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Scattering number and modular decomposition

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Abstract

The scattering number of a graph G equals $\max\{c(G\backslash S) - |S| \mid S$ is a cutset of $G\}$ where $c(G\backslash S)$ denotes the number of connected components in $G\backslash S$. Jung (1978) has given for any graph having no induced path on four vertices (P_4-free graph) a correspondence between the value of its scattering number and the existence of Hamiltonian paths or Hamiltonian cycles. Hochstättler and Tinhofer (to appear) studied the Hamiltonicity of P_4-sparse graphs introduced by Hoàng (1985).

In this paper, using modular decomposition, we show that the results of Jung and Hochstättler and Tinhofer can be generalized to a subclass of the family of semi-P_4-sparse graphs introduced in Fouquet and Giakoumakis (to appear).

1. Introduction

1.1. Motivations

Jung in [14] studied the existence of a hamiltonian path or a hamiltonian cycle in a graph G without chordless path of four vertices (*P_4-free graph*), by examining the value of the *scattering number* of G.

The class of P_4-free graphs or *cographs*, introduced in the early 1970s by Lerch in [16,17] has been discovered independently in different areas of Mathematics and Computer Sciences. Corneil et al. [4] proposed a linear (in the number of edges of G) recognition algorithm obtained from a unique tree representation of a cograph G (a *cotree* associated with G).

The numerous structural properties of P_4-free graphs motivated various researchers to define classes of graphs obtained as extensions of cographs. In [10], Hoàng introduced the class of P_4-sparse graphs as the graphs for which every set of five vertices induces at most one P_4.

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By extending the notion of cotree, Jamison and Olariu proposed, in [12], a unique tree representation for P_4-sparse graphs, used as framework to a linear recognition algorithm for them in [13]

This tree representation is also the underlying data structure in [11] for studying the hamiltonicity of P_4-sparse graphs.

Using the modular decomposition of graphs, Giakoumakis in [8] and Fouquet and Giakoumakis in [6] studied two classes of graphs strictly containing the class of P_4-sparse graphs. In [8,6] linear algorithms for the recognition as well as for classical optimization problems, are obtained from the unique (up to isomorphism) *modular decomposition* tree associated with any graph G.

In this paper we introduce a new method for studying hamiltonicity of graphs, by associating the modular decomposition tree of a graph G with its scattering number. To illustrate our technique, we apply this method to a class of graphs: the Jung semi-P_4-sparse graphs (see the definition below).

1.2. Definitions and notations

As usual, for any undirected graph G, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges (or simply by V and E if there is no confusion, and we shall write $G=(V,E)$). For any set of vertices A of G, the subgraph induced by A is denoted $G[A]$, while the subgraph $G[V\backslash A]$ is simply denoted by $G\backslash A$. The *complement* $\bar{G}$ of G is the graph $(V,\bar{E})$ where $\bar{E}$ is the set $\{xy \mid x\in V, y\in V, x \neq y \text{ and } xy \notin E\}$. A connected component of a graph is simply said to be a *component* and the number of components of a graph G is denoted by $c(G)$. For any path P, the *length* of P is the number of its edges. A chordless path on k vertices shall be denoted by P_k. If $V(P_k)=\{v_1,\dots,v_k\}$ and $E(P_k)=\{v_iv_{i+1} \mid i\in\{1,\dots,k-1\}\}$, P_k is also denoted by $[v_1,\dots,v_k]$. Vertices v_1 and v_k are called *end-vertices*, while $v_2,\dots,v_{k-1}$ are *internal* vertices. If u and v are vertices of a path P then $P[u,v]$ denotes the subpath of P whose end-vertices are u and v. In a P_4, $[a,b,c,d]$, the two internal vertices b and c are referred to as *midpoints* while the end-vertices a and d as *endpoints*. A hamiltonian path whose end-vertices are x and y is said to be a *hamiltonian path joining x and y*.

Let H be a simple graph with vertices $\{v_1,\dots,v_n\}$ and let $\{G_1,\dots,G_n\}$ be a family of vertex-disjoint simple graphs. The *join of* $\{G_1,\dots,G_n\}$ *over* H (or *composition of* $\{G_1,\dots,G_n\}$ *over* H) is the graph denoted by $J_H(G_1,\dots,G_n)$ having $V(G_1)\cup\dots\cup V(G_n)$ as vertices and a pair $\{u,v\}$, with $u\in V(G_i)$ and $v\in V(G_j)$, is an edge of the join if either $i=j$ and $\{u,v\}$ is an edge of G_i, or $i\neq j$ and $\{v_i,v_j\}$ is an edge of H.

Let $\mathscr{X}$ be a set of graphs. We shall say that a graph G is *$\mathscr{X}$-free* if no induced subgraph of G is isomorphic to a graph of $\mathscr{X}$. A set of graphs $\mathscr{F}$ will be *$\mathscr{X}$-free* if every graph of $\mathscr{F}$ is $\mathscr{X}$-free. The subset $\mathscr{F}$ of all $\mathscr{X}$-free graphs of a set of graphs $\mathscr{G}$ is said to be *defined by the forbidden configurations $\mathscr{X}$*.

Let $k \geqslant 3$ be an integer. A *k-sun* is a graph obtained from a chordless cycle $[x_1, \ldots, x_k, x_1]$ by adding k new vertices $y_1, \ldots, y_k$ and the edges $x_1 y_1, \ldots, x_k y_k$. We shall be interested in families of graphs defined by forbidden configurations including a 3-sun.

For terms not defined in this paper the reader can refer to [9].

2. Scattering number and Jung's conditions

2.1. Scattering number

Let G be a graph and let Σ be the family of the subsets S of $V(G)$ such that the number of components in $G \backslash S$ does not equal 1. The *scattering number* of G is the number $s(G) = \max\{s \mid \exists S \in \Sigma \text{ and } s = c(G \backslash S) - |S|\}$.

A set S such that $c(G \backslash S) \neq 1$ and $c(G \backslash S) - |S| = s(G)$ is called a *scattering set* of G. Note that for a graph G of order n, $s(G) = -n$ if and only if G is isomorphic to the complete graph K_n, $s(G) = n$ if and only if G is isomorphic to the stable graph $S_n = \bar{K}_n$.

Remark 1. If G is not connected then $s(G) \geqslant c(G)$.

The following lemma is implicit in [14].

Lemma 1. *Let $G = (V, E)$ be a graph and S be a scattering set of G. Then, for any subset A of V, $s(G \backslash A) \leqslant s(G) + |A|$. Moreover, if A is a subset of S then $s(G \backslash A) = s(G) + |A|$.*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. We recall that the *disjoint union* of G_1 and G_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and the *disjoint sum* is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{xy \mid x \in V_1, y \in V_2\}$. We shall denote the disjoint union of G_1 and G_2 by G_1 ⓪ G_2 and the disjoint sum by G_1 ① G_2.

Lemma 2 (Jung [14]). *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. Then*

(1) $s(G_1$ ⓪ $G_2) = \max(1, s(G_1)) + \max(1, s(G_2))$,

(2) $s(G_1$ ① $G_2) = \max(s(G_1) - |V_2|, s(G_2) - |V_1|)$.

The scattering number $s(G)$ of a graph $G = (V, E)$ distinct from a complete graph is closely related to the *toughness* $t(G) = \min\{s \mid \exists S \subset V,\ c(G \backslash S) > 1 \text{ and } s = |S|/c(G \backslash S)\}$ introduced by Chvátal [2] in order to study hamiltonicity. More precisely, if S_0 and S_1 are subsets of V such that $c(G \backslash S_0) - |S_0| = s(G)$ (a scattering set) and $c(G \backslash S_1) t(G) = |S_1|$ (a tough cutset) then it is easy to show that $c(G \backslash S_1)(1 - t(G)) \leqslant s(G) \leqslant c(G \backslash S_0)(1 - t(G))(*)$.

Remark 2. By the preceding inequalities (∗), for any graph G distinct from a complete graph, we see that $s(G) > 0$ if and only if $t(G) < 1$, and $s(G) = 0$ if and only if $t(G) = 1$. Since for any proper subset S of vertices of a hamiltonian graph G the inequality $c(G\backslash S) \leqslant |S|$ holds, for such a graph we have $s(G) \leqslant 0$ (or equivalently $t(G) \geqslant 1$). We note that the problem: '*Given a graph* G *and an integer* k, *decide whether* $s(G) \geqslant k$' is NP-complete (see [16]).

For a graph G we shall denote by $\rho(G)$ the minimum number of elementary disjoint paths which cover $V(G)$ (i.e. the *minimum path partition number* of G, or simply the *path number* of G). Skupień [23] studied some graphs whose scattering number is the number $\rho(G)$ and Jung [14] studied relationships between minimum path partition (in particular, hamiltonicity) and scattering number in P_4-free graphs. Namely, he proved the following result.

Theorem 1 (Jung [14]). *Let* $G = (V,E)$ *be a* P_4*-free graph. Then*

(1) $\rho(G) = \max(1, s(G))$,

(2) *G is hamiltonian if and only if* $s(G) \leqslant 0$ *and* $|V| \geqslant 3$,

(3) *G is hamilton-connected if and only if* $s(G) < 0$.

Since the scattering number of a P_4-free graph is computable in linear time (see [3]), by Theorem 1, we see that the Hamiltonian Decision Problem for cographs is linear.

2.2. Jung graphs

Note that

- for an arbitrary graph G, $\rho(G) \geqslant \max(1, s(G))$,
- for any hamiltonian graph G, $s(G) \leqslant 0$ and
- for any hamilton-connected graph G, $s(G) < 0$.

Definition 1. A graph $G = (V,E)$ is said to be a *Jung graph* if it verifies the following conditions:

(1) $\rho(G) = \max(1, s(G))$,

(2) if $s(G) = 0$ then G is hamiltonian,

(3) if $s(G) < 0$ then G is hamilton-connected.

A given class of Jung graphs is said to be a *Jung's family*.

A 3-sun H is not a Jung graph (because $s(H) = 1$ and $\rho(H) = 2$). By Theorem 1, the class of P_4-free graphs is an example of Jung family. The following result is implicit in [14, 3, 11].

Proposition 1. *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two Jung graphs with* $V_1 \cap V_2 = \emptyset$. *Then the disjoint union* $G_1 ⓪ G_2$ *and the disjoint sum* $G_1 ① G_2$ *are Jung graphs.*

It is quite clear that the disjoint union of two Jung graphs is a Jung graph. For the disjoint sum see, for instance, [14] (the proof is given for P_4-free graphs but can be easily generalized).

Lemma 3. *Let G_1, G_2, G_3 and G_4 be four nonempty graphs such that G_1 and G_4 are Jung graphs, and G_2 and G_3 are stable graphs, and let $P = [v_1, v_2, v_3, v_4]$ be a P_4. Then the join of $\{G_1, G_2, G_3, G_4\}$ over P, $G = J_P(G_1, G_2, G_3, G_4)$, is a Jung graph.*

The proof does not present any particular difficulty (see [21] for a complete proof). It consists in showing that if $s(G) = p \geqslant 1$ then $V(G)$ has a path partition of p paths, if $s(G) = 0$ then G has a hamiltonian cycle and if $s(G) < 0$ then for any pair of vertices $\{x, y\}$ G has a hamiltonian path of end-vertices x and y.

3. Modular decomposition of Jung semi-P_4-sparse graphs

3.1. Modular decomposition

Let $G = (V, E)$ be an arbitrary graph. A set M of vertices is called a *module* if every vertex in $V \backslash M$ is either adjacent to all the vertices in M, or to none of them. Hence, a module M of G is also a module of $\bar{G}$. The empty set, the singletons and V are the *trivial* modules of G. Many other names had been given for module; for instance, closed set [7], partitive subset [24], autonomous set [20], etc. A module M is called a *strong module* if, for any other module A, the intersection of M and A is empty or one module is contained into the other. A graph having only trivial modules is called *prime* or *indecomposable*. Let us remark that any prime graph G distinct from K_1, K_2 and S_2 is connected, $\bar{G}$ is also connected and G has at least four vertices. The *modular decomposition* is a form of decomposition of a graph G that associates with G a unique modular decomposition tree $T(G)$. The leaves of $T(G)$ are the vertices of G and a set of leaves of $T(G)$ having the same least common ancestor in $T(G)$ is a strong module of G. The internal nodes of $T(G)$ are labelled by P, S or N. Let f be an internal node of $T(G)$, $M(f)$ be the set of leaves of the subtree of $T(G)$ rooted on f, and $V(f) = \{f_1, \ldots, f_k\}$ be the set of children of f in $T(G)$. If $G[M(f)]$ is disconnected then f is labeled by P (for parallel module) and $G[M(f_1)], \ldots, G[M(f_k)]$ are its components. If $\bar{G}[M(f)]$ is disconnected then f is labeled by S (series module) and $\bar{G}[M(f_1)], \ldots, \bar{G}[M(f_k)]$ are its components. Finally, if both graphs $G[M(f)]$ and $\bar{G}[M(f)]$ are connected then f is labelled by N (neighbourhood module) and $M(f_1), \ldots, M(f_k)$ is the unique set of maximal strong submodules of $M(f)$ (see [7,1]). The decomposition of graphs following modules (called X-join by Sabidussi [22]) has been discovered independently by many people (see [19,20] for surveys on different applications). The efficient construction of the modular decomposition tree $T(G)$ had been extensively studied. McConnel and Spinrad in [18] and independently Cournier and Habib in [5] gave

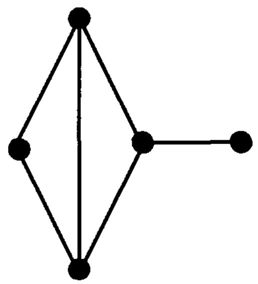

Fig. 1. Kite.

linear algorithms for this purpose ('linear' means $O(m+n)$ with $m = |E(G)|$ and $n = |V(G)|$).

As we pointed out in the introduction of this paper, we shall propose new techniques for studying hamiltonicity of graphs based on the method introduced in [8] and reemployed in [6]. This method uses, as basic data structure for recognition and optimization graph problems, the modular decomposition tree $T(G)$ of a graph G. Concerning the recognition problem of G, the key idea of this method is to transform this problem into that of recognizing a set of prime graphs associated with G. More precisely, let f be an internal node of $T(G)$ and $V(f) = \{f_1, \dots, f_k\}$ the set of children of f in $T(G)$. Then the *representative* graph $G(f)$ of the module $M(f)$ is the graph whose vertex set is $V(f)$ and whose edge set is obtained by adding an edge f_if_j in $G(f)$ if and only if there is a vertex of $M(f_i)$ that is adjacent to a vertex of $M(f_j)$. Note that, by the definition of a module, if a vertex of $M(f_i)$ is adjacent to a vertex of $M(f_j)$, then every vertex of $M(f_i)$ will be adjacent to every vertex of $M(f_j)$. Thus, $G(f)$ is isomorphic to the graph induced by a subset of $M(f)$ consisting of a single vertex from each maximal strong submodule of $M(f)$ in the modular decomposition of G. It is easy to see that if f is an S-node then $G(f)$ is a clique, if f is a P-node then $G(f)$ is a stable set and if f is an N-node then $G(f)$ is a prime graph on at least four vertices. Let us denote by $\pi(G)$ the set of prime graphs $\{G(N_1), \dots, G(N_s)\}$, where $\{N_1, \dots, N_s\}$ is the set of N-nodes of $T(G)$.

Theorem 2 (Giakoumakis [8]). *Let Z be a prime graph; then a graph G is Z-free iff every graph in $\pi(G)$ is Z-free.*

3.2. Jung semi P_4-sparse graphs

The class of semi-P_4-sparse graphs strictly containing the class of P_4-sparse graphs was defined in [6] by three forbidden configurations, namely the P_5, the $\bar{P}_5$ and the *kite* depicted in Fig. 1.

Definition 2. A *Jung semi-P_4-sparse graph G* (or JSP$_4$S graph for short) is a semi-P_4-sparse that does not contain a 3-sun as induced subgraph. In other words, a JSP$_4$S

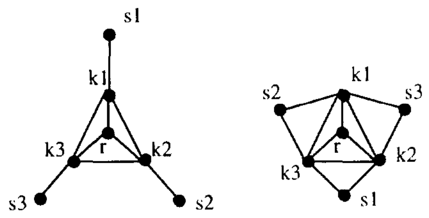

Fig. 2. An urchin and a starfish.

graph G is defined by four forbidden configurations namely the P_5, the $\bar{P}_5$, the kite and the 3-sun.

We recall that the 3-sun is not a Jung graph (see Section 2.2). In this paper we will show that any JSP$_4$S graph is a Jung graph.

Definition 3. An *urchin* is a special prime graph $G = K + S$, where K is a clique and S a stable set, such that $2 \leqslant |S| \leqslant |K| \leqslant |S| + 1$ and such that every vertex of S has exactly one neighbour in K. A *starfish* is the complementary graph of an urchin.

Notations. If $|K| = |S| + 1$ (respectively, $|K| = |S|$) we shall denote by $\{k_1, \ldots, k_l, r\}$ (respectively, $\{k_1, \ldots, k_l\}$ the vertices of K, by $\{s_1, \ldots, s_l\}$ the vertices of S. For every $i \in \{1, \ldots, l\}$, the neighbourhood of s_i in an urchin will be $N(s_i) = \{k_i\}$, and in a starfish $N(s_i) = K \backslash \{k_i, r\}$ (respectively, $N(s_i) = K \backslash \{k_i\}$); see Fig. 2 for examples with $l = 3$.

Theorem 3 (Fouquet and Giakoumakis [6]). *Let G be a prime semi-P_4-sparse graph: then G is bipartite or isomorphic to one of the graphs: C_5, starfish or urchin.*

Theorem 4 (Fouquet and Giakoumakis [6]). *A bipartite graph G is a prime P_5-free graph if and only if the degree sequence s of vertices of G in increasing order is* $s = (1, 1, 2, 2, \ldots, n/2, n/2)$.

Notations. In such a balanced bipartite graph G of bipartition (X, Y) (with $|X| = |Y| = k \geqslant 2$) we will denote by $y_1, \ldots, y_k$ the vertices of Y, those of X by $x_1, \ldots, x_k$, such that for every $i \in \{1, \ldots, k\}$ $N(x_i) = \{y_i, \ldots, y_k\}$ and $N(y_i) = \{x_1, \ldots, x_i\}$.

Consider now a graph G and the set of prime graphs $\pi(G)$ defined in Section 3.1. By Theorem 2 we clearly have that G is $(P_5, \bar{P}_5)$-free if and only if each graph of $\pi(G)$ is $(P_5, \bar{P}_5)$-free. Moreover, since every graph of $\pi(G)$ is isomorphic to an induced subgraph of G, if G is kite-free, $\pi(G)$ is kite-free. But the converse is not necessarily true.

In order to transform the recognition problem of a semi-P_4-sparse graph G into that of the associated set of prime graphs $\pi(G)$, following the method introduced in [6], we mark every vertex x of any graph of $\pi(G)$ whose corresponding module $M(x)$ in $T(G)$ does not induce a stable set in G. Indeed, two adjacent vertices of M are likely to be the two vertices that form the unique module of an induced kite in G. Then, any graph G_i of $\pi(G)$ must verify the following.

(i) If G_i is isomorphic to a C_5, no vertex of G_i is marked.

(ii) If G_i is a bipartite P_5-free graph, no vertex of degree greater than 1 is marked.

(iii) If G_i is isomorphic to a starfish or an urchin with vertex set $K+S$, no vertex of the complete graph K having a neighbour in S is marked.

If G_i verifies conditions (ii) or (iii), G_i is said to be *weak-marked*. Then, we have the following characterization of semi-P_4-sparse graphs.

Theorem 5 (Fouquet and Giakoumakis [6]). *A graph G is semi-P_4-sparse if and only if $\pi(G)$ contains unmarked C_5 or weak-marked P_5-free bipartite graphs or weak-marked urchins or weak-marked starfishes.*

This theorem leads to the following characterization of JSP4S graphs.

Theorem 6. *A graph G is Jung semi-P_4-sparse if and only if $\pi(G)$ contains unmarked C_5 or weak-marked P_5-free bipartite graphs or weak-marked starfishes.*

Definition 4 (and notations). (1) A *$\mathscr{C}$-graph* is a graph defined as the join $J_H(A_1,A_2,A_3,A_4,A_5)$ of stable graphs $A_1,\dots,A_5$ over a C_5 $H=[a_1,a_2,a_3,a_4,a_5]$ (see Fig. 3).

(2) A *$\mathscr{B}$-graph* is a graph defined as the join $J_H(X_1,\dots,X_{k-1},X_k,Y_1,Y_2,\dots,Y_k)$ of stable graphs $X_1,\dots,X_{k-1},Y_2,\dots,Y_k$ and of JSP4S graphs X_k, Y_1 over a bipartite P_5-free prime graph H with set of vertices $\{x_1,\dots,x_k,y_1,\dots,y_k\}$ (see Fig. 4).

(3) An *$\mathscr{S}$-graph* is a graph defined as the join $J_H(K_1,\dots,K_l,R,S_1,\dots,S_l)$ of stable graphs $K_1,\dots,K_l$ and of JSP4S graphs $R,Y_1,\dots,Y_l$ over a starfish H with set of vertices $\{k_1,\dots,k_l,r,s_1,\dots,s_l\}$ (see Fig. 5).

In Figs. 3–5, a bold line connecting two sets of vertices means that there exist all possible edges between these sets.

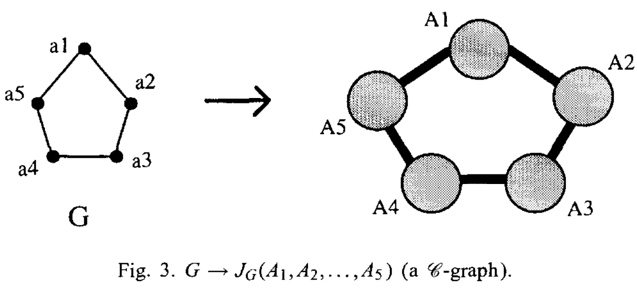

Fig. 3. $G \rightarrow J_G(A_1,A_2,\dots,A_5)$ (a $\mathscr{C}$-graph).

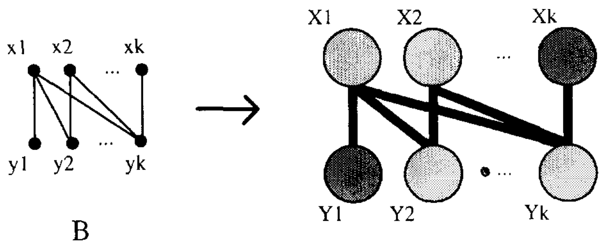

Fig. 4. $B \rightarrow J_B(X_1,\ldots,X_k,Y_1,\ldots,Y_k)$ (a $\mathscr{B}$-graph).

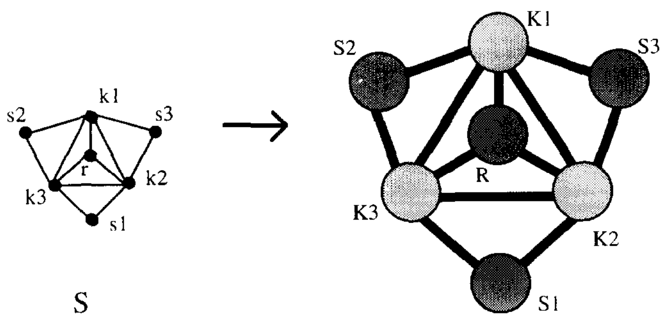

Fig. 5. $S \rightarrow J_S(K_1,\ldots,K_l,R,S_1,\ldots,S_l)$ (a $\mathscr{S}$-graph).

Remark 3. By Theorem 6, a neighbourhood module of a JSP$_4$S graph is a $\mathscr{C}$-graph or a $\mathscr{B}$-graph or an $\mathscr{S}$-graph.

4. Hamiltonicity of JSP$_4$S graphs

In this section we shall show that the 3-sun-free semi-P_4-sparse graphs are Jung graphs, which is the main result of the paper, by proving that $\mathscr{C}$-graphs, $\mathscr{B}$-graphs and $\mathscr{S}$-graphs are Jung graphs. In Sections 4.1–4.3 we describe, respectively, the structures of such graphs. The main result is explained in Section 4.4 and the proofs, rather technical, are given in Section 4.5. We note here that the scattering number of a JSP$_4$S graph is computable in linear time (see [21]). We can easily deduce, from the (constructive) proofs, an efficient algorithmic construction of a minimum path partition (or of a hamiltonian cycle) of such a graph.

4.1. Hamiltonicity of $\mathscr{C}$-graphs

The following three lemmas precise the structure of scattering sets and scattering number of $\mathscr{C}$-graphs. Their proofs are detailed in [21] and do not require complicated argumentations. Notations are those given in Definition 4.

Lemma 4. *Let G be a $\mathscr{C}$-graph and for every $i \in \{1,\dots,5\}$, let $W_i = A_i \cup A_{i+2}$, $T_i = A_i \cup A_{i+2} \cup A_{i+3}$ (where subscript i is to be taken modulo 5 in $\{1,\dots,5\}$). Let S be a scattering set of G. Then there exists $i \in \{1,\dots,5\}$ such that $S = W_i$ or $S = T_i$ and we have $s(G) = |A_{i+1}| + \max(1, ||A_{i+3}| - |A_{i+4}||) - |A_i| - |A_{i+2}|$. More precisely,*
— *W_i is a scattering set and T_i, T_{i+2} are not if and only if $|A_{i+3}| = |A_{i+4}|$,*
— *W_i and T_i are scattering sets if and only if $|A_{i+4}| = |A_{i+3}| + 1$,*
— *W_i and T_{i+2} are scattering sets if and only if $|A_{i+3}| = |A_{i+4}| + 1$,*
— *T_i is a scattering set and W_i is not if and only if $|A_{i+4}| > |A_{i+3}| + 1$, and*
— *T_{i+2} is a scattering set and W_i is not if and only if $|A_{i+3}| > |A_{i+4}| + 1$.*

Lemma 5. *Let G be a $\mathscr{C}$-graph such that there exist distinct i and j with $|A_i| = |A_j| = 1$. Then $s(G) \geqslant 0$.*

Lemma 6. *Let G be a $\mathscr{C}$-graph such that, for every $i \in \{1,\dots,5\}$, $|A_i| \geqslant 2$. Let $a_i \in A_i$, for every i, and let $C = [a_1, a_2, a_3, a_4, a_5, a_1]$ be a chordless cycle of length 5. Then $G \backslash C$ is a $\mathscr{C}$-graph and $s(G \backslash C) = s(G) + 1$.*

The proof of the following result is given below in Section 4.5. It uses the three preceding lemmas.

Proposition 2. *$\mathscr{C}$-graphs are Jung graphs.*

4.2. Hamiltonicity of $\mathscr{B}$-graphs

Let us consider a $\mathscr{B}$-graph G. We use notations introduced in Section 3.2 (see Definition 4 and notations). Since a scattering set S of G is a cutset, S contains X_1 or Y_k. Moreover, for $i \in \{2,\dots,k-1\}$, if $X_i \subseteq S$ (respectively, $Y_i \subseteq S$) then $X_{i-1} \subseteq S$ and $Y_i \not\subseteq S$ (respectively, $Y_{i+1} \subseteq S$ and $X_i \not\subseteq S$). Then, we deduce easily the following fact.

Fact 1. *Any scattering set of a $\mathscr{B}$-graph G is exactly one of the three following sets:*
(i) $X_1 \cup \dots \cup X_l \cup Y_{l'} \cup \dots \cup Y_k \cup S_1 \cup S_k$ *with* $1 \leqslant l < l' \leqslant k$ *with*

$$S_1 = \begin{cases} \emptyset & \text{if } s(Y_1) \leqslant 0, \\ \text{a scattering set of } Y_1 & \text{if } s(Y_1) \geqslant 1, \end{cases}$$

$$S_k = \begin{cases} \emptyset & \text{if } s(X_k) \leqslant 0, \\ \text{a scattering set of } X_k & \text{if } s(X_k) \geqslant 1. \end{cases}$$

(ii) $X_1 \cup \dots \cup X_l \cup S_1$ *with* $1 \leqslant l \leqslant k$.
(iii) $Y_{l'} \cup \dots \cup Y_k \cup S_k$ *with* $1 \leqslant l' \leqslant k$.

By using Fact 1, we can prove the following result (see [21] for a complete proof).

Lemma 7. *Let G be a $\mathscr{B}$-graph with $k \geqslant 3$. For any $i \in \{2,\dots,k-1\}$ and for any pair $\{x, y\}$ such that $x \in X_i$ and $y \in Y_i$, $G \backslash \{x, y\}$ is a $\mathscr{B}$-graph and $s(G \backslash \{x, y\}) \leqslant s(G)$.*

The preceding lemma is the most important statement used in the proof of the next result.

Proposition 3. *$\mathscr{B}$-graphs are Jung graphs.*

Proof. See Section 4.5. □

4.3. Hamiltonicity of $\mathscr{S}$-graphs

The following three lemmas will be used in the proof of Proposition 4 below. Their (straight-forward) proofs can be found in [21]. We use notations introduced in Section 3.2.

Lemma 8. *Let G be an $\mathscr{S}$-graph such that $s(G) < 0$ and let $z \in K_1 \cup \cdots \cup K_l$ such that there exists a hamiltonian path P of $G\backslash\{z\}$ joining x and y. Then there exists a hamiltonian path of G joining x and y.*

Lemma 9. *Let G be an $\mathscr{S}$-graph such that $s(G) < 0$. Let $x \in K_i$ with $i \in \{1, \ldots, l\}$ such that there exists a hamiltonian cycle C of $G\backslash\{x\}$ and let $y \in V(C)$. If there exists an edge cd of C with $c \in K_j$, $j \neq i$, and $d \in S_k \cup R$, $k \neq i$, then there exists a hamiltonian path of G joining x and y.*

Lemma 10. *Let G be an $\mathscr{S}$-graph such that $s(G) < 0$ and let $\{x, y\} \subseteq S_1 \cup \cdots \cup S_l \cup R$. Let $z \in K_i$ with $i \in \{1, \ldots, l\}$ such that z is adjacent to x or y, and let C be a hamiltonian cycle of $G\backslash\{z\}$. If there exists an edge cd of C with $c \in K_j$, $j \neq i$, and $d \in S_k \cup R$, $k \neq i$, then there exists a hamiltonian path of G joining x and y.*

Note that in Lemmas 9 and 10, R is possibly empty.

Proposition 4. *$\mathscr{S}$-graphs are Jung graphs.*

Proof. See Section 4.5. □

4.4. Main result

By Proposition 1 we know that the disjoint union and the disjoint sum of Jung graphs are Jung graphs. By Propositions 2–4 we know that $\mathscr{C}$-graphs, $\mathscr{B}$-graphs and $\mathscr{S}$-graphs are Jung graphs. Hence, by Theorem 6 (Remark 3) and using modular decomposition, we have

Theorem 7. *The 3-sun-free semi-P_4-sparse graphs are Jung graphs.*

4.5. *Proofs*

In this part, we prove that $\mathscr{C}$-graphs, $\mathscr{B}$-graphs and $\mathscr{S}$-graphs are Jung graphs.

Proof of Proposition 2 (*$\mathscr{C}$-graphs are Jung graphs*). For every $i \in \{1,\dots,5\}$, let $A_i = \{a_1^i, a_2^i, \dots, a_{q_i}^i\}$ with $q_i \geqslant 1$. Without loss of generality, we suppose that a scattering set of G contains $A_1 \cup A_3$. Then, by Lemma 4,

$$s(G) = |A_2| + \max(1, ||A_4| - |A_5||) - |A_1| - |A_3|$$

Case 1: $s(G) = p,\ p \geqslant 1$. We shall prove that $\rho(G) = \max(1, s(G)) = s(G)$. Since for any graph $\rho(G) \geqslant \max(1, s(G))$, it is sufficient to show that there exists a path partition of $V(G)$ with $s(G)$ paths.

1.1: $|A_4| = |A_5|$ (that is $q_4 = q_5$). By Lemma 4, $A_1 \cup A_3$ is a scattering set. Then, $s(G) = |A_2| + 1 - |A_1| - |A_3|$, that is $q_2 = q_1 + q_3 + p - 1$. Let P be the following path: $[a_1^5, a_1^4, \dots, a_{q_5}^5, a_{q_5}^4, a_1^3, a_1^2, a_2^3, a_2^2, \dots, a_{q_3}^3, a_{q_3}^2, a_1^1, a_{q_3+1}^2, a_2^1, a_{q_3+2}^2, \dots, a_{q_1}^1, a_{q_3+q_1}^2]$. This path contains the vertices of $A_1 \cup A_3 \cup A_4 \cup A_5$ and $q_1 + q_3$ vertices of A_2. Thus, the other $p - 1$ vertices of A_2 plus P are a path partition of $V(G)$.

1.2: $|A_5| > |A_4|$ ($q_5 > q_4$). By Lemma 4, $A_1 \cup A_3 \cup A_4$ is a scattering set. Then, $s(G) = |A_2| + |A_5| - |A_4| - |A_1| - |A_3|$. Since $c(G \backslash (A_1 \cup A_4)) - |A_1 \cup A_4| \leqslant s(G)$, $|A_3| < |A_2|$. Then, $q_3 < q_2$ and $q_2 = q_1 - (q_5 - q_4) + q_3 + p$.

1.2.1: $|A_1| \geqslant |A_5| - |A_4|$ ($q_1 \geqslant q_5 - q_4$). Let $P = [a_1^5, a_1^4, a_2^5, a_2^4, \dots, a_{q_4}^5, a_{q_4}^4, a_{q_4+1}^5, a_1^1, a_{q_4+2}^5, a_2^1, \dots, a_{q_5}^5, a_{q_5-q_4}^1, a_1^2, a_{q_5-q_4+1}^1, a_2^2, a_{q_5-q_4+2}^1, \dots, a_{q_1-q_5+q_4}^2, a_{q_1}^1, a_{q_1-q_5+q_4+1}^2, a_3^1, \dots, a_{q_1-q_5+q_4+q_3}^2, a_{q_3}^3, a_{q_1-q_5+q_4+q_3+1}^2]$. Since $q_1 - q_5 + q_4 + q_3 + 1 = q_2 - p + 1$, all the vertices of $A_1 \cup A_3 \cup A_4 \cup A_5$ and $q_2 - p + 1$ vertices of A_2 belong to P. Then, P and the remaining vertices of A_2 are a path partition of $V(G)$.

1.2.2: $|A_1| < |A_5| - |A_4|$ ($q_1 < q_5 - q_4$). We consider the following two paths:

$$P_1 = [a_1^5, a_1^4, a_2^5, a_2^4, \dots, a_{q_4}^5, a_{q_4}^4, a_{q_4+1}^5, a_1^1, a_{q_4+2}^5, a_2^1, a_{q_4+3}^5, \dots, a_{q_1}^1, a_{q_4+q_1+1}^5]$$

$$P_2 = [a_1^2, a_1^3, a_2^2, a_2^3, \dots, a_{q_3}^2, a_{q_3}^3, a_{q_3+1}^2] \quad \text{(we recall that } q_3 < q_2\text{)}.$$

P_1 contains the vertices of $A_1 \cup A_4$ and $q_1 + q_4 + 1$ vertices of A_5, and P_2 contains the vertices of A_3 and $q_3 + 1$ vertices of A_2. Then, $|A_5 \backslash V(P_1)| = q_5 - q_1 - q_4 - 1$ and $|A_2 \backslash V(P_2)| = q_2 - q_3 - 1$. Since $q_5 - q_1 - q_4 - 1 + q_2 - q_3 - 1 = p - 2$, these $p - 2$ vertices of $A_2 \cup A_5$ plus P_1 and P_2 are a path partition of $V(G)$.

1.3: $|A_5| < |A_4|$ ($q_5 < q_4$). Symmetrically, this case is analogous to Case 1.2.

Case 2: $s(G) = 0$. We shall prove that G has a hamiltonian cycle.

2.1: $|A_4| = |A_5|$ ($q_4 = q_5$). By Lemma 4, $A_1 \cup A_3$ is a scattering set. Then, $s(G) = |A_2| + 1 - |A_1| - |A_3|$, that is $q_2 = q_1 + q_3 - 1$. The following cycle C,

$$[a_1^5, a_1^4, \dots, a^5{}_{q_4}, a_{q_4}^4, a_1^3, a_1^2, \dots, a_{q_3}^3, a_{q_3}^2, a_1^1, a_{q_3+1}^2, a_2^1, a_{q_3+2}^2, \dots, a_{q_1-1}^1, a_{q_2}^2, a_{q_1}^1, a_1^5]$$

is a hamiltonian cycle of G.

2.2: $|A_5| > |A_4|$ ($q_5 > q_4$). By Lemma 4, $A_1 \cup A_3 \cup A_4$ is a scattering set. Then, $s(G) = |A_2| + |A_5| - |A_4| - |A_1| - |A_3|$. Since $c(G \backslash (A_1 \cup A_4)) - |A_1 \cup A_4| \leqslant s(G)$, $|A_3| < |A_2|$. Then, $q_3 < q_2$ and $q_2 = q_1 + q_3 + q_4 - q_5$. Since $|A_2| - |A_3| = |A_1| + |A_4| - |A_5|$, we have $|A_1| > |A_5| - |A_4|$ that is $q_1 > q_5 - q_4$.

The following cycle,

$$[a^5_1, a^4_1, \ldots, a^5_{q_4}, a^4_{q_4}, a^5_{q_4+1}, a^1_1, a^5_{q_4+2}, a^1_2, \ldots, a^5_{q_5}, a^1_{q_5-q_4}, a^2_1, \\ a^3_1, \ldots, a^2_{q_3}, a^3_{q_3}, a^2_{q_3+1}, a^1_{q_5-q_4+1}, \ldots, a^2_{q_2}, a^1_{q_1}, a^5_1]$$

is a hamiltonian cycle of G.

2.3: $|A_5| < |A_4|$ ($q_5 < q_4$). Symmetrically, this case is analogous to Case 1.2.

Case 3: $s(G) < 0$. It is easy to prove that the smallest $\mathscr{C}$-graph G such that $s(G) < 0$ is the $\mathscr{C}$-graph verifying for every $i \in \{1, \ldots, 5\}$ $|A_i| = 2$ and that this $\mathscr{C}$-graph is hamilton-connected. We shall prove by induction on $|V(G)|$ that a $\mathscr{C}$-graph such that $s(G) < 0$ is hamilton-connected.

Let x and y be two arbitrary vertices of G.

3.1: $x \in A_i$ and $y \in A_i$. $G \backslash \{x\}$ is a $\mathscr{C}$-graph and by Lemma 1, $s(G \backslash \{x\}) \leqslant s(G) + 1 \leqslant 0$. If $s(G \backslash \{x\}) = 0$ then, by Case 2, $G \backslash \{x\}$ has a hamiltonian cycle. If $s(G \backslash \{x\}) < 0$ then, by induction, $G \backslash \{x\}$ is hamilton-connected. Let C be a hamiltonian cycle of $G \backslash \{x\}$ and let z be a neighbour of y in C. Then, $P = C \backslash \{zy\}$ is a hamiltonian path of $G \backslash \{x\}$ joining y and z. Since $z \in A_{i-1} \cup A_{i+1}$, $zx \in E(G)$ and $P \cup [z,x]$ is a hamiltonian path of G joining x and y.

3.2: $x \in A_i$ and $y \in A_{i+2}$.

3.2.1: $|A_i| \geqslant 2$. As in Case 3.1 we consider a hamiltonian cycle C of the $\mathscr{C}$-graph $G \backslash \{x\}$. If a neighbour z of y in C belongs to A_{i+1} then $P = C \backslash \{yz\}$ is a hamiltonian path of $G \backslash \{x\}$ joining y and z. Then $P \cup [z,x]$ is a hamiltonian path of G joining x and y. If the two neighbours z and t of y in C belong to A_{i+3} then C has an edge ab with $a \in A_{i+1}$ and $b \in A_{i+2}$ (indeed, if no such edge exists then $|A_i| \geqslant |A_{i+1}| + 2$ and $|A_{i+3}| \geqslant |A_{i+2}| + 1$; more precisely, if $r = |A_i| - |A_{i+1}| - 2$ and $q = |A_{i+3}| - |A_{i+2}| - 1$ then $|A_{i+4}| = r + q + 2$; let $S = A_{i+1} \cup A_{i+2} \cup A_{i+4}$; then, $s(G) \geqslant c(G \backslash S) - |S| = |A_i| + |A_{i+3}| - |A_{i+1}| - |A_{i+2}| - |A_{i+4}| = 1$, a contradiction). Let c be the other neighbour of b in C (that is, $c \in A_{i+1} \cup A_{i+3}$). By deleting the edges ab, bc, ty and zy of C and adding the edges ay, cy, tb and zb, we obtain a new hamiltonian cycle C' such that y has a neighbour belonging to A_{i+1}. This is the preceding case.

3.2.2: $|A_i| = 1$. By Lemma 5, for every $j \neq i$ $|A_j| \geqslant 2$. By permuting x and y and examining the $\mathscr{C}$-graph $G \backslash \{y\}$, we see that we are in Case 3.2.1.

3.3: $x \in A_i$ and $y \in A_{i+1}$. By Lemma 5, there is at most one j such that $|A_j| = 1$.

3.3.1: For every $j \in \{1, \ldots, 5\}$, $|A_j| \geqslant 2$. Let C_0 be a cycle of length 5 containing x and y and set $C_0 = [x, y, a_{i+2}, a_{i+3}, a_{i+4}]$. $G \backslash C_0$ is a $\mathscr{C}$-graph. By Lemma 6, $s(G \backslash C_0) = s(G) + 1 \leqslant 0$. If $s(G \backslash C_0) = 0$ then by Case 2 $G \backslash C_0$ is hamiltonian. If $s(G \backslash C_0) < 0$ then, by induction, $G \backslash C_0$ is hamilton-connected. Let C be a hamiltonian cycle of $G \backslash C_0$. Then, there exists an edge uv of C such that $u \in A_{i+3}$ and $v \in A_{i+4}$, or there exists an edge uw of C such that $u \in A_{i+3}$ and $w \in A_{i+2}$. If uv exists, let $P_0 = C \backslash \{uv\}$; the path $P = [x, a_{i+4}, u] \cup P_0 \cup [v, a_{i+3}, a_{i+2}, y]$ is a hamiltonian path of G joining x and y. If uw exists, let $P_1 = C \backslash \{uw\}$; the path $P' = [x, a_{i+4}, a_{i+3}, w] \cup P_1 \cup [u, a_{i+2}, y]$ is a hamiltonian path of G joining x and y.

3.3.2: There exists $j \in \{1, \dots, 5\}$ such that $|A_j| = 1$ and, for every $k \neq j$, $|A_k| \geqslant 2$. Let $T_1 = A_j \cup A_{j+1} \cup A_{j+3}$ and $T_2 = A_j \cup A_{j+2} \cup A_{j+4}$. Since $c(G \backslash T_1) - |T_1| \leqslant s(G)$ and $c(G \backslash T_2) - |T_2| \leqslant s(G)$, we have

(1) $|A_{j+2}| + |A_{j+4}| \leqslant 1 + |A_{j+1}| + |A_{j+3}| + s(G)$

and

(2) $|A_{j+1}| + |A_{j+3}| - 1 - s(G) \leqslant |A_{j+2}| + |A_{j+4}|$.

Since $s(G) < 0$, by (1) and (2) we have $s(G) = -1$ and

(3) $|A_{j+2}| + |A_{j+4}| = |A_{j+1}| + |A_{j+3}|$.

We have

$$c(G \backslash (A_j \cup A_{j+2})) - |A_j \cup A_{j+2}| = |A_{j+1}| - |A_{j+2}| \leqslant s(G) < 0$$

and

$$c(G \backslash (A_j \cup A_{j+3})) - |A_j \cup A_{j+2}| = |A_{j+4}| - |A_{j+3}| \leqslant s(G) < 0.$$

Then,

(4) $|A_{j+1}| < |A_{j+2}|$ and $|A_{j+4}| < |A_{j+3}|$.

As previously, if for every $i \in \{1, \dots, 5\}$ we denote by $a^i_1, a^i_2, \dots, a^i_{q_i}$ the set of vertices of A_i (with, for every $i \in \{1, \dots, 5\}$, $q_i \geqslant 1$, and $q_j = 1$), we have, by (3) and (4),

(5) $q_{j+2} = q_{j+1} + r$ and $q_{j+3} = q_{j+4} + r$, with $r \geqslant 1$.

Let $k = q_{j+1}$ and $l = q_{j+4}$, and let the cycle

$$C = [a^j_1, a_1^{j+1}, a_1^{j+2}, \dots, a_k^{j+1}, a_k^{j+2}, a_1^{j+3}, a_{k+1}^{j+2}, \dots, a_r^{j+3}, a_{k+r}^{j+2}, a_{r+1}^{j+3}, a_1^{j+4}, \dots, a_{r+l}^{j+3}, a_l^{j+4}, a^j_1]$$

Clearly, C is hamiltonian. We see that it is always possible to number the vertices of A_i and A_{i+1} such that xy is an edge of C. Since C is a hamiltonian cycle, $P = C \backslash \{xy\}$ is a hamiltonian path of G. □

Proof of Proposition 3 (*$\mathscr{B}$-graphs are Jung graphs*). Let G be a $\mathscr{B}$-graph. If $k = 1$ or 2 then, by Proposition 1 and Lemma 3, G is a Jung graph. By induction, suppose that for every $\mathscr{B}$-graph H with $|V(H)| < |V(G)|$, H is a Jung graph.

Case 1: $s(G) \leqslant -1$. Let a and b two arbitrary vertices of G.

1.1: There exists $i \in \{2, \dots, k-1\}$ with $|X_i| \geqslant 2$ and $|Y_i| \geqslant 2$.

1.1.1: $X_i \neq \{a, b\}$ and $Y_i \neq \{a, b\}$. Let $x \in X_i \backslash \{a, b\}$ and $y \in Y_i \backslash \{a, b\}$. Let $G' = G \backslash \{x, y\}$. By Lemma 7, $s(G') \leqslant s(G) \leqslant -1$. By induction, there exists a hamiltonian path P joining a and b in G'. Let c be a vertex of $X_i \backslash \{x\}$ (possibly, $c = a$) and let d a neighbour of c on the path P. Since $xd \in E$, $yc \in E$ and $xy \in E$, $(P \backslash \{cd\}) \cup [c, y, x, d]$ is a hamiltonian path of G joining a and b.

1.1.2: $X_i = \{a,b\}$ (or symmetrically $Y_i = \{a,b\}$). Let y be any vertex of Y_i. By Lemma 7, $s(G\backslash\{a,y\}) \leqslant s(G) \leqslant -1$. Let c be a neighbour of y in $G\backslash\{a,y\}$. By induction, there exists a hamiltonian path P joining b and c. Thus, $P \cup [c,y,a]$ is a hamiltonian path joining a and b.

1.2: There exists $i \in \{2,\ldots,k-1\}$ with $|X_i| \geqslant 2$ and $|Y_i| = 1$ (we set $Y_i = \{y\}$) (or symmetrically $|X_i| = 1$ and $|Y_i| \geqslant 2$).

1.2.1: $X_i \neq \{a,b\}$ and $y \notin \{a,b\}$. Let $x \in X_i\backslash\{a,b\}$. By Lemma 7, $s(G\backslash\{x,y\}) \leqslant s(G) \leqslant -1$. There exists a hamiltonian path P joining a and b in $G\backslash\{x,y\}$. Let $c \in X_i\backslash\{x\}$ (c can be equal to a or b), and d be a neighbour of c in P. Since $yc \in E$, $xd \in E$ and $xy \in E$, $(P\backslash\{cd\}) \cup [c,y,x,d]$ is a hamiltonian path of G joining a and b.

1.2.2: $X_i = \{a,b\}$ or $Y_i = \{a\}$ (or symmetrically $Y_i = \{b\}$). If $X_i = \{a,b\}$ then put $z = y$ else choose $z \in X_i\backslash\{b\}$. Let $G' = G\backslash\{a,z\}$. By induction there exists a hamiltonian path P of G' joining b and c, where c is a neighbour of z in G. Then $P \cup [c,z,a]$ is a hamiltonian path of G joining a and b.

1.3: For every $i \in \{2,\ldots,k-1\}$, $|X_i| = 1$ and $|Y_i| = 1$. Let us choose an arbitrary i and put $X_i = \{x\}$ and $Y_i = \{y\}$.

1.3.1: $x \notin \{a,b\}$ and $y \notin \{a,b\}$. Let P be a hamiltonian path joining a and b in $G' = G\backslash\{x,y\}$. Since every vertex of G' belongs to P, clearly there exists an edge $cd \in E(P)$ with $c \in X_1 \cup \cdots \cup X_{i-1}$ and $d \in Y_{i+1} \cup \cdots \cup Y_k$. Clearly, $(P\backslash\{cd\}) \cup [c,y,x,d]$ is a hamiltonian path of G joining a and b.

1.3.2: $x \notin \{a,b\}$ and $y \in \{a,b\}$ (or symmetrically $x \in \{a,b\}$ and $y \notin \{a,b\}$). Suppose, without loss of generality, that $y = a$. In $G' = G\backslash\{x,a\}$, there exists a hamiltonian path P joining b and a neighbour c of x in G. Thus, $P \cup [c,x,a]$ is a hamiltonian path.

1.3.3: $x = a$ and $y = b$ (or symmetrically $x = b$ and $y = a$). Let c be a neighbour of a distinct from b (for example, $c \in Y_k$) and d a neighbour of b distinct from a (for example, $d \in X_1$). By induction, there exists a hamiltonian path in $G' = G\backslash\{a,b\}$ joining c and d. Thus, $[a,c] \cup P \cup [d,b]$ is a hamiltonian path of G joining a and b.

Case 2: $s(G) \leqslant 0$.

2.1: There exists $i \in \{2,\ldots,k-1\}$ with $|X_i| \geqslant 2$ (or symmetrically $|Y_i| \geqslant 2$). Let $x \in X_i$ and $y \in Y_i$. By Lemma 7, $G\backslash\{x,y\} \leqslant s(G)$. By induction, there exists a hamiltonian cycle C in $G\backslash\{x,y\}$. Let $c \in X_i\backslash\{x\}$ and d be a neighbour of c in C. Since $yc \in E(G)$, $xd \in E(G)$ and $xy \in E(G)$, $(C\backslash\{cd\}) \cup [c,y,x,d]$ is a hamiltonian cycle of G.

2.2: For every $i \in \{2,\ldots,k-1\}$, $|X_i| = |Y_i| = 1$. Choose arbitrarily i and set $X_i = \{x\}$, $Y_i = \{y\}$ and $G' = G\backslash\{x,y\}$. By Lemma 7, $s(G') \leqslant 0$. Let C be a hamiltonian cycle of G'. There exists an edge $cd \in E(C)$ with $c \in X_1 \cup \cdots \cup X_{i-1}$ and $d \in Y_{i+1} \cup \cdots \cup Y_k$. Since $yc \in E(G)$, $xd \in E(G)$ and $xy \in E(G)$, $(C\backslash\{cd\}) \cup [c,y,x,d]$ is a hamiltonian cycle of G.

Case 3: $s(G) = p \geqslant 1$.

3.1: There exists $i \in \{2,\ldots,k-1\}$ such that $|X_i| \geqslant 2$ (or symmetrically $|Y_i| \geqslant 2$). Let $x \in X_i$, $y \in Y_i$ and $G' = G\backslash\{x,y\}$. By Lemma 7, $s(G') \leqslant s(G) = p$. Thus, by induction, we can find a minimum path partition of $V(G')$ containing $q = \rho(G') = \max(1, s(G')) \leqslant p$ paths. If necessary, we can break some paths to obtain a partition of $V(G')$ having exactly p paths. Let $c \in X_i\backslash\{x\}$. If c is an end-vertex of a path P of the path partition, then $P \cup [c, y, x]$ with the other paths of the partition of $V(G')$ are a path partition of $V(G)$. If c is an internal vertex of a path P_1 then we choose a neighbour d of c in P_1. Since $xd \in E(G)$, $yc \in E(G)$ and $xy \in E(G)$, $(P_1 \backslash \{cd\}) \cup [c, y, x, d]$ with the other paths of the partition of $V(G')$ are a path partition of $V(G)$ with p paths.

3.2: For every $i \in \{2, \ldots, k-1\}$, $|X_i| = |Y_i| = 1$. Choose arbitrarily i and set $X_i = \{x\}$, $Y_i = \{y\}$ and $G' = G \backslash \{x, y\}$. By Lemma 7, $s(G') \leqslant s(G)$. As in 3.1 we consider a path partition of $V(G')$ having exactly p paths. There are three cases to consider:

3.2.1: A path P has an end-vertex c in $X_1 \cup \cdots \cup X_{i-1} \cup Y_{i+1} \cup \cdots \cup Y_k$. Clearly, $P \cup [x, y]$ (or $P \cup [y, x]$) is a path.

3.2.2: There exists a path P with $cd \in E(P)$ such that $c \in X_1 \cup \cdots \cup X_{i-1}$ and $d \in Y_{i+1} \cup \cdots \cup Y_k$. We consider the path $P_1 = (P \backslash \{cd\}) \cup [c, y, x, d]$.

3.2.3: Every path partition of $V(G')$ having p paths is the union of a path partition of $X_1 \cup Y_1$ having p' paths with their end-vertices in $Y_1 \cup \cdots \cup Y_{i-1}$ and a path partition of $V(G_{i+1,k})$ having p'' paths with their end-vertices in $X_{i+1} \cup \cdots \cup X_k$ $(p' + p'' = p)$.

If $3 \leqslant i \leqslant k-2$, since $|X_{i-1}| = |Y_{i-1}| = 1$ or $|Y_{i+1}| = |X_{i+1}| = 1$, the vertex of X_{i-1} or Y_{i+1} is necessary an end-vertex of a path (Type 1) or is adjacent, in a path of the partition, to a vertex of $Y_{i+1} \cup \cdots \cup Y_k$ (Type 2). Thus, we can suppose that $k = 3$ and $i = 2$. Consider a path partition of $X_1 \cup Y_1$ having p' paths, as previously, and suppose that the trace of this partition on Y_1 is the path partition $\{P_1, \ldots, P_q\}$. We can suppose that one of the end-vertices of P_q is end-vertex of a path of the path partition of $X_1 \cup Y_1$. If $q > \rho(Y_1)$, we partition Y_1 in $q-1$ paths $Q_1, \ldots, Q_{q-1}$ and we replace the paths $P_1, \ldots, P_{q-1}$ by the paths $Q_1, \ldots, Q_{q-1}$. Then, either P_q is a path of the initial path partition of $X_1 \cup Y_1$ and we obtain a new path partition of $X_1 \cup Y_1$ having $p'-1$ paths, or we obtain a new path partition of $X_1 \cup Y_1$ having p' paths, one of which has an end-vertex in X_1. Thus, either by adding $[x, y]$ we obtain a path partition of $V(G)$ having p paths, or we are in Type 1.

Hence, we can suppose that $q = \rho(Y_1)$. Analogously, we can suppose that the number of paths of a path partition of $X_3 \cup Y_3$ is $\rho(X_3)$. Then, we can see that $\rho(Y_1) - |X_1| = p'$ and $\rho(X_3) - |Y_3| = p''$. If we consider $T = X_1 \cup S_1 \cup Y_3 \cup S_3$, we obtain $c(G-T) - |T| = \rho(Y_1) - |X_1| + \rho(X_3) - |Y_3| + 1 = p + 1$, a contradiction. □

Proof of Proposition 4 (*$\mathscr{S}$-graphs are Jung graphs*). Let G be a $\mathscr{S}$-graph. We can suppose that if R is empty then $l \geqslant 3$ (otherwise, by Lemma 3 G is a Jung graph and we are done). Let S be a scattering set of G. It is easy to see that S contains at least $l-1$ sets of $\{K_1, \ldots, K_l\}$ (else, $G \backslash S$ is a connected graph). Without loss of generality, we suppose that $S \supseteq K_2 \cup \cdots \cup K_l$. Note that if S contains a vertex of K_1 then it must contain all the vertices of K_1. Then, we can see that
either

$$S = K_1 \cup K_2 \cup \cdots \cup K_l \cup A_1 \cup A_2 \cup \cdots \cup A_l \cup A_R,$$

or

$$S = K_2 \cup \cdots \cup K_l \cup S_2 \cdots \cup S_l \cup R \cup A_1,$$

or

$$S = K_2 \cup \cdots \cup K_l \cup A_1,$$

with, for every $i \in \{1,\ldots,l\}$,

$$A_i = \begin{cases} \emptyset & \text{if } s(S_i) \leqslant 0, \\ \text{a scattering set of } S_i & \text{if } s(S_i) \geqslant 1, \end{cases}$$

and if $R \neq \emptyset$

$$A_R = \begin{cases} \emptyset & \text{if } s(R) \leqslant 0, \\ \text{a scattering set of } R & \text{if } s(R) \geqslant 1. \end{cases}$$

In this proof we suppose that if R is empty then $A_R = \emptyset$ and $\rho(R) = 0$. We shall distinguish between two cases.

Case 1: $s(G) \geqslant 0$.

1.1: $S = K_1 \cup K_2 \cdots \cup K_l \cup A_1 \cup A_2 \cdots \cup A_l \cup A_R$. Since R and each S_i are Jung graphs by Lemma 1 and 2, $s(G) = \rho(S_1) + \cdots + \rho(S_l) + \rho(R) - |K_1| - \cdots - |K_l|$. Let G' be the graph obtained from G by replacing the subgraph R by a stable R' on $\rho(R)$ vertices and, for every $i \in \{1,\ldots,l\}$, the subgraph S_i by a stable S'_i on $\rho(S_i)$ vertices. For $i \in \{1,\ldots,l\}$ every vertex of S'_i corresponds to a path of a minimum path partition of S_i and if $R \neq \emptyset$ then every vertex of R' corresponds to a path belonging to a minimum path partition of R. Set $K = K_1 \cup \cdots \cup K_l$ and $S' = S'_1 \cup \cdots \cup S'_l \cup R'$; then $s(G) = |S'| - |K|$.

Let $i \in \{1,\ldots,l\}$ and let $T = (K \backslash K_i) \cup A_i$. Then, $|S'| - |K| = s(G) \geqslant c(G-T) - |T| = \rho(S_i) + 1 - |K| + |K_i| = |S'_i| + 1 - |K| + |K_i|$. Thus,

(A) For every $i \in \{1,\ldots,l\}$, $|K_i| < |S'| - |S'_i|$.

Let P be a path defined in the following way: P has an end-vertex a in S' and the other end-vertex b in K. It uses alternatively vertices of S' and vertices of K. Moreover, if P contains a vertex of K_j for $j \in \{2,\ldots,l\}$ then it contains all the vertices of K_{j-1} (all these vertices appear in P before those of K_j when moving on P from a to b). We suppose that P is a longest path with this previous properties and that b belongs to K_i.

Let B_1 (respectively, B_2) be the set of vertices of $S' \cap V(P)$ which are not adjacent in P (respectively, are adjacent in P) to a vertex of $K_i \cap V(P)$. Clearly, $S'_i \cap V(P) \subseteq B_1$ and $|B_2| = |K_i \cap V(P)| \leqslant |K_i|$.

Claim. *The path P contains all vertices of K and $|K|$ vertices of S'.*

Proof. We must prove that $i = l$ and $K_i \subset V(P)$.

First, assume that there exists a vertex $c \in K_i \backslash V(P)$. If there exists a vertex $d \in S' \backslash (V(P) \cup S'_i)$ then $P \cup [b,d,c]$ extends P, a contradiction with the choice of P. Thus, $(S' \backslash S'_i) \subset V(P)$. Clearly, $|S' \cap V(P)| = |K \cap V(P)|$ and $K \backslash V(P) = (K_i \backslash V(P)) \cup K_{i+1} \cup \cdots \cup K_l$. Since $|S'| - |K| \geqslant 0$, we have $|S'_i \backslash V(P)| = |S' \backslash V(P)| \geqslant |K_i \backslash V(P)| + |K_{i+1}| + \cdots +$

$|K_l| > 0$. Thus, there exists a vertex $d \in S_i'\backslash V(P)$. If there exists a vertex $e \in B_1\backslash S_i'$ then, by permuting e and d, we obtain a new path P' such that $P' \cup [b,e,c]$ is longer than P, a contradiction. Then, $B_1 = S_i' \cap V(P)$ and $S'\backslash S_i' = B_2$. Since $|K_i| \geqslant |B_2|$, we have $|K_i| \geqslant |S'\backslash S_i'|$, that is, by (A), a contradiction. Thus, $K_i \subset V(P)$ (that is, $V(P) \cap K = K_1 \cup \cdots \cup K_i$) and $|B_2| = |K_i|$.

Now, assume that $i < l$. Then, $K_{i+1} \cap V(P) = \emptyset$.
We see that $S'\backslash(S_i' \cup S_{i+1}') \subset V(P)$ (otherwise, it is easy to extend P).

Case a: $S_{i+1}'\backslash V(P) \neq \emptyset$. Let d be a vertex of $S_{i+1}'\backslash V(P)$ and let c be a vertex of K_{i+1} (d is adjacent to every vertex of $V(P) \cap K$). If there exists a vertex e in $V(P) \cap (S'\backslash(S_i' \cup S_{i+1}'))$, then by permuting e and d we obtain a new path P' such that $P' \cup [b,e,c]$ is longer than P, a contradiction. Then, $V(P) \cap S' \subset (S_i' \cup S_{i+1}')$ and $S'\backslash(S_i' \cup S_{i+1}') = \emptyset$. Thus, $S' = S_1' \cup S_2'$ and necessarily $V(G) = K_1 \cup K_2 \cup S_1 \cup S_2$. But we have supposed that if R is empty then $l \geqslant 3$, a contradiction.

Case b: $S_{i+1}' \subset V(P)$. Since $|S'| - |K| \geqslant 0$ and $|S' \cap V(P)| = |K \cap V(P)|$, then $|S_i'\backslash V(P)| \geqslant |K_{i+1}| + \cdots + |K_l| > 0$. Let d be a vertex of $S_i'\backslash V(P)$ and let c be a vertex of K_{i+1}. If there exists a vertex $e \in B_1 \cap S_{i+1}'$, then, by permuting e and d, we obtain a new maximum path P' such that $S_{i+1}'\backslash V(P') \neq \emptyset$ and we are in Case a. Thus, we can suppose that $B_1 \cap S_{i+1}' = \emptyset$, that is $S_{i+1}' \subset B_2$. If there exists a vertex $f \in B_1\backslash S_i'$ ($f \notin S_{i+1}'$), then, by permuting f and d, we obtain a new path P' such that $P' \cup [b,f,c]$ is longer than P, a contradiction. Thus, $B_1 \subseteq S_i'$ and $S'\backslash S_i' = B_2$. Since $|B_2| \leqslant |K_i|$, we have $|K_i| \geqslant |S'\backslash S_i'|$, and this contradicts (A). Thus, $i = l$. □

1.1.1: $s(G) = p \geqslant 1$. Let $B = S'\backslash V(P)$. Then, $|B| = |S'| - |K| = s(G) \geqslant 1$. If there exists $c \in B$ such that $bc \in E(G')$, then the vertices of $B\backslash\{c\}$ and the path $P \cup [b,c]$ is a path partition of $V(G')$ with $s(G) = |S'| - |K|$ paths. Then, by replacing every vertex of S' by its corresponding path in S, we obtain a path partition of $V(G)$ with $s(G)$ paths.

If there is no such vertex c then $B \subset S_l'$. If there exists a vertex e of $B_1\backslash S_l'$, then by permuting e and any vertex of B we obtain a new path P and a new set B for which c exists. Otherwise, $|S'\backslash S_l'| = |B_2| \leqslant |K_l|$, that is, by (A), a contradiction.

1.1.2: $s(G) = 0$. Since $|S'| - |K| = 0$, then $G'\backslash P = \emptyset$. If a does not belong to S_l' then $P \cup [a,b]$ is a hamiltonian cycle of G'. Then, by replacing every vertex of S' by its corresponding path in S, we obtain a hamiltonian cycle of G. If a belongs to S_l', we see that there exists an edge de of P with the properties $d \in S'\backslash S_l'$, $e \in K\backslash K_l$ and $P = [a,\ldots,d,e,\ldots,b]$ (otherwise, $S'\backslash S_l' \subseteq B_2$ and then $|K_l| \geqslant |B_2| \geqslant |S'\backslash S_l'|$, a contradiction). Then, $[a,\ldots,d,b,\ldots,e,a]$ is a hamiltonian cycle of G' from which we deduce a hamiltonian cycle of G.

Remark 4. In Case 1.1.2, since $G \neq K_1 \cup K_2 \cup S_1 \cup S_2$, we have $\forall i \in \{1,\ldots,l\}$, there exists an edge cd of the hamiltonian cycle with $c \in K_j$, $j \neq i$, and $d \in S_k \cup R$, $k \neq i$.

1.2: $S = K_2 \cup \cdots \cup K_l \cup S_2 \cdots \cup S_l \cup R \cup A_1$. Then, $s(G) = \max(1, s(K_1)) + \max(1, s(S_1)) - |K_2| - \cdots - |K_l| - |S_2| - \cdots - S_l| - |R|$. That is,

(B) $s(G) = |K_1| + \rho(S_1) - |K_2| - \cdots - |K_l| - |S_2| - \cdots - |S_l| - |R|$.
If $T = A_1 \cup K_2 \cup \cdots \cup K_l$ then $s(G) \geqslant c(G-T) - |T| = 1 + \rho(S_1) - |K_2| - \cdots - |K_l|$. Thus,

(C) $|K_1| \geqslant |S_2| + \cdots + |S_l| + |R| + 1$.
By using (C) we easily construct a path which contains all the vertices of $S' = S_2 \cup \cdots \cup S_l \cup R$ and whose end-vertices (a and b) are in K_1 (this path contains alternatively vertices of K_1 and S'). We denote this path by R_1. Then, there are $|K_1| - |S'| - 1$ vertices of K_1 not belonging to R_1.

Let $K' = K_2 \cup \cdots \cup K_l$ and $r = \rho(S_1)$. Let $Q_1, \ldots, Q_r$ be a path partition of S_1. We remark that the vertices of K' are adjacent to all the vertices of $Q_1 \cup \cdots \cup Q_r$. We can also construct a path R_2 beginning in c, an end-vertex of Q_1, and containing $Q_1 \cup \cdots \cup Q_j$ (with $j \leqslant r$ and j as great as possible). The other end-vertex of R_2 is d, end-vertex of Q_j. R_2 contains $j-1$ vertices of K'. Thus, $|K'| - j + 1$ vertices of K' does not belong to R_2.

1.2.1: $s(G) = p \geqslant 1$. Then by (B), $|K_2| + \cdots + |K_l| + p = |K'| + p = |K_1| - |S'| + r$ and by (C), $|K_1| - |S'| \geqslant 1$.

1.2.1.a: $|K'| \geqslant r$. Then, it is easy to see that $j = r$. Since $|K'| - r + 1 \geqslant 1$, there exists a vertex e of $K' \backslash R_2$ adjacent to d. Since ea is an edge, $R_3 = R_1 \cup [a,e,d] \cup R_2$ is a path. Moreover $|K_1| - |S'| = |K'| - r + p$. Then, it is possible to construct a path R_4 containing $K' \backslash V(R_3)$ and $|K'| - r$ vertices of $K_1 \backslash V(R_3)$ (such that the vertices of R_4 belong alternatively to K_1 and K'). Let f and g be the end-vertices of R_4 (f belongs to K_1 and g belongs to K'). Let $R_5 = R_3 \cup [b,g] \cup R_4$. There are $|K_1| - |S'| - (|K'| - r + 1) = |K_1| - |S'| - |K'| + r - 1 = p - 1$ vertices of K_1 which are not in R_5. Then we obtain p paths partitioning $V(G)$.

1.2.1.b: $r > |K'|$. We can see that $K' \subset R_2$. Then we have two paths and $|K_1| - |S'| - 1$ vertices of $K_1 \backslash V(R_1)$ and $r - |K'| - 1$ paths ($\{Q_{j+1}, \ldots, Q_r\}$) disjoint from R_2. Then we have $|K_1| - |S'| - 1 + r - |K'| - 1 + 2 = p$ disjoint paths covering $V(G)$.

1.2.2: $s(G) = 0$. By (B), $|K'| = |K_1| - |S'| + r$ and by (C), $|K_1| - |S'| \geqslant 1$, thus $|K'| > r$. Then, $j = r$ and there exist two vertices e and f of $K' \backslash V(R_2)$, respectively, adjacent to c and d. Since $|K'| - r - 1 = |K_1| - |S'| - 1$, if $|K'| - r - 1 > 0$ then it is possible to construct a path R_3 containing $(K' \backslash V(R_2)) \cup (K_1 \backslash V(R_1))$. Moreover, one end-vertex g of R_3 is in K' and the other one h is in K_1. Thus, $R_1 \cup [a,e,d] \cup R_2 \cup [c,f,h] \cup R_3 \cup [g,b]$ is a hamiltonian cycle of G. If $|K'| - r - 1 = 0$ then $R_1 \cup [a,e,d] \cup R_2 \cup [c,f,b]$ is a hamiltonian cycle of G.

Remark 5. In Case 1.2.2, since $G \neq K_1 \cup K_2 \cup S_1 \cup S_2$, we have $\forall i \in \{2, \ldots, l\}$, there exists an edge cd of the hamiltonian cycle with $c \in K_1$ and $d \in R \cup S_k$ with $k \neq i$.

1.3: $S = K_2 \cup \cdots \cup K_l \cup A_1$. Set $K' = K_2 \cup \cdots K_l$. Then, $s(G) = 1 + \max(1, s(S_1)) - |K'| = 1 + \rho(S_1) - |K'|$. Moreover, we have

$$\rho(S_2) + \cdots + \rho(S_l) + \rho(R) \leqslant |K_1|, \quad \text{(D)}$$

$$|K_1| \leqslant |S_2| + \cdots + |S_l| + |R|. \quad \text{(E)}$$

If (D) is false then set $T = K_1 \cup K' \cup A_1 \cdots \cup A_l \cup A_R$.

Then, $c(G\backslash T)-|T| = \rho(S_1)+\cdots+\rho(S_l)+\rho(R)-|K_1|-|K'|\geqslant 1+\rho(S_1)-|K'| = s(G)$. Then T is a scattering set and we are in Case 1.1. If (E) is false then set $T = K'\cup S_2\cup\cdots\cup S_l\cup R\cup A_1$. Then, $c(G-T)-|T| = |K_1|+\rho(S_1)-|K'|-|S_2|-\cdots-|S_l|-|R|\geqslant s(G)$. Then, T is a scattering set and we are in Case 1.2.

By (D) and (E), we can construct a path R_1 using all vertices of $S_2\cup\cdots\cup S_l\cup R$ and all the vertices of K_1. Let a and b be the two end-vertices of R_1. There are two possibilities (i) and (ii):

(i) if $|K_1| > \rho(S_2)+\cdots+\rho(S_l)+\rho(R)$ then a and b are in K_1.

(ii) if $|K_1| = \rho(S_2)+\cdots+\rho(S_l)+\rho(R)$ then a is in K_1 and b is in $S_2\cup\cdots\cup S_l\cup R$. Since $R\neq\emptyset$ then we choose b in R. (Note that for the proof with $R=\emptyset$ we choose b in S_l.)

We remark that every vertex of K_1 is adjacent to all the vertices of $S_2\cup\cdots\cup S_l\cup R$. Let $\mathscr{P}$ be a minimum path partition of S' (that is the union of minimum path partitions of $S_1,\ldots,S_l,R$). To construct R_1 we use the following idea: we begin at a, we use $\rho(S_2)$ paths partitioning $V(S_2)$, $\rho(S_3)$ paths partitioning $V(S_3),\ldots,$ $\rho(S_l)$ paths partitioning $V(S_l)$ and $\rho(R)$ paths partitioning R (this is possible by (D)).

In Case (ii), all the vertices of K_1 belong to R_1 and b is in $S_2\cup\cdots\cup S_l\cup R$. In Case (i), $|K_1\backslash V(R_1)|\neq\emptyset$. By (E), by conveniently broking and reconnecting some paths of $\mathscr{P}$ we can use all the vertices of K_1 not yet in the path. Moreover, $b\in K_1$.

Set $r=\rho(S_1)$ and let $Q_1,\ldots,Q_r$ be a path partition of S_1. We remark that the vertices of K' are adjacent to all the vertices of $Q_1\cup\cdots\cup Q_r$. We can also construct a path R_2 beginning in c, an end-vertex of Q_1, and containing $Q_1\cup\cdots\cup Q_j$ (with $j\leqslant r$ and j as great as possible). The other end-vertex of R_2 is d, end-vertex of Q_j. R_2 contains $j-1$ vertices of K'. Thus, $|K'|-j+1$ vertices of K' are not in R_2.

1.3.1: $s(G)=p\geqslant 1$. Then $p=\rho(S_1)+1-|K'|$, thus $|K'|+p-1=\rho(S_1)$.

1.3.1.a: $p=1$. Then $|K'|=\rho(S_1)$, thus $j=r$ and there exists one vertex $e\in(K'\backslash V(R_2))$. Then $R_1\cup[a,e,c]\cup R_2$ is a hamiltonian path of G.

1.3.1.b: $p>1$. Then, it is easy to see that $j<r$. Moreover, $j=|K'|+1$. Since we have R_1, R_2 and $\{Q_{j+1},\ldots,Q_r\}$ partitioning $V(G)$, then we have $2+r-|K'|-1=p$ paths partitioning $V(G)$.

1.3.2: $s(G)=0$. Then $\rho(S_1)+1-|K'|=0$. Since $\rho(S_1)+1=|K'|$, then it is easy to see that $j=r$ and $|K'\backslash V(R_2)|=2$. Let e and f be the two vertices of $K'\backslash V(R_2)$ (note that for the proof with $R=\emptyset$, $K'=K_2\cup\cdots\cup K_l$ with $l\geqslant 3$, if $K_{l-1}\backslash V(R_2)\neq\emptyset$ then we choose f in K_{l-1} else we modify R_2 by permuting a vertex of R_2 and a vertex of $K'\backslash V(R_2)$ to obtain a new path R_2 such that we can choose f in K_{l-1}). Then in Cases (i) and (ii), $R_1\cup[a,e,c]\cup R_2\cup[d,f,b]$ is a hamiltonian cycle of G.

Remark 6. In Case 1.3.2, since $G\neq K_1\cup K_2\cup S_1\cup S_2$, we have $\forall i\in\{2,\ldots,l\}$, there exists an edge cd of the hamiltonian cycle with $c\in K_1$ and $d\in R\cup S_k$ with $k\neq i$.

Case 2. $s(G)\leqslant -1$. Let G' be an $\mathscr{S}$-graph with $|V(G')|<|V(G)|$; then we suppose by induction that G' is a Jung graph.

Let x and y be two arbitrary vertices of G, let $z \in K_i$ and let $G' = G\backslash\{z\}$. Remark that it is always possible to choose z such that $zx \in E$ or $zy \in E$. G' is an $\mathscr{S}$-graph. (If $K_i = \{z\}$, then we have $G \neq K_1 \cup K_2 \cup S_1 \cup S_2 \cup R$. Otherwise, we can suppose without loss of generality that $K_i = K_1$; then $s(G) \geqslant c(G\backslash(K_1 \cup A_2)) - |K_1 \cup A_2| = \rho(S_2) \geqslant 1$, a contradiction. Thus, $G' = K_1 \cup \cdots \cup K_{i-1} \cup K_{i+1} \cup \cdots \cup K_l \cup S_1 \cup \cdots \cup S_{i-1} \cup S_{i+1} \cup \cdots \cup S_l \cup (R \cup S_i)$ is an $\mathscr{S}$-graph.) Moreover, G' is not of the form $K_1' \cup K_2' \cup S_1' \cup S_2'$, $s(G') \leqslant s(G) + 1$ and by induction, G' is a Jung graph.

2.1: $x \in K_j$ or $y \in K_j$. Without loss of generality, suppose that $x \in K_j$. Let $z = x$; then $G' = G\backslash\{x\}$ and G' is a Jung graph. If $s(G') \leqslant -1$ then there exists a hamiltonian path P of G' joining y and a, a neighbour of x in G. Then, $P \cup [a,x]$ is a hamiltonian path of G joining x and y. If $s(G') = 0$, then in Case 1.1.2, by Remark 4 and by Lemma 9 there exists a hamiltonian path of G joining x and y. In Case 1.2.2 (respectively, Case 1.3.2), we see that $i \neq 1$; else we have $s(G' \cup \{x\}) = 1 > s(G)$ (respectively, $s(G' \cup \{x\}) = 0 > s(G)$), a contradiction. By Remark 5 (respectively, Remark 6) and by Lemma 9, there exists a hamiltonian path of G joining x and y.

2.2: $\{x, y\} \subset S_1 \cup \cdots \cup S_l \cup R$. If $s(G') \leqslant -1$ then there exists a hamiltonian path P of G' joining y and x. Then, by Lemma 8, there exists a hamiltonian path of G joining x and y. If $s(G') = 0$, then in Case 1.1.2, by Remark 4 and by Lemma 10 there exists a hamiltonian path of G joining x and y. In Case 1.2.2 (respectively, Case 1.3.2), we see that $i \neq 1$ else we have $s(G' \cup \{z\}) = 1 > s(G)$ (respectively, $s(G' \cup \{z\}) = 0 > s(G)$), a contradiction. By Remark 5 (respectively, Remark 6) and by Lemma 10, there exists a hamiltonian path of G joining x and y. □

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