian in [17]. The latest algorithm is based on the primal-dual technique and has a 3-approximation guarantee. Their approach is scenario-based, i.e. in each scenario all the data are known, including the probability with which each scenario takes place. In [12], Gupta et al. present an approximation algorithm for a variant of the stochastic facility location problem where only the probability distribution of the clients is known and the facilities can be opened in two stages: before the actual demand points are known, at a lower price, or later, after the actual demand points are known, at a higher price. In [5,6] approximation algorithms were proposed for a facility location problem with stochastic demands and periodically replenished inventory.

This paper is organized as follows. In Section 2 we introduce the class of facility location problems with subadditive costs and present three examples with stochastic data. In Section 3 we present two reductions of the facility location problem with subadditive cost functions to facility location problems with known approximation algorithms. The first one is a reduction to the soft capacitated facility location problem which will, based on results in [20], imply the existence of a $2(1 + \varepsilon)$-approximation algorithm for the facility location problem with subadditive costs. For a special class of subadditive functions, we present a reduction to the linear cost facility location problem. Combining this result with the approximation algorithm for the linear cost facility location problem presented in [20], we infer that a 2-approximation algorithm exists. We conclude with some remarks on the class of facility location problems we have analyzed.

2. The facility location problem with subadditive costs

The facility location problem with subadditive costs can be formulated as follows. There is a set of demand points D, at which r_j requests ($j \in D$) are generated, and a set of locations F, where facilities may be opened. The transportation cost per unit of demand from location $i \in F$ to demand point $j \in D$ is c_{ij}. The transportation costs between demand points and locations are assumed to be proportional to the distances and to form a metric. At every open facility, certain resources have to be purchased in order to satisfy the demand. The costs incurred at each open facility $i \in F$ are variable costs for purchasing the necessary resources $v_i(d_i)$, where d_i is the demand served by facility i and v_i is a non-decreasing function, $v_i : \mathbb{R}^+ \to \mathbb{Q}^+$. We assume that for each $i \in F$, the function v_i is subadditive, i.e. $v_i(x + y) \leqslant v_i(x) + v_i(y)$ for each $x, y \in \mathbb{R}^+$, left continuous and that $s_i > 0$, where

$$s_i \stackrel{\text{def}}{=} \sup\{x | v_i(y) = v_i(x) \text{ for } 0 < y \leqslant x\}. \tag{1}$$

Subadditivity implies that sharing resources by demand points is not less profitable than having separate resources for each demand point. We assume that the functions v_i are not a part of the input.

The goal is to decide where to open facilities, how to assign demand points to facilities and what is the necessary quantity of resources at each open facility such that the total cost (transportation costs and opening facilities cost) is minimized.

Remark 1. Note that if the v_i's were concave, one could use the 1.861 approximation algorithm proposed in [13,18] for solving the above problem. This algorithm, based on the dual fitting and factor revealing LP techniques relies heavily on the concavity of the cost function, and therefore is not suitable for subadditive cost functions.

Examples of facility location problems with subadditive costs are frequently met in the stochastic OR literature. Below we will present three of them.

2.1. Examples of facility location problems with stochastic demand

In the examples we present, at each demand point $j \in D$ requests are generated according to a Poisson process with rate ρ_j, independent of the processes at other demand points. As the incurred costs, we consider the expected costs made during an arbitrary unit period. For example, we define the transportation costs from demand point j to location i as the transportation cost of a single item, multiplied by the expected number of generated requests during a unit period.

Note that, since the requests are generated according to independent Poisson processes, the arrival processes of requests at facilities, is a Poisson process as well.

Next we present the specific details to each example.

2.2. Facilities with periodically replenished inventory (from [5,6])

In this variant of the facility location problem, at each open facility an inventory is kept such that arriving requests find a zero inventory (and are lost), with probability at most α. We then say that $(1 - \alpha)$ is the *fill rate* of the system. The inventories at the open facilities are restored only at equidistant points in time and the period between two such points is called a *reorder period*. The holding cost per unit of inventory at an open facility $i \in F$ is c_i and the cost of keeping a facility open at location $i \in F$ during a reorder period is f_i. Since the requests at demand point j are generated according to a Poisson process, the number of requests during reorder periods are independent and have a Poisson distribution (see e.g. [26, p. 70]).

Denote by V_i the inventory order up to level at facility $i \in F$, i.e. the inventory level at the beginning of a reorder period. The constraint on the fill rate is written as

$$P\begin{pmatrix} \text{an arbitrary request arriving at facility } i \\ \text{with inventory level } V_i \text{ is lost} \end{pmatrix} \leqslant \alpha, \quad i \in F. \tag{2}$$

Next we will give an equivalent formulation of constraints (2). Let D_i be the set of demand points that is assigned to location i and let $\tilde{X}_i$ be the total demand assigned to location i during a reorder period. Clearly, $\tilde{X}_i$ has a Poisson distribution with mean $E(\tilde{X}_i) = \sum_{j \in D_i} \rho_j$. From the theory of regenerative processes (see e.g. [26]), it follows that for location i, the following holds:

$$P\begin{pmatrix} \text{an arbitrary request arriving at facility } i \\ \text{with inventory level } V_i \text{ is lost} \end{pmatrix} = \frac{E(\max\{0, \tilde{X}_i - V_i\})}{E(\tilde{X}_i)}.$$

Condition (2) can be rewritten as

$$E(\max\{0, \tilde{X}_i - V_i\}) \leqslant \alpha E(\tilde{X}_i).$$

For a Poisson distributed random variable Y with $E(Y) = \rho$, define the inventory $V_\alpha(\rho)$ by

$$V_\alpha(\rho) = \min\{n | E(\max\{0, Y - n\}) \leqslant \alpha\rho\}.$$

Note that $V_\alpha(\rho)$ is the minimal order up to level to satisfy condition (2) at a location when the expected demand per reorder period at that location is ρ. Thus, for an expected demand ρ at a location i, the costs that are made are $v_i(\rho) = f_i + V_\alpha(\rho)c_i$. Clearly, v_i satisfies condition (1). In the following lemma, we show that $V_\alpha(\rho)$ is subadditive, which implies that the cost functions for facilities are subadditive.

Lemma 2. *The function V_α satisfies*

$$V_\alpha(\rho_1 + \rho_2) \leqslant V_\alpha(\rho_1) + V_\alpha(\rho_2).$$

Proof. Suppose that two independent Poisson streams with rate ρ_1, respectively ρ_2, arrive at a location i and that the inventory level at location i is $V_\alpha(\rho_1) + V_\alpha(\rho_2)$. Let Y_1 and Y_2 be the number of arrivals in the first, respectively in the second stream. Since, for $y_1, y_2 \in \mathbb{R}$,

$$\max\{0, y_1 + y_2 - (V_\alpha(\rho_1) + V_\alpha(\rho_2))\} \leqslant \max\{0, y_1 - V_\alpha(\rho_1)\} + \max\{0, y_2 - V_\alpha(\rho_2)\},$$

it is readily seen that

$$E(\max\{0, Y_1 + Y_2 - (V_\alpha(\rho_1) + V_\alpha(\rho_2))\}) \leqslant E(\max\{0, Y_1 - V_\alpha(\rho_1)\}) + E(\max\{0, Y_2 - V_\alpha(\rho_2)\}) \leqslant \alpha(\rho_1 + \rho_2).$$

Hence, $V_\alpha(\rho_1 + \rho_2) \leqslant V_\alpha(\rho_1) + V_\alpha(\rho_2)$. □

2.3. Facilities with exponential servers

In this problem, at each location i, one may install servers with exponential service time. The cost of installing servers at an open facility $i \in F$ is linear in the number of servers. The necessary number of servers at a facility is influenced not only by the demand served by an open facility, but also by a prespecified upper bound τ on the expected waiting time of a customer.

The incurred costs are the expected transportation costs and the facility costs; the cost at a facility is composed from a fixed cost f_i and the cost of installing the necessary servers for satisfying the demand (cost c_i per server). The goal is to decide where to open facilities, how many servers to install at each open facility and an assignment of demand points to the open facilities such that the total cost is minimized and no customer has an expected waiting time larger then τ.

We model a facility as an $M_\lambda/M/k$ queue. In this model, customers arrive according to a Poisson process with rate λ. They are served by one of the k servers and have independent service times which are exponentially distributed with expectation 1. We assume that at time 0 the system is empty. The queueing discipline is first come first served.

Let $W(\lambda, k)$ denote the expected waiting time at such a queue. At an open facility i with arrival rate λ_i and k_i servers, the constraint on the waiting time becomes $W(\lambda_i, k_i) \leqslant \tau$ (an explicit expression for this expectation can be found in e.g. [9, p. 71]). Define $N_\tau(\lambda) = \min\{k | W(\lambda, k) \leqslant \tau\}$. The following result can be easily proven (see Lemma 18 in Appendix)

$$N_\tau(\lambda_1 + \lambda_2) \leqslant N_\tau(\lambda_1) + N_\tau(\lambda_2)$$

which implies that in this facility location problem with exponential servers, the cost functions for facilities, defined by $v_i(\lambda) = f_i + N_\tau(\lambda)c_i$, are subadditive. Clearly, the v_i's also satisfy condition (1).

2.4. Facilities with inventory replenished by exponential servers

In this variant of the facility location problem, inventory is kept at each open facility. An arriving request finding a zero inventory (and has to wait), is backlogged. The inventories at the open facilities are restored by servers installed at the facilities. The costs during a unit period at an open facility at location $i \in F$ are g_i per unit inventory, p_i per backlogged customer, c_i per installed server and f_i for keeping the facility open.

The goal is to decide at which locations to open facilities, the level of inventory, the number of servers to be installed at each open facility and an assignment of demand points to facilities such that the average total cost per unit period is minimized. The total cost is the sum of the expected transportation costs, the facility costs and the costs for backlogged customers.

As in the previous example, we model a facility with k servers as an $M_\lambda/M/k$ queue. The queueing discipline is again first come first served. We assume that the maximum number of stored items is m. Let $L_\infty(\lambda, k)$ denote the number of customers at such a station in steady state. The number of backlogged requests can now be written as $\max\{0, L_\infty(\lambda, k) - m\}$ (an explicit expression for the distribution of $L_\infty(\lambda, k)$ can be found in e.g. [9, p. 71]).

At an open facility i with arrival rate λ, the cost function can be written as

$$v_i(\lambda) = f_i + \min\{kc_i + mg_i + p_i E(\max\{0, L_\infty(\lambda, k) - m\}) | k \in \mathbb{N}^+, m \in \mathbb{N}\}.$$

From Lemma 19 (see Appendix) follows that v_i is subadditive. It can be easily verified that v_i also satisfy condition (1).

3. Approximation algorithms

The facility location problem with subadditive cost function is a generalization of the UFLP and therefore an NP-hard problem. For this reason, we are interested in approximation algorithms for this problem. Our analysis is based on reducing the facility location problem with subadditive costs to other types of facility location problems for which approximation algorithms are known.

To categorize the reductions, we introduce some notations.

For a facility location problem P and an instance $\mathcal{I}$, a feasible solution $\mathcal{S}$ is formed by a set of open facilities and an assignment of demand points to the open facilities. We denote by $cost_{f,\mathcal{I}(P)}(\mathcal{S})$ the cost incurred for opening facilities, including the cost for purchasing resources and by $cost_{c,\mathcal{I}(P)}(\mathcal{S})$ the transportation cost incurred by $\mathcal{S}$.

Definition 3. We call a polynomial time reduction $\mathcal{R}$ from problem P_1 to P_2 a (σ_f, σ_c)-reduction if $\mathcal{R}$ maps an instance $\mathcal{I}$ of P_1 to an instance $\mathcal{R}(\mathcal{I})$ of P_2 and it has the following properties:

(a) For any feasible solution $\mathcal{S}_1$ for the instance $\mathcal{I}$ of P_1 there is a corresponding solution $\mathcal{S}_2$ for the instance $\mathcal{R}(\mathcal{I})$ of P_2 with

$$cost_{f,\mathcal{R}(\mathcal{I})}(\mathcal{S}_2) \leqslant \sigma_f cost_{f,\mathcal{I}}(\mathcal{S}_1),$$

and

$$cost_{c,\mathcal{R}(\mathcal{I})}(\mathcal{S}_2) \leqslant \sigma_c cost_{c,\mathcal{I}}(\mathcal{S}_1).$$

(b) For any feasible solution $\mathcal{S}_2$ for the instance $\mathcal{R}(\mathcal{I})$ of P_2, there is a feasible solution $\mathcal{S}_1$ for the instance $\mathcal{I}$ of P_1 with

$$cost_{f,\mathcal{I}}(\mathcal{S}_1) + cost_{c,\mathcal{I}}(\mathcal{S}_1) \leqslant cost_{f,\mathcal{R}(\mathcal{I})}(\mathcal{S}_2) + cost_{c,\mathcal{R}(\mathcal{I})}(\mathcal{S}_2).$$

Definition 4. An algorithm is called an (α, β)-approximation algorithm for a facility location problem P, if for any instance $\mathcal{I}$ of P, and for any solution $\mathcal{S}$ for $\mathcal{I}$ the cost of the solution found by the algorithm is at most $\alpha cost_{f,P}(\mathcal{S}) + \beta cost_{c,P}(\mathcal{S})$.

Remark 5. Note that combining a (σ_f, σ_c)-reduction from P_1 to P_2 and an (α, β)-approximation algorithm for P_2 gives an $(\alpha\sigma_f, \beta\sigma_c)$-approximation algorithm for P_1. Moreover, the approximation guarantee of the algorithm for P_1 is $\max\{\alpha\sigma_f, \beta\sigma_c\}$.

Finally, we introduce some functions related to the subadditive cost functions v_i and analyse their properties.

Define the generalized inverse function of v_i by

$$v_i^{-1}(u) = \sup\{x | v_i(x) \leqslant u\} \quad \text{for } u \in \mathbb{R}^+,$$

the inverse function maps the cost of a resource level to the maximal demand that can be served at this cost.

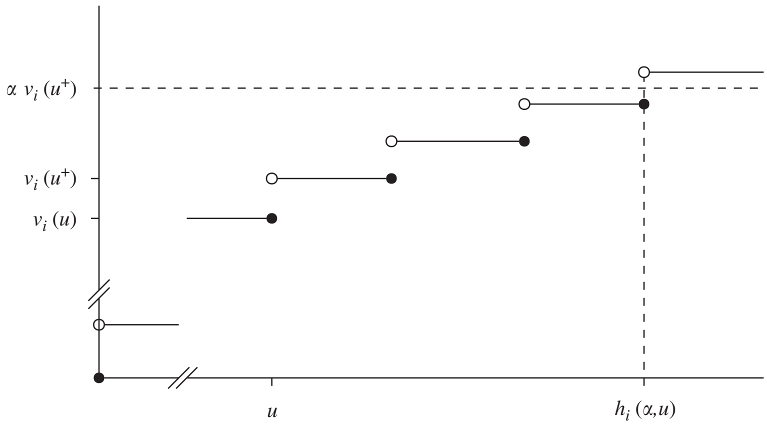

Fig. 1. The subadditive function v_i.

Denote by $v_i(x^+) \overset{\text{def}}{=} \lim_{y \downarrow x} v_i(y)$. For each $i \in F$, let

$$h_i(\alpha, u) = v_i^{-1}(\alpha v_i(u^+)) \quad \text{for } \alpha \geqslant 1 \text{ and } u \in \mathbb{R}^+.$$

The function $h_i(\alpha, u)$ maps the demand u to the maximal demand u' such that $v_i(u') \leqslant \alpha v_i(u^+)$ (see also Lemma 6(d)). In Fig. 1, the relation between v_i and h_i is depicted for a step function v_i.

Denote the n-fold composition of h_i with itself by $h_i^{n*}(\alpha, u) = h_i(\alpha, h_i^{(n-1)*}(\alpha, u))$ where $h_i^{0*}(\alpha, u) = v_i^{-1}(v_i(u))$.

In the next section, we use the functions v_i and h_i to construct soft capacitated facility location problems to which our original facility location problem can be reduced. In doing this, we use properties of these function stated in the following lemma.

Lemma 6. *The functions v_i and h_i have the following properties*:
(a) $v_i^{-1}(v_i(x)) \geqslant x$ *and* $v_i(v_i^{-1}(x)^+) > x$;
(b) $v_i(h_i^{0*}(\alpha, u)) = v_i(u)$ *for* $\alpha \geqslant 1$ *and* $u \geqslant 0$;
(c) $\alpha v_i(u^+) < v_i(h_i(\alpha, u)^+)$;
(d) $v_i(h_i(\alpha, u)) \leqslant \alpha v_i(d)$ *for* $d \in (u, h_i(\alpha, u)]$;
(e) *The sequence* $h_i^{n*}(1, 0)$ *for* $n = 0, 1, \ldots$ *and* $v_i(0) = 0$ *gives all the jump points of* v_i *and the sequence* $v_i(h_i^{n*}(1, 0))$ *for* $n = 0, 1, \ldots$ *and* $v_i(0) = 0$ *gives all the function values of* v_i;
(f) $\alpha^n v_i(u^+) < v_i(h_i^{(n+1)*}(\alpha, u))$ *for* $n = 1, 2, \ldots$.

Proof. (a)–(c) follow from the definition of v_i^{-1} and h_i;

(d) from the definition of h_i and the fact that v_i is non-decreasing follows that $v_i(h_i(\alpha, u)) \leqslant \alpha v_i(u^+) \leqslant \alpha v_i(d)$ for $d > u$;

(e) follows directly from (a) and the definitions of v_i and h_i;

(f) follows from $v_i(h_i^{(n+1)*}(\alpha, u)) \geqslant v_i(h_i^{n*}(\alpha, u)^+)$ and then repeatedly using that $v_i(h_i^{n*}(\alpha, u)^+) > \alpha v_i(h_i^{(n-1)*}(\alpha, u)^+)$ (use (c)). □

The facility location problem with subadditive cost functions can easily be reduced to the following soft capacitated facility location problem. At each location i, facilities with capacity $h_i^{l*}(1, 0)$ and corresponding costs $v_i(h_i^{l*}(1, 0))$ for $l = 0, 1, \ldots, \min\{n | h_i^{n*}(1, 0) \geqslant \sum_{j \in D} r_j\}$ may be opened. Although more facilities may be opened at a location, this will not occur in the optimal solution due to the subadditivity of the cost function. By the direct identification of solutions in the original and the soft capacitated facility location problem (same assignment of demand points to locations), we see that the cost for both solutions is equal. Note that this reduction runs only in pseudopolynomial time. If its running time of was polynomial, then it would have been a $(1, 1)$-reduction.

However, due to the subadditivity of v_i, it is possible to reduce the problem to facility location problems for which approximation algorithms have been developed. First we will show that a $2(1+\varepsilon)$-approximation algorithm exists, based on a reduction to a soft capacitated facility location problem. For facility location problems with subadditive cost functions satisfying $v_i(d) \geqslant \gamma_i d$, for some $\gamma_i > 0$, we propose a 2-approximation algorithm based on a reduction to a facility location problem with linear cost function. These reductions are the topic of the next sections.

The results in the next two subsections rely on the following theorem proven in [20].

Theorem 7 (*Mahdian et al. [20]*).
(a) *There exists a* $(2,2)$*-approximation algorithm for the facility location problem with soft capacities and unit demands.*
(b) *There exists a* $(1,2)$*-approximation algorithm for the facility location problem with linear costs* (*the cost for opening a facility is linear in the demand served by that facility*).

Remark 8. The $(2,2)$-approximation algorithm for the facility location problem with soft capacities described in [20] can be easily extended to general demands (see Lemma 20 in Appendix).

3.1. A $2(1+\varepsilon)$*-approximation algorithm*

In this section, we show that a $2(1+\varepsilon)$-approximation algorithm for the facility location problem with subadditive costs exists. First we propose a $(2,1)$-reduction to a soft capacitated facility location problem. Then we refine the soft capacitated problem and show that a $(1+\varepsilon, 1)$-reduction is possible. The $2(1+\varepsilon)$-approximation algorithm will follow by combining the reduction with Theorem 7(a) and Remark 8.

Let $\mathbf{I}$ be an instance of a facility location problem with subadditive cost functions. Construct the following instance $\mathbf{I}_2$ of the soft capacitated facility location problem. The demand points, their requests and facility locations are the same as in $\mathbf{I}$. For $i \in F$, let $M_i = \lceil \log_2(\sum_{j\in D} r_j/s_i)\rceil + 1$, where s_i is the step width of the first step of v_i (see (1)). At each location $i \in F$, M_i types of facilities may be opened. A facility of type ℓ at location i is denoted by the pair (i,ℓ). Facility (i,ℓ) has capacity $u_{i1} = s_i$ for $\ell = 1$, capacity $u_{i\ell} = h_i^{0*}(1, 2u_{i\ell-1})$ for $\ell \in \{2,\dots,M_i\}$ and cost $f_{i\ell} = v_i(u_{i\ell})$. At each location $i \in F$, at most one facility of each type may be opened. The goal is to decide where to open facilities and of which type, and how to assign the demand points to facilities, such that the total cost, i.e., the cost for opening facilities and the transportation cost, is minimized.

Remark 9. (a) From Lemma 6(b) and the subadditivity of v_i, follows that $v_i(u_{i\ell+1}) = v_i(2u_{i\ell}) \leqslant 2v_i(u_\ell)$.

(b) Suppose that in the optimal solution for $\mathbf{I}_2$, at location i a facility of type $k < M_i$ is opened. Since opening a facility of type $k+1$ is not more expensive and has at least the same capacity as two facilities of type k, only one facility of type k is opened. Suppose now, that in the optimal solution, a facility of type M_i is opened. Since this facility already can handle all the demand, no other facility will be opened at location i. Thus, in the optimal solution, at most one facility of a type is opened.

In the following lemma we prove that the construction described above leads to a $(2,1)$-reduction of the facility location problem with subadditive costs to the soft capacitated facility location problem.

Lemma 10. (i) *For each feasible solution* $\mathcal{S}_1$ *for* $\mathbf{I}$, *there exists a feasible solution* $\mathcal{S}_2$ *corresponding to* $\mathbf{I}_2$ *with* $cost_{c,\mathbf{I}_2}(\mathcal{S}_2) = cost_{c,\mathbf{I}}(\mathcal{S}_1)$ *and* $cost_{f,\mathbf{I}_2}(\mathcal{S}_2) \leqslant 2cost_{f,\mathbf{I}}(\mathcal{S}_1)$.

(ii) *For each feasible solution* $\mathcal{S}_2$ *for* $\mathbf{I}_2$, *there exists a feasible solution* $\mathcal{S}_1$ *for* $\mathbf{I}$ *of lower cost.*

(iii) *There exists a* $(2,1)$*-reduction of the facility location problem with subadditive costs to the soft capacitated facility location problem.*

Proof. (i) Let $\mathcal{S}_1$ be a solution for $\mathbf{I}$. We construct the following solution $\mathcal{S}_2$ for $\mathbf{I}_2$. Consider a location $i \in F$. If the facility at location i is not opened in $\mathcal{S}_1$, it will not be opened in $\mathcal{S}_2$ either. If the facility at location i is opened in $\mathcal{S}_1$, let d_i be the demand assigned to it. In $\mathcal{S}_2$, open the facility (i,ℓ), where $\ell = \min\{n | d_i \leqslant u_{in}\}$ and assign all the demand d_i to it. Clearly, the transportation costs incurred by $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same. Moreover, at each location i where facilities are opened, the opening costs incurred by $\mathcal{S}_1$, respectively $\mathcal{S}_2$ satisfy

$$f_{i\ell} = v_i(u_{i\ell}) \leqslant 2v_i(d_i),$$

where the inequality follows from the fact that v_i is not decreasing, $u_{i\ell-1} < d < u_{i\ell}$ and $v_i(u_{i\ell}) \leqslant 2v_i(u_{i\ell-1})$ (see Remark 9(a)). Hence, $cost_{f,\mathbf{I}_2}(\mathcal{S}_2) \leqslant 2cost_{f,\mathbf{I}}(\mathcal{S}_1)$.

(ii) Consider now a feasible solution $\mathcal{S}_2$ for $\mathbf{I}_2$. We construct a feasible solution $\mathcal{S}_1$ for $\mathbf{I}$ of lower cost as follows. If at a location i is a facility opened by $\mathcal{S}_2$, open a facility in $\mathcal{S}_1$ as well. Let F_i denote the set of the types of facilities opened at location i in $\mathcal{S}_1$ and $d_{i\ell}$ the demand served by the facilities $\ell \in F_i$. In $\mathcal{S}_1$, assign to facility i the demand $d_i = \sum_{\ell \in F_i} d_{i\ell}$. The transportation costs incurred by $\mathcal{S}_1$ and $\mathcal{S}_2$ are again the same. Since v_i is subadditive and non-decreasing, $v_i(d_i) = v_i(\sum_{\ell \in F_i} d_{i\ell}) \leqslant \sum_{\ell \in F_i} v_i(d_{i\ell}) \leqslant \sum_{i \in F_i} f_{il}$. Therefore, $cost_{f,\mathbf{I}}(\mathcal{S}_1) \leqslant cost_{f,\mathbf{I}_2}(\mathcal{S}_2)$.

(iii) Follows from (i) and (ii) of this lemma. □

In the following, we refine the (2, 1)-reduction of the facility location problem with subadditive cost function to the soft capacitated facility location problem to a $(1+\varepsilon, 1)$-reduction.

Let $\mathbf{I}_{1+\varepsilon}$ be the following instance of the soft capacitated facility location problem. The demand characteristics and the locations of the facilities are the same as in $\mathbf{I}$. At each location $i \in F$, at most M_i types of facilities may be opened (and only at most one facility of each type). A facility of type ℓ at location i, has capacity $u_{i\ell}$ and opening cost $v_i(u_{i\ell})$, where M_i and the capacities $u_{i\ell}$ are defined below.

For $\varepsilon > 0$, construct iteratively the sequence $u_{i,ml}$ as follows: $u_{i,m0} = h_i^{m*}(1+\varepsilon, 0)$ and $u_{i,mk} = h_i^{0*}(1, 2u_{im,k-1})$ for $m = 1, 2, \ldots$ and $k = 1, 2, \ldots$. The capacities $u_{i\ell}$ are now defined by $u_{i0} = 0$ and $u_{i\ell} = \min\{u_{i,mk} > u_{i,\ell-1} | m = 1, 2, \ldots \text{ and } k = 0, 1, \ldots\}$ for $\ell = 1, 2, \ldots$.

For each $i \in F$, let $M_i = \min\{\ell | u_{i\ell} \geqslant \sum_{j \in D} r_j)\}$.

Remark 11. (a) Note that from Lemma 6(f) follows that $v_i(u_{i,m+1,0}) \geqslant (1+\varepsilon)^m v(0^+)$. Hence, for $\varepsilon \in (0, 1)$,

$$M_i \leqslant \lceil 1 \rceil + \log_{(1+\varepsilon)} \left(\frac{v_i(\sum_{j \in D} r_j)}{v_i(0^+)} \right) \left\lceil 1 + \log_2 \left(\frac{\sum_{j \in D} r_j}{s_i} \right) \right\rceil$$

$$\leqslant \frac{4}{\varepsilon} \left\lceil 1 + \log_2 \left(\frac{v_i(\sum_{j \in D} r_j)}{v_i(0^+)} \right) \right\rceil \left\lceil 1 + \log_2 \left(\frac{\sum_{j \in D} r_j}{s_i} \right) \right\rceil,$$

with s_i as defined in (1). This shows that M_i is bounded by a linear function in $1/\varepsilon$.

(b) For d with $u_{i\ell} < d \leqslant u_{i,\ell+1}$, it follows from the construction of the sequence $u_{i\ell}$ and from Lemma 6(d), that $v_i(u_{i\ell}^+) \leqslant v_i(d) \leqslant v_i(u_{i,\ell+1}) \leqslant (1+\varepsilon) v_i(u_{i\ell}^+)$.

As in Remark 9(b), we will show that in an optimal solution for $\mathbf{I}_{1+\varepsilon}$, at most one facility of a type is opened at every location. Assume that in the optimal solution, at least one facility of type k at location i is opened. Suppose that u_{ik}, the capacity of facility (i, k) satisfies $v_i(u_{ik}) < v_i(u_{iM_i})/2$. Let (i, k') be the facility with capacity $h_i^{0*}(1, 2u_{ik})$ (the existence of such a facility follows from the definition of $u_{i\ell}$). The cost of facility (i, k') is at most twice the cost of facility (i, k). If $v_i(u_{ik}) \geqslant v_i(u_{iM_i})/2$, then opening facility (i, M_i) (which can handle all demands) is not more expensive than opening two facilities (i, k). Hence, in the optimal solution for $\mathbf{I}_{1+\varepsilon}$, at every location at most one facility of type k will be opened.

Lemma 12. *For any* $\varepsilon > 0$, *the facility location problem with subadditive cost functions can be* $(1+\varepsilon, 1)$*-reduced to a soft capacitated facility location problem.*

Proof. We follow the proof of Lemma 10. Consider a feasible solution $\mathcal{S}_1$ for $\mathbf{I}$ and construct a feasible solution $\mathcal{S}_2$ for $\mathbf{I}_{1+\varepsilon}$ as follows. Open a facility at location i in $\mathcal{S}_2$ only if a facility is opened at i in $\mathcal{S}_1$. The type of the open facility is $\ell = \min\{n | d_i \leqslant u_{in}\}$, where d_i is the demand assigned to i in $\mathcal{S}_1$. By Remark 11(b) and the definition of ℓ, it follows that the cost of opening facilities in $\mathbf{I}_{1+\varepsilon}$ is at most $(1+\varepsilon)$ times the facility costs in $\mathbf{I}$. The transportation costs incurred by $\mathcal{S}_1$ and $\mathcal{S}_2$ are clearly the same.

Now consider a solution $\mathcal{S}_2$ for $\mathbf{I}_{1+\varepsilon}$ and construct a corresponding solution $\mathcal{S}_1$ of $\mathbf{I}$, of lower cost, as follows. Open a facility at location i in $\mathcal{S}_1$ only if a facility of any type was opened at i in $\mathcal{S}_2$. Assign to i all the demand assigned to all facilities opened at i in $\mathcal{S}_2$. As in Lemma 10, one can show that $\mathcal{S}_1$ incurs the same transportation cost and lower opening facility cost than the costs incurred by $\mathcal{S}_2$. □

Theorem 13. *There exists a* $2(1+\varepsilon)$*-approximation algorithm for the facility location problem with subadditive cost functions.*

Proof. Recall that $\mathbf{SP}_{1+\varepsilon}$ is a soft capacitated facility location problem with general demands. The lemma follows by combining Remark 8, Lemma 12 and Remark 5. □

Next we refine the approximation guarantee for a specific class of subadditive functions.

3.2. A 2-approximation algorithm

In this section we assume that the functions v_i satisfy

Condition 14.
(a) *In any finite interval the number of jump points of* v_i *is finite.*

(b) $$\gamma_i = \inf \left\{ \frac{v_i(d)}{d} \,\middle|\, d \in \mathbb{R}^+ \right\} > 0. \tag{3}$$

These properties enable a 2-approximation algorithm for our problem. The algorithm is based on a (2, 1)-reduction of the facility location problem with subadditive cost functions to the facility location problem with linear costs and general demands. This reduction, together with Theorem 7 will lead to a (2, 2)-approximation algorithm for the special class of facility location problems with subadditive costs under consideration. The main idea of the reduction is that $v_i(d_i)$ can be approximated within a factor of 2 by a piecewise linear function with the number of segments independent of the input size.

Before presenting the reduction, we introduce some notations.

Let $\mathbf{I}$ be an instance of the facility location problem with subadditive costs satisfying the extra Condition 14.

Let $u_{i\ell} = h_i^{\ell *}(1, 0)$, the jump points of v_i (cf. Lemma 6(e)). Define $\ell_i^* = \min\{\ell | v_i(u_{i\ell})/u_{i\ell} \leqslant 4\gamma_i/3\}$, $d_i^* = 2u_{i,\ell_i^*}$ and $L_i^* = \min\{\ell | u_{i\ell} \geqslant d_i^*\}$. From the definition of γ_i (see 3) and Condition 14(a) follows that L_i^* is finite and does not depend on the input size.

Denote by $\mathbf{I}_L$ the following instance of the facility location problem with linear costs. The demand points and the locations where facilities may be built are the same as in $\mathbf{I}$. At each location, L_i^* types of facilities may be opened. Facility (i, ℓ), at location i and of type ℓ, has costs $(1 + d_{i\ell}/u_{i\ell})v_i(u_{i\ell})$, where $d_{i\ell}$ is the demand that is assigned to facility(i, ℓ).

We proceed with the (2, 1)-reduction to the linear cost facility location problem with general demands.

Lemma 15. *If the subadditive cost functions satisfy the extra Condition* 14, *there exists a* (2, 1)*-reduction of the stochastic facility location problem with subadditive costs to the linear facility location problem with general demands.*

Proof. Let $\mathcal{S}_1$ be a solution for $\mathbf{I}$. We construct the following solution $\mathcal{S}_2$ for $\mathbf{I}_L$. Consider a location i where a facility is opened by $\mathcal{S}_1$. Let d_i be the demand assigned to it. If $d_i > d_i^*$ then open in $\mathcal{S}_2$, at location i, a facility of type $\ell = \ell_i^*$. Otherwise, open a facility of type ℓ, where ℓ is chosen such that $u_{i,\ell-1} < d_i \leqslant u_{i\ell}$. Note that such an ℓ always exists and is unique by the definition of $u_{i\ell}$. Assign to facility (i, ℓ) all the demand d_i.

Clearly, the transportation costs incurred by $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same. Between the costs of opening the facilities, the following relationship exists.

If $d_i \leqslant d_i^*$,

$$\left(1 + \frac{d_i}{u_{i\ell}}\right) v_i(u_{i\ell}) \leqslant 2v_i(u_{i\ell}) = 2v_i(d_i),$$

where the last equality follows from the fact that $u_{i,\ell-1} < d_i \leqslant u_{i\ell}$.

If $d_i > d_i^*$,

$$\left(1 + \frac{d_i}{u_{i,\ell_i^*}}\right) v_i(u_{i,\ell_i^*}) \leqslant \frac{4}{3}\gamma_i u_{i,\ell_i^*} \left(1 + \frac{d_i}{u_{i,\ell_i^*}}\right) \leqslant 2\gamma_i d_i \leqslant 2v_i(d_i),$$

where the first inequality follows from the definition of ℓ_i^*, the second from $u_{i\ell_i^*} \leqslant 3d_i/2$ and the last from the definition of γ_i. Hence, $cost_{f,\mathbf{I}_L}(\mathcal{S}_2) \leqslant 2cost_{f,\mathbf{I}}(\mathcal{S}_1)$.

Let $\mathcal{S}_2$ be a solution for $\mathbf{I}_L$. We construct a solution $\mathcal{S}_1$ for $\mathbf{I}$ of lower cost than $\mathcal{S}_2$ as follows. Denote by F_i the set of facilities opened by $\mathcal{S}_2$ at location i. If $F_i \neq \emptyset$, open a facility at location i in $\mathcal{S}_1$ as well. Let $d_{i\ell}$ be the demand assigned to the facility $\ell \in F_i$ by $\mathcal{S}_2$. In $\mathcal{S}_1$, assign to facility i demand $d_i = \sum_{\ell \in F_i} d_{i\ell}$.

The transportation costs of $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same. We compare next the costs for opening facilities. Since v_i is subadditive, the following relations hold:

$$v_i(d_i) \leqslant \sum_{\ell \in F_i} v_i(d_{i\ell}) \tag{4}$$

and

$$v_i(d_{i\ell}) = v_i\left(\frac{d_{i\ell}}{u_{i\ell}}u_{i\ell}\right) \leqslant \left\lceil \frac{d_{i\ell}}{u_{i\ell}} \right\rceil v_i(u_{i\ell}) \leqslant \left(1 + \frac{d_{i\ell}}{u_{i\ell}}\right) v_i(u_{i\ell}). \tag{5}$$

Combining (4) and (5), it follows that $cost_{f,\mathbf{I}}(\mathcal{S}_1) \leqslant cost_{f,\mathbf{I}_L}(\mathcal{S}_2)$. □

From Theorem 7 and Lemma 15 the next theorem follows.

Theorem 16. *There exists a 2-approximation algorithm for the facility location problem with stochastic demands and inventory.*

4. Conclusions

In this paper we have considered facility location problems with shared resources, which are modelled via subadditive costs for opening facilities. As examples of such problems, we have presented three facility location problems with stochastic demand and exponential servers, respectively inventory. We have proposed a $2(1 + \varepsilon)$-approximation algorithm for this model based on a $(1 + \varepsilon, 1)$-reduction to a soft capacitated facility location problem with general demands. For the special case where the subadditive cost functions have a positive linearly increasing lower bound, we have proved the existence of a 2-approximation algorithm.

Appendix A

In this appendix we prove some results that are used in this paper. We present three lemmas on queues that are used in the examples at the end of Section 2. The appendix ends with a lemma on the existence of an approximation algorithm for the soft capacitated facility location problem.

In the sequel, we focus on $M_\lambda/M/k$ queueing systems. In these models, customers arrive according to a Poisson process with rate λ. They are served by one of the k servers and have independent service times which are exponentially distributed with expectation 1. We assume that at time 0 the system is empty. Denote the number of customers in the system at time t by $L_t(\lambda, k)$.

Lemma 17. *For all* $t \geqslant 0$

$$Pr(L_t(\lambda_1 + \lambda_2, k_1 + k_2) < m_1 + m_2) \geqslant P(L_t(\lambda_1, k_1) + L_t(\lambda_2, k_2) < m_1 + m_2).$$

Proof. Consider an $M_{\lambda_1}/M/k_1$ queue and an $M_{\lambda_2}/M/k_2$ queue in isolation. We say that the servers and customers served in the $M_{\lambda_1}/M/k_1$ system are of type I and the servers and customers served in the $M_{\lambda_2}/M/k_2$ system are of type II.

Consider a new $M_{\lambda_1+\lambda_2}/M/k_1 + k_2$ queue, formed by "merging" the $M_{\lambda_1}/M/k_1$ and $M_{\lambda_2}/M/k_2$ queues with the following working discipline. Each server basically serves clients of the same type. However, if the server is idle, it will serve a client of a different type if such a client is present. If during such an event a customer of the same type as the server arrives, the service of the customer of different type is interrupted and this customer goes back in queue. The service of the newly arrived customer (of the same type as the server) begins immediately. Since the service times are exponentially distributed, the number of customers in the new queue is the same as in an $M_{\lambda_1+\lambda_2}/M/k_1+k_2$ with a standard first come first served working discipline and less than the total number of customers in the $M_{\lambda_1}/M/k_1$ and $M_{\lambda_2}/M/k_2$ systems in isolation. This proves the lemma. $\square$

Lemma 18. $N_\tau(\lambda_1+\lambda_2) \leqslant N_\tau(\lambda_1)+N_\tau(\lambda_2)$.

Proof. From Little's Law (see e.g. [26, p. 235]) we know that in steady state, the expected sojourn time of a customer is linear in the expected number of customers in the system that is $W(\lambda,k)=E(L_\infty(\lambda,k))/\lambda$. From the definition of $N_\tau(\lambda)$ and the previous lemma follows that

$$\begin{aligned} W(\lambda_1+\lambda_2, N_\tau(\lambda_1)+N_\tau(\lambda_2)) &= \frac{E(L_\infty(\lambda_1+\lambda_2, N_\tau(\lambda_1)+N_\tau(\lambda_2)))}{\lambda_1+\lambda_2} \\ &\leqslant \frac{E(L_\infty(\lambda_1, N_\tau(\lambda_1)))+E(L_\infty(\lambda_2, N_\tau(\lambda_2)))}{\lambda_1+\lambda_2} \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{E(L_\infty(\lambda_1, N_\tau(\lambda_1)))}{\lambda_1} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{E(L_\infty(\lambda_2, N_\tau(\lambda_2)))}{\lambda_2} \\ &\leqslant \frac{\lambda_1}{\lambda_1+\lambda_2}W(\lambda_1, N_\tau(\lambda_1)) + \frac{\lambda_2}{\lambda_1+\lambda_2}W(\lambda_2, N_\tau(\lambda_2)) \leqslant \tau. \end{aligned}$$

This proves the lemma. $\square$

Lemma 19. *The expected numbers of backlogged customers in $M_\lambda/M/k$ queues with an initial inventory m (denoted by $E(\max\{0, L_\infty(\lambda,k)-m\})$) satisfy*

$$\begin{aligned} &E(\max\{0, L_\infty(\lambda_1+\lambda_2, k_1+k_2)-m_1+m_2\}) \leqslant E(\max\{0, L_\infty(\lambda_1,k_1)-m_1\}) \\ &\quad +E(\max\{0, L_\infty(\lambda_2,k_2)-m_2\}). \end{aligned}$$

Proof. Since $\max\{0, \ell_1+\ell_2-m_1+m_2\} \leqslant \max\{0, \ell_1-m_1\}+\max\{0, \ell_2-m_2\}$, and in a system where the servers are shared, the number of customers is not more than in a system without shared servers (see Lemma 17), the lemma follows. $\square$

Lemma 20. *There exists a* $(2,2)$*-approximation algorithm for the soft capacitated facility location problem with general demands.*

Proof. In [20], the authors first present a (2, 1)-reduction of the soft capacitated facility location problem with unit demands to a linear cost facility location problem. Then the authors prove that the algorithm proposed by Jain et al. [14] for the UFLP can be used for deriving an (1, 2)-approximation algorithm for the linear cost facility location problem.

The same ideas can be applied to design a (2,2)-approximation algorithm for the soft capacitated problem with general demands. We will just describe the reduction of a soft capacitated facility location problem with general demands to a linear cost facility location problem, the rest of the proof being identical to the one in [20]. In general, we denote by f_i the cost of opening a facility at location i and by u_i the capacity of this facility. Consider an instance **I** of a soft capacitated facility location problem with general demands. Construct an instance of a linear cost facility location problem as follows: the demand points, demands, set of locations, transportation costs are the same as in **I**. The costs of opening facilities are given by $f'_i = f_i(1+d_i u_i^{-1})$, where d_i is the demand served by the facility at location i. Note that $\lceil du_i^{-1}\rceil \leqslant 1+du_i^{-1} \leqslant 2\lceil du_i^{-1}\rceil$ for $d>0$ and $u_i>0$. Hence, this reduction is a (2, 1)-reduction of a soft capacitated facility location problem with general demands to a linear cost facility location problem with general demands. $\square$

References

[1] V. Arya, N. Garg, R. Khandekar, V. Pandit, A. Meyerson, K. Munagala, Local search heuristics for k-median and facility location problems, in: Proc. 33rd ACM Symp. on Theory of Computing, 2001, pp. 21–29.

[2] R. Batta, A queuing location model with expected service time dependent queuing disciplines, European J. Oper. Res. 39 (1989) 192–205.

[3] O. Berman, R. Larson, S. Chiu, Optimal server location on a network operating as a $M/G/1$ queue, Oper. Res. 12 (1985) 746–771.

[4] O. Berman, K. Sapna, Optimal control of service for facilities holding inventory, Comput. Oper. Res. 28 (2001) 429–441.
[5] A.F. Bumb, J.C.W. van Ommeren, An approximation algorithm for a facility location problem with inventories and stochastic demands, in: N. Megiddo, Y. Xu, B. Zhu (Eds.), Proc. First Internat. Conf. on Algorithmic Applications in Management, Springer, Lecture Notes in Computer Science, Vol. 3521, 2005, pp. 330–339.
[6] A.F. Bumb, J.C.W. van Ommeren, An approximation algorithm for a facility location problem with stochastic demands and inventory, Oper. Res. Lett. 34 (3) (2006) 257–263.
[7] F.A. Chudak, D.B. Shmoys, Improved approximation algorithms for the uncapacitated facility location problem, SIAM J. Comput. 33 (1) (2003) 1–25.
[8] G. Cornuejols, G.L. Nemhauser, L.A. Wolsey, The uncapacitated facility location problem, in: P. Mirchandani, R. Francis (Eds.), Discrete Location Theory, Wiley, New York, 1990, pp. 119–171.
[9] D. Gross, C.M. Harris, Fundamentals of Queueing Theory, third ed., Wiley, New York, 1998.
[10] S. Guha, S. Khuller, Greedy strikes back: improved facility location algorithms, J. Algorithms 31 (1) (1999) 228–248.
[11] S. Guha, A. Meyerson, K. Munagala, A constant factor approximation algorithm for the fault-tolerant facility location problem, J. Algorithms 48 (2) (2003) 429–440.
[12] A. Gupta, M. Pal, R. Ravi, A. Sinha, Boosted sampling: approximation algorithms for stochastic optimization, in Proc. 36th Ann. ACM Symp. on Theory of Computing, 2004, pp. 417–426.
[13] M.G. Hajiaghayi, M. Mahdian, V. S Mirrokni, The facility location problemwith general cost functions, Networks 42 (1) (2003) 42–47.
[14] K. Jain, M. Mahdian, E. Markakis, A. Saberi, V. Vazirani, Greedy facility location Algorithms analyzed using dual fitting with factor-revealing LP, J. ACM 50 (6) (2003) 795–824.
[15] K. Jain, V. Vazirani, Approximation algorithms for metric facility location and *k*-median problems using the primal-dual schema and Lagrangian relaxation, J. ACM 48 (2001) 274–296.
[16] M.R. Korupolu, C.G. Plaxton, R. Rajaraman, Analysis of a local search heuristic for facility location problems, J. Algorithms 37 (1) (2000) 146–188.
[17] M. Mahdian, Facility location and the analysis of algorithms through factor-revealing programs, Ph.D. Thesis, MIT, June 2004, available at ⟨http://www-math.mit.edu/∼mahdian/phdthesis.pdf⟩.
[18] M. Mahdian, M. Pal, Universal facility location, ESA 2003, 2003, pp. 409–421.
[19] M. Mahdian, Y. Ye, J. Zhang, A 1.52 approximation algorithm for the uncapacitated facility location problem, in: Proc. Fifth Internat. Workshop on Approximation Algorithms for Combinatorial Optimization, Springer, Lecture Notes on Computer Science, Vol. 2462, 2002, pp. 229–242.
[20] M. Mahdian, Y. Ye, J. Zhang, A 2-approximation algorithm for the soft-capacitated facility location problem, RANDOM-APPROX 2003, 129–140.
[21] V. Marianov, D. Serra, Location-allocation of multiple-server service centers with constrained queues or waiting times, Ann. Oper. Res. 111 (2002) 35–50.
[22] R. Ravi, A. Sinha, Hedging uncertainty: approximation algorithms for stochastic optimization problems, IPCO 2004, 2004, pp. 101–115.
[23] D. Shmoys, E. Tardos, K. Aardal, Approximation algorithms for facility location problems, in: Proc. 29th ACM Symp. on Theory of Computing, 1997, pp. 265–274.
[24] M. Sviridenko, Personal communication. Cited in S. Guha, Approximation algorithms for facility location problems, P.h.D. Thesis, Stanford, 2000, (Downloadable from website ⟨http://Theory.Stanford.EDU/∼sudipto⟩).
[25] Q. Wang, R. Batta, C. Rump, Algorithms for a facility location problemwith stochastic customer demand and immobile servers, Ann. Oper. Res. 111 (2002) 17–34.
[26] R. W. Wolf, Stochastic Modeling and the Theory of Queues, Prentice-Hall International Series in Industrial and System Engineering, 1989.
[27] J. Zhang, B. Chen, Y. Ye, A Multi-exchange local search algorithm for the capacitated facility location problem, IPCO 2004, 2004, pp. 219–233.