JOURNAL OF COMBINATORIAL THEORY (B) 20, 265–282 (1976)

On the Number of 1-Factorizations of the Complete Graph

CHARLES C. LINDNER

Department of Mathematics, Auburn University, Auburn, Alabama 36830

ERIC MENDELSOHN

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada

AND

ALEXANDER ROSA

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada L8S-4K1

Communicated by W. T. Tutte

Received August 23, 1974

1. INTRODUCTION

It is well known that for every positive integer n there exists a 1-factorization of the complete graph K_{2n}. (For this result and for undefined graph-theoretical notions and standard notation, see [12].) Although the question about the existence of 1-factorizations of K_{2n} is answered easily, the problem of determining the number $N(2n)$ of pairwise nonisomorphic 1-factorizations of K_{2n} appears to be a difficult one. Known results on $N(2n)$ can be summarized as follows: $N(2) = N(4) = N(6) = 1$ (this is easily obtained). Further, $N(8) = 6$ (proved by Safford [7] in 1906 and again by Wallis [18] in 1972). Gelling ([9]; see also [10]) used a computer to obtain $N(10) = 396$ (he also determined the orders of the groups of the respective 1-factorizations). Finally, a recent result of Wallis [19] states that $N(2n) \geqslant 2$ for $n \geqslant 4$.

The main purpose of this paper is to improve this last result. We show in Section 3, among other things, that the number $N(2n)$ goes to infinity with n, by making use of the relationship between 1-factorizations and quasigroups satisfying certain identities (this relationship has apparently been noticed also in [13, 14]). The same result is proved again in Section 5 where we use two recursive constructions to show that the number $A(2n)$

Research supported by N.R.C. Grant No. A7268.

of pairwise nonisomorphic automorphism-free 1-factorizations of K_{2n} goes to infinity with n. Finally, some results concerning embeddings of 1-factorizations and Steiner triple systems are obtained in Sections 4 and 6.

2. Preliminaries

Throughout this paper, all quasigroups are understood to be finite. The reader is referred to [6] for basic notions in the theory of quasigroups and latin squares used in what follows.

A commutative quasigroup $(V, \circ)$ satisfying the identity $x \circ x = y \circ y$ will be called a *CC-quasigroup*. An idempotent commutative quasigroup $(V, \circ)$ satisfying the identity $x \circ (x \circ y) = y$ is called a *Steiner quasigroup* (also an idempotent totally symmetric quasigroup). A commutative loop $(V, \circ)$ satisfying the identities $x \circ x = e$, $x \circ (x \circ y) = y$ (where e is the identity element) is called a *Steiner loop* (or totally symmetric loop). It is well known that there is a one–one correspondence between Steiner quasigroups of order n and Steiner loops of order $n + 1$ [4].

A *Steiner triple system* (briefly STS) is a pair $(S, \mathscr{B})$ where S is a finite set and $\mathscr{B}$ is a collection of 3-subsets of S (called triples) such that every pair of distinct elements of S belongs to exactly one triple of $\mathscr{B}$. The number $|S|$ is called the order of $(S, \mathscr{B})$. It is well-known that there is a Steiner triple system of order n if and only if $n \equiv 1$ or $3 \pmod 6$. It is also well known that the theory of Steiner triple systems is coextensive with that of Steiner quasigroups. Therefore, a Steiner quasigroup [loop] of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$ [$n \equiv 2$ or $4 \pmod 6$].

Two quasigroups $(V, \circ)$ and $(W, \otimes)$ are *isotopic* if there exist three bijections $\alpha, \beta, \gamma\colon V \to W$ such that $(x\alpha \otimes y\beta) = (x \circ y)\gamma$ for all $x, y \in V$. If $\alpha = \beta$ then $(V, \circ)$ and $(W, \otimes)$ are *rc-isotopic*, and if $\alpha = \beta = \gamma$ then $(V, \circ)$ and $(W, \otimes)$ are *isomorphic*.

A 1-factorization of K_{2n} (briefly $OF(K_{2n})$) will be denoted by a pair $(V, \mathscr{F})$ where $V = V(K_{2n})$ is the vertex-set of K_{2n} and $\mathscr{F} = \{F_i\}_{i \in I_{\mathscr{F}}}$ is the set of 1-factors which can be indexed by any $(2n - 1)$-subset $I_{\mathscr{F}}$ of V.

THEOREM 1. *There is a one–one correspondence between the 1-factorizations of K_{2n} and the CC-quasigroups of order $2n$.*

Proof. Let $(V, \mathscr{F})$ be a 1-factorization of K_{2n}. Define a binary operation $\circ$ on the set V by

$x \circ x = u,$ where u is the unique element of $V \backslash I_{\mathscr{F}}$,

and

$x \circ y = z$ if $x \neq y$ and the edge $[x, y]$ belongs to the factor F_z of $\mathscr{F}$.

Obviously, $(V, \circ)$ is a CC-quasigroup of order $2n$.

Conversely, let $(V, \circ)$ be a CC-quasigroup of order $2n$. Then for all $x \in V$, $x \circ x = u$ for some $u \in V$. Put $V(K_{2n}) = V$ and for all $z \in V \backslash \{u\}$ define a factor F_z of K_{2n} to contain all edges $[x, y]$, $x \neq y$, such that $x \circ y = z$. Clearly, $(V, \mathscr{F})$, where $\mathscr{F} = \{F_z\}_{z \in V \backslash \{u\}}$, is an $OF(K_{2n})$.

Under our convention, two 1-factorizations $(V, \mathscr{F})$ and $(W, \mathscr{G})$ of K_{2n} (where $\mathscr{F} = \{F_i\}_{i \in I_{\mathscr{F}}}$, $\mathscr{G} = \{G_i\}_{i \in I_{\mathscr{G}}}$) are *isomorphic* if there exist two bijections

$$\alpha\colon V \to W \qquad \gamma'\colon I_{\mathscr{F}} \to I_{\mathscr{G}}$$

such that $[x, y] \in F_i \leftrightarrow [x\alpha, y\alpha] \in F_{i\gamma'}$.

Obviously one can extend γ' to $\gamma\colon V \to W$ uniquely by putting

$$x\gamma = x\gamma', \qquad \text{for } x \in I_{\mathscr{F}},$$

and

$$u\gamma = v, \qquad \text{where } \{u\} = V \backslash I_{\mathscr{F}}, \{v\} = V \backslash I_{\mathscr{G}}.$$

This observation results in the following theorem:

THEOREM 2. *Let $(V, \mathscr{F})$ and $(W, \mathscr{G})$ be two $OF(K_{2n})$ and let $(V, \circ)$ and $(W, \otimes)$ be the corresponding CC-quasigroups of order $2n$ (under the correspondence established by Theorem 1). Then $(V, \mathscr{F})$ and $(W, \mathscr{G})$ are isomorphic if and only if $(V, \circ)$ and $(W, \otimes)$ are rc-isotopic.*

An $OF(K_{2n})$ isomorphic to an $OF(K_{2n})$ for which the corresponding CC-quasigroup is a Steiner loop will be called a *Steiner 1-factorization.* Thus, Steiner 1-factorizations of K_{2n} exist if and only if $n \equiv 1$ or 2 (mod 3).

THEOREM 3. *Two Steiner 1-factorizations of K_{2n} are isomorphic if and only if the corresponding Steiner loops are isomorphic.*

Proof. It is well known [3] that any two isotopic Steiner loops are necessarily isomorphic.

3. NONISOMORPHIC 1-FACTORIZATIONS OF K_{2n}

The best results to-date on the number of pairwise nonisomorphic STS are due to Wilson [20]. In view of Theorem 3, Wilson's bounds yield the following:

LEMMA 4. *Let $n \equiv 1$ or 2 (mod 3), and let $S(2n)$ denote the number of pairwise nonisomorphic Steiner 1-factorizations of K_{2n}. Then*

$$\exp\left(\frac{(2n-1)^2}{12}(\log(2n-1)-5)\right) \leq S(2n) \leq \exp\left(\frac{(2n-1)^2}{6}\left(\log(2n-1)-\frac{1}{2}\right)\right). \blacksquare$$

Denote by $C(t)$ the number of distinct symmetric latin squares of order t with constant diagonal (= the number of distinct CC-quasigroups of order t), by $D(t)$ the number of distinct diagonalized symmetric latin squares of order t, and by $E(t)$ the number of distinct latin squares of order t with constant diagonal. Trivially, $C(t) \geqslant t$ and $D(t) = 0$ for t even, $C(t) = 0$ and $D(t) \geqslant t!$ for t odd, and $E(t) \geqslant t$ for all t. The number of pairwise nonisomorphic $OF(K_{2n})$ is denoted by $N(K_{2n})$.

LEMMA 5. *Let $n = ks$ where k and s are positive integers. Then*

(a) $$N(K_{2n}) \geqslant (C(2k)\, C(s)/[(2ks)!]^2) \max_{\substack{k,s\\ n=ks}} [s!(s-1)!\dots 2!1!]^{k(2k-1)},$$ if s is even, and

(b) $$N(K_{2n}) \geqslant (C(2k)\, D(s)[E(s)]^k/[(2ks)!]^2\, s^{k-1}) \max_{\substack{k,s\\ n=ks}} [s!(s-1)!\dots 2!1!]^{2k(k-1)},$$ if s is odd.

Proof. Let k and s be positive integers such that $n \geqslant ks$, and let $L = \| 1_{ij} \|$ be a symmetric latin square of order $2k$ with constant diagonal based on $K = \{1, 2,\dots, 2k\}$. Let $S = \{1, 2,\dots, s\}$ and construct a latin square V or order $2n = 2ks$ based on $S \times K$ as follows:

Case 1. s is even. Let $M = \| m_{xy} \|$ be a symmetric latin square of order s with constant diagonal based on S, and let $P^{ij} = \| p_{xy}^{ij} \|$; $i, j = 1, 2,\dots, 2k$, $i < j$; be a set of $k(2k-1)$ (not necessarily distinct) latin squares of order s, all based on S. Define V as follows: the element $v_{xi,yj}$ (= the entry in the cell $((x, i), (y, j))$ of V) is

$$\begin{aligned} v_{xi,yj} &= (m_{xy}, 1_{ij}) && \text{if } i = j,\\ &= (p_{xy}^{ij}, 1_{ij}) && \text{if } i < j,\\ &= (p_{yx}^{ji}, 1_{ij}) && \text{if } i > j. \end{aligned}$$

Obviously, V is a symmetric latin square of order $2ks$ with constant diagonal. It is well known [11] that there are at least $s!\,(s-1)!\dots 2!\,1!$ distinct latin squares of order s. As we have $C(2k)$ choices for L, $C(s)$ choices for M and at least $s!\,(s-1)!\dots 2!\,1!$ choices for each p^{ij} we obtain that there are at least $[C(2k)\, C(s)\, s!\,(s-1)!\dots 2!\,1!]^{k(2k-1)}$ distinct latin squares of order $2ks$ with constant diagonal (based on the same set) when s is even.

Case 2. s is odd. Let $Q = \| q_{xy} \|$ be a diagonalized symmetric latin square of order s, let $R^i = \| r_{xy}^i \|$, $i = 1, 3,\dots, 2k-1$ be a set of k (not necessarily distinct) latin squares of order s with constant diagonal. Further, let $P^{ij} = \| p^{ij}_{xy} \|$, $i, j = 2,..., 2k$, $i < j$, $j \neq i + 1$ for i odd, be a set of $2k(k-1)$ (not necessarily distinct) latin squares of order s (all of Q, R^i, P^{ij} based on S). Define a latin square W of order $2ks$ based on $S \times K$ as follows: The element $w_{xi,yj}$ (= the entry in the cell $((x, i), (y, j))$ of W is

$$\begin{aligned} w_{xi,yj} &= (q_{xy}, 1_{ij}) && \text{if } i = j \\ &= (r^i_{xy}, 1_{ij}) && \text{if } j = i + 1, i = 1, 3,..., 2k-1 \\ &= (r^j_{yx}, 1_{ij}) && \text{if } i = j + 1, j = 1, 3,..., 2k-1 \\ &= (p^{ij}_{xy}, 1_{ij}) && \text{if } i < j, i + 1 \neq j \text{ for } i \text{ odd} \\ &= (p^{ji}_{yx}, 1_{ij}) && \text{if } i > j, i \neq j + 1 \text{ for } i \text{ even.} \end{aligned}$$

Obviously, W is a symmetric latin square but its diagonal is not constant. However, for any $x \in S$ and $i \in \{1, 3,..., 2k-1\}$ the entries in the four cells $((x, i), (x, i))$, $((x, i+1), (x, i+1))$, $((x, i), (x, i+1))$, and $((x, i+1), (x, i))$ form a sublatin square of W of order 2. By interchanging the two elements in this square, and by performing this interchange for all $x \in S$ and $i \in \{1, 3,..., 2k-1\}$ one obtains from W a symmetric latin square V with constant diagonal. Now we have $C(2k)$ choices for L, $D(s)$ choices for Q, $E(s)$ choices for one of the R^i's, and $E(s)/s$ choices for each of the remaining $k-1$ R^i's (since the diagonal of every R^i has to be occupied by the same element), and at least $s!\,(s-1)!... 2!\,1!$ choices for each P^{ij}. Thus there are at least $C(2k)\,D(s)\,E(s)^k\,[s!\,(s-1)!... 2!\,1!]^{2k(k-1)}/s^{k-1}$ distinct latin squares of order $2ks$ with constant diagonal (based on the same set) when s is odd.

Since each equivalence class (under *rc*-isotopy) of latin squares of order $2n$ contains at most $[(2n)!]^2$ distinct latin squares the inequalities (a) and (b) follow.

One finds easily $C(6) = 4320$; thus, we have the following corollary:

COROLLARY 6. *Let* $n \equiv 0 \pmod 3$, $n = 3s$. *Then*

$$N(K_{2n}) \geqslant \begin{cases} \dfrac{4320s[s!...2!1!]^{15}}{[(6s)!]^2} & \text{for } s \text{ even} \\ \geqslant \dfrac{4320s.s![s!(s-1)!...2!1!]^{12}}{[(6s)!]^2} & \text{for } s \text{ odd.} \end{cases}$$

THEOREM 7. $\lim_{n\to\infty} N(K_{2n}) = \infty$.

Proof. For $n \equiv 1$ or 2 (mod 3) the statement follows from Lemma 4, and for $n \equiv 0$ (mod 3) from Corollary 6.

4. Embeddings of 1-Factorizations

Given a 1-factor F of K_{2n}, any nonempty set of edges from F will be called a subfactor of F. Given two 1-factors F and F' of K_{2n}, $F \cap F'$ denotes the set of edges contained in both F and F', thus $F \cap F'$ is either empty or is a subfactor of both F and F'. Given two sets $\mathscr{F}$ and $\mathscr{F}'$ of 1-factors of K_{2n}, we denote $\mathscr{F} \cap \mathscr{F}' = \{F_i \cap F_j' \mid F_i \in \mathscr{F}, F_j' \in \mathscr{F}'\}$.

An $OF(K_{2s})$ $(W, \mathscr{G})$ is said to be a *sub-1-factorization* (briefly sub-OF) of an $OF(K_{2n})$ $(V, \mathscr{F})$ if (1) $W \subseteq V$ and (2) there exists a (one-to-one) mapping γ: $I_{\mathscr{G}} \to I_{\mathscr{F}}$ such that for each $j \in I_{\mathscr{G}}$, the 1-factor $G_j \in \mathscr{G}$ is a subfactor of $F_{j\gamma} \in \mathscr{F}$.

The number n/s is said to be the *index* of $(W, \mathscr{G})$ in $(V, \mathscr{F})$. If $(W, \mathscr{G})$ is a sub-OF of (V, F) and $W \subsetneq V$ will also say that $(W, \mathscr{G})$ is *embedded* into (V, F).

THEOREM 8. *An $OF(K_{2s})$ can be embedded into an $OF(K_{2n})$ if and only if $n \geqslant 2s$.*

Proof. It has been proved in [5] that every symmetric diagonalized latin square of order k can be (properly) embedded into a symmetric diagonalized latin square of order t if and only if $t \geqslant 2k + 1$ (both k and t are necessarily odd). Since there is an obvious one–one correspondence between symmetric diagonalized latin squares of order k and symmetric latin squares with constant diagonal of order $k + 1$, this is equivalent to saying that a symmetric latin square with constant diagonal of order $2s$ can be embedded into a symmetric latin square with constant diagonal or order $2n$ if and only if $n \geqslant 2s$.

If (V, F) and (W, G) are $OF(K_{2s})$ and $OF(K_{2t})$, respectively, and if both are sub-OF of an $OF(K_{2n})$ $(Z, \mathscr{H})$ and $V \cap W = \varnothing$ then $(V, \mathscr{F})$ and $(W, \mathscr{G})$ are said to be *disjointly embedded* into $(Z, \mathscr{H})$.

THEOREM 9. *Let $(V, \mathscr{F})$ and $(W, \mathscr{G})$ be $OF(K_{2s})$ and $OF(K_{2t})$, respectively, and let $V \cap W = \varnothing$. Then $(V, \mathscr{F})$ and $(W, \mathscr{G})$ can be disjointly embedded into an $OF(K_{2n})$ for every $n \geqslant 4 \max(s, t)$.*

Proof. Let $n \geqslant 4 \max(s, t)$ and let Z_1, Z_2 be any two disjoint n-sets such that $V \subseteq Z_1$ and $W \subseteq Z_2$. By Theorem 8, $(V, \mathscr{F})$ can be embedded into an $OF(K_n)$ $(Z_1, \mathscr{H}^1)$ and $(W, \mathscr{G})$ can be embedded into an $OF(K_n)$ $(Z_2, \mathscr{H}^2)$. Let $\mathscr{H}^1 = \{H_i^1\}_{i \in I_{\mathscr{H}^1}}$, $H^2 = \{H_i^2\}_{i \in I_{\mathscr{H}^2}}$, and let

$\mathscr{X} = \{X_1, X_2, ..., X_n\}$ be the set of 1-factors in any 1-factorization of the complete bipartite graph $K_{n,n}$ [12] with the vertex-set $Z_1 \cup Z_2$ (where the subgraph induced by Z_i is null). Let α be any bijection $\alpha: I_{\mathscr{H}^1} \to I_{\mathscr{H}^2}$, and put

$$H_i = H_i^1 \cup H_{i\alpha}^2 \qquad \text{for all } i \in I_{\mathscr{H}^1}, \qquad \mathscr{H} = \{H_i\}_{i \in I_{\mathscr{H}^1}}.$$

Then $(Z_1 \cup Z_2, \mathscr{H} \cup \mathscr{X})$ is an $OF(K_{2n})$ with $(V, \mathscr{F})$ and $(W, \mathscr{G})$ disjointly embedded into $(Z_1 \cup Z_2, \mathscr{H} \cup \mathscr{X})$.

COROLLARY 10. *Let $(V, \mathscr{F})$ and $(W, \mathscr{G})$ be two $OF(K_{2n})$ such that $V \cap W = \varnothing$. Then $(V, \mathscr{F})$ and $(W, \mathscr{G})$ can be disjointly embedded into an $OF(K_{4n})$.*

We conclude this section with a lemma which will be needed in subsequent sections.

LEMMA 11. *Let $(V_1, \mathscr{F}_1)$ and $(V_2, \mathscr{F}_2)$ be two sub-OF of an $OF(V, \mathscr{F})$. Then either $\mathscr{F}_1 \cap \mathscr{F}_2 = \varnothing$ or $(V_1 \cap V_2, \mathscr{F}_1 \cap \mathscr{F}_2)$ is a sub-OF of $(V, \mathscr{F})$.*

Proof. If $\mathscr{F}_1 \cap \mathscr{F}_2 \neq \varnothing$, then there are two distinct vertices $a, b \in V_1 \cap V_2$. Then the edge $[a, b]$ belongs to some 1-factor $F_i \in \mathscr{F}$. Obviously, the subfactor $F_i \mid V_1 = F_i^1 \in \mathscr{F}_1$ and the subfactor $F_i \mid V_2 = F_i^2 \in \mathscr{F}_2$, and $[a, b] \in F_i^1$, $[a, b] \in F_i^2$ as well. Let c be any vertex in $V_1 \cap V_2$ other than a, b (if it exists), and let d be the vertex in V such that $[c, d] \in F_i$. Then we must have $d \in V_1$ since $(V_1, \mathscr{F}_1)$ is a sub-*OF* of $(V, \mathscr{F})$ and $d \in V_2$ since $(V_2, \mathscr{F}_2)$ is a sub-*OF* of $(V, \mathscr{F})$. Thus $d \in V_1 \cap V_2$ which in turn implies that $(V_1 \cap V_2, \mathscr{F}_1 \cap \mathscr{F}_2)$ is a sub-*OF* of $(V, \mathscr{F})$.

5. AUTOMORPHISM-FREE 1-FACTORIZATIONS

An automorphism of a 1-factorization $(V, \mathscr{F})$ is an isomorphism of $(V, \mathscr{F})$ with itself. An automorphism of $(V, \mathscr{F})$ corresponds to an *rc*-autotopy of the *CC*-quasigroup $(V, \circ)$, i.e., to a pair of bijections $\alpha, \gamma: V \to V$ such that (α, γ) is an *rc*-isotopy of $(V, \circ)$ with itself.

An $OF(K_{2n})$ is said to be *automorphism-free* if it has only the trivial automorphism. Obviously, an $OF(K_{2n})$ is automorphism-free (briefly AF) if and only if the corresponding *CC*-quasigroup has only the trivial *rc*-autotopy (i.e., both α and γ are identity mappings).

Let $A(2n)$ denote the number of pairwise nonisomorphic AF $OF(K_{2n})$. It is known that $A(2) = A(4) = A(6) = A(8) = 0$, $A(10) = 298$ [9]. Recently, it has been shown [15] that an automorphism-free Steiner triple system of order n exists if and only if $n \geqslant 15$ (and $n \equiv 1$ or 3 (mod 6), of course), and that the number of nonisomorphic AF STS of order n goes to infinity with n. In view of Theorem 3, the following theorem is immediate.

THEOREM 12. *Let* $n \equiv 1$ *or* 2 (mod 3). *Then* $A(2n) \geqslant 1$ *for* $n \geqslant 8$, *and* $\lim_{n\to\infty} A(2n) = \infty$.

Unfortunately, one cannot use AF STS directly to show the existence of an AF $OF(K_{2n})$ for every n. For this we have to use a different method. Below two recursive constructions are given which enable us to build AF $OF(K_{2n})$ from "smaller" AF OF.

In what follows we denote by $GF(K_{2n})$ the particular series of 1-factorizations which is probably the best known and has been discovered and studied by many authors (see, e.g., [1, 2, 12, 14]).

THEOREM 13. *If there exists an* AF $OF(K_{2n})$ *then there exists an* AF $OF(K_{4n})$.

Proof. Let $(V_1, \mathscr{F}_1)$ be any AF $OF(K_{2n})$ and let $(V_2, \mathscr{F}_2)$ be the $GF(K_{2n})$. By Corollary 10, $(V_1, \mathscr{F}_1)$ and $(V_2, \mathscr{F}_2)$ can be disjointly embedded into an $OF(K_{4n})$. Let $(V, \mathscr{F})$ (where $V = V_1 \cup V_2$) be any $OF(K_{4n})$ containing $(V_1, \mathscr{F}_1)$ and $(V_2, \mathscr{F}_2)$ as (disjoint) sub-OF of index 2. We will show that $(V, \mathscr{F})$ is an AF $OF(K_{4n})$. Observe that if (α, γ') is an automorphism of $(V, \mathscr{F})$, then $\gamma': I_{\mathscr{F}} \to I_{\mathscr{F}}$ is induced by $\alpha: V \to V$, thus it is enough to consider just one mapping α.

I. Assume first that α is a nontrivial automorphism of $(V, \mathscr{F})$ which maps $(V_1, \mathscr{F}_1)$ onto itself. Then necessarily $x\alpha = x$ for every $x \in V_1$, every 1-factor of $\mathscr{F}$ is fixed under α and consequently every 1-factor of $\mathscr{F}_2$ must be fixed under α. But it is well known that $GF(K_{2n})$ has no nontrivial automorphism fixing all its factors.

II. Assume now that α is a nontrivial automorphism of $(V, \mathscr{F})$ which maps $(V_1, \mathscr{F}_1)$ onto $(V', \mathscr{F}')$ where $(V', \mathscr{F}')$ is another sub-OF of $(V, \mathscr{F})$ of index 2, A simple numberical argument shows then that $| V' \cap V_1 | = | V' \cap V_2 | = n$ and $(V' \cap V_1, \mathscr{F}' \cap \mathscr{F}_1)$ $[(V' \cap V_2, \mathscr{F}' \cap \mathscr{F}_2)]$ is a sub-OF of $(V_1, \mathscr{F}_1)$ [sub-OF of $(V_2, \mathscr{F}_2)$] of index 2. However, $GF(K_{2n})$ cannot have a sub-OF of index 2 (cf. [15, Theorem 3.1]). This completes the proof.

COROLLARY 14. $A(4n) \geqslant (2n-3)!\, A(2n)$.

Proof. Let $(V_1, \mathscr{F}_1)$ and $(V_2, \mathscr{F}_2)$ be as in Theorem 13. To obtain $(V, \mathscr{F})$, we have $(2n-1)!$ choices for the bijection α^* (cf. proof of

Theorem 9), and therefore $(2n-1)!$ distinct AF $OF(K_{4n})$ corresponding to a fixed AF $OF(K_{2n})$ (and to a fixed 1-factorization of $K_{2n,2n}$). On the other hand, the order of the automorphism group of $GF(K_{2n})$ does not exceed $(2n-1)(2n-2)$ ([2]; cf. also [15]) thus there are at least $(2n-3)!$ nonisomorphic $OF(K_{4n})$ obtained from a given AF $OF(K_{2n})$ $(V_1, \mathscr{F}_1)$. Obviously, any two AF $OF(K_{4n})$ constructed as in Theorem 13 and obtained from two nonisomorphic AF $OF(K_{2n})$ are also nonisomorphic and the Corollary follows.

In order to prove the next theorem, we need one more auxilliary device. The following definitions are taken from [16] (cf. also [15]).

An (A, k)-system is a set of k disjoint pairs (p_r, q_r) covering the elements of $\{1, 2,..., 2k\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2,..., k$. Similarly, a (B, k)-system is a set of k disjoint pairs (p_r, q_r) covering the elements of $\{1, 2,..., 2k-1, 2k+1\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2,..., k$. It is known (see, e.g., [16]) that an (A, k)-system exists if and only if $k \equiv 0$ or $1 \pmod 4$, and a (B, k)-system exists if and only if $k \equiv 2$ or $3 \pmod 4$. Observe that an (A, k) system and a (B, k)-system is essentially the same thing as what has been called by several authors a Skolem $(2, k)$-sequence and a hooked Skolem $(2, k)$-sequence (cf. [15]).

THEOREM 15. *If there exists an* AF $OF(K_{2n})$ *then there exists an* AF $OF(K_{4n+6})$.

Proof. Let $(V, \mathscr{F})$ be any AF $OF(K_{2n})$, $V = \{a_1, a_2, ..., a_{2n-1}, a^*\}$, $\mathscr{F} = \{F_k\}_{k \in I_{\mathscr{F}}}$. Further let $U = \{b_i \mid i = 1, 2,..., 2n-1\}$, $X = \{\infty_i \mid i = 1, 2,..., 7\}$ and let $(X \cup \{a^*\}, \mathscr{D})$, where $\mathscr{D} = \{D_i \mid i = 1, 2,..., 7\}$ be the $GF(K_8)$. Let $L = \{(p_r, q_r) \mid q_r - p_r = r,\ r = 1, 2,..., n-1\}$ be an $(A, n-1)$-system or $(B, n-1)$-system according to whether $n \equiv 1, 2 \pmod 4$ or $n \equiv 0, 3 \pmod 4$. Denote further $Y = U - W$ where $W = \{b_i \mid i = p_r \text{ or } q_r, r = 4, 5,..., n-1; (p_r, q_r) \in L\}$. Obviously $|Y| = 7$. Now let $Y = \{b_{j_i} \mid i = 1, 2,..., 7\}$. Put $S = V \cup U \cup X$ and $\mathscr{H} = \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$ where $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are the following sets of 1-factors:

$$\mathscr{A} = \{A_k \mid k = 1, 2,..., 2n-1\}, \quad A_k = A_k' \cup F_{k\beta},$$

where β is any bijection from $\{1, 2,..., 2n-1\}$ into $I_{\mathscr{F}}$, $A_k' = \{[\infty_i, b_{j_i+k-1}] \mid i = 1, 2,..., 7\} \cup \{[b_{p_r+k-1}, b_{q_r+k-1}] \mid r = 4, 5,..., n-1\}$,

$$\mathscr{B} = \{B_k \mid k = 1, 2,..., 2n-1\}, \quad B_k = B_k' \cup B_k'' \cup B_k''' \cup \{[b_k, a^*]\},$$

$$B_k' = \{[b_{k+1}, b_{k+3}], [b_{k-1}, b_{k+2}], [b_{k-3}, b_{k-2}]\}$$

$$B_k'' = \{[\infty_i, a_{k-j_i+1}] \mid i = 1, 2,..., 7\}$$

$$B_k''' = \{[a_{k-p_r+1}, b_{k+r}] \mid r = 4, 5,..., n-1\},$$

and

$$\mathscr{C} = \{C_i \mid i = 1, 2,..., 7\},$$
$$C_i = C_i' \cup D_i\,,\ C_i' = \{[a_k\,, b_{j_i+k-1}] \mid k = 1, 2,..., 2n - 1\}$$

with subscripts reduced modulo $2n - 1$ to the range $\{1, 2,..., 2n - 1\}$ whenever necessary. It is readily verified that $(S, \mathscr{H})$ is an $OF(K_{4n+6})$.

In order to show that $(S, \mathscr{H})$ is automorphism-free we show first that $(V, \mathscr{F})$ is the unique sub-OF of $(S, \mathscr{H})$ of index $2 + 3/n$. Assume that $(V', \mathscr{F}')$ is another sub-OF of $(S, \mathscr{H})$ with $|V'| = 2n$. Distinguish the following cases (in the discussion below, we refer to a 1-factor from the set $\mathscr{A}$ ($\mathscr{B}$ and $\mathscr{C}$) as an $\mathscr{A}$-factor ($\mathscr{B}$-factor and $\mathscr{C}$-factor)).

Case 1. $V \cap V' = \varnothing$. Then no 1-factor F_i' of $\mathscr{F}'$ can be a subfactor of a $\mathscr{B}$-factor or of a $\mathscr{C}$-factor, as such a 1-factor contains at most three edges joining vertices from $U \cup X$. However, we have by our assumption $2n \geqslant 10$, and thus $\frac{1}{2}n > 3$. Therefore all 1-factors of $\mathscr{F}'$ are subfactors of $\mathscr{A}$-factors. Since $|V'| = 2n$, among the $2n$ vertices of V' chosen in any way from $2n + 6$ vertices of $U \cup X$ there must be two vertices b_x, b_y such that $x - y \equiv 1$ or 2 or 3 (mod $2n - 1$). But no $\mathscr{A}$-factor contains an edge joining any two such vertices b_x and b_y which contradicts the fact that $(V', \mathscr{F}')$ is an $OF(K_{2n})$.

Case 2. $|V \cap V'| = 1$. This case is similar to case 1.

Case 3. $|V \cap V'| \geqslant 2$, and thus $\mathscr{F} \cap \mathscr{F}' \neq \varnothing$. By Lemma 11, $(V \cap V', \mathscr{F} \cap \mathscr{F}')$ is a sub-OF of $(S, \mathscr{H})$, and also of $(V, \mathscr{F})$ and of $(V', \mathscr{F}')$. Therefore $|V \cap V'| \leqslant n$ and consequently $|V' \cap (U \cup X)| \geqslant n$. Distinguish the following subcases:

Case 3*a*. $|V' \cap X| \geqslant 2$. Then $\mathscr{F}'$ must contain at least one 1-factor which is a subfactor of a $\mathscr{C}$-factor, and it follows that either $|V' \cap U| = |V' \cap V|$ or $|V' \cap U| + 1 = |V' \cap V|$. This implies that $\mathscr{F}'$ must contain at least $|V' \cap U|$ 1-factors which are subfactors of $\mathscr{A}$-factors (since these are the only 1-factors containing edges joining vertices from U to vertices in X). Since $(V \cap V', \mathscr{F} \cap \mathscr{F}')$ is a sub-OF of $(V, \mathscr{F})$ it follows that the case $|V' \cap U| = |V' \cap V|$ is impossible and so $|V' \cap U| + 1 = |V' \cap V|$. This in turn implies that the number of vertices in $V' \cap X$ is odd, giving $|V' \cap X| \geqslant 3$. Since $(V' \cap (X \cup \{a^*\}), \mathscr{F}' \cap \mathscr{D})$ is also a sub-OF of $(S, \mathscr{H})$, and $GF(K_8)$ does not contain any sub-OF of index 2, it follows that $V' \cap X = X$ and therefore $|V' \cap V| = n - 3$, $|V' \cap U| = n - 4$. Since $\mathscr{B}$-factors are the only 1-factors containing edges joining vertices from X to vertices in $V \backslash \{a^*\}$ and since there are $n - 4$ vertices in $V' \cap (V \backslash \{a^*\})$, there must be $n - 4$ 1-factors in $\mathscr{F}'$ which are subfactors of $\mathscr{B}$-factors, $n-4$ 1-factors which are subfactors of $\mathscr{A}$-factors, and 7 1-factors which are subfactors of $\mathscr{C}$-factors. As there are altogether $\binom{n-4}{2}$ edges joining vertices from $V' \cap U$ and there are $\frac{1}{2}(n-4)(n-11)$ edges joining vertices from $V' \cap U$ in all $n-4$ 1-factors of $\mathscr{F}'$ which are subfactors of $\mathscr{A}$-factors, there are exactly $3(n-4)$ edges joining vertices from $V' \cap U$ in the $n-4$ 1-factors of $\mathscr{F}'$ which are subfactors of the $\mathscr{B}$-factors. This implies that each 1-factor of $\mathscr{F}'$ which is a subfactor of a $\mathscr{B}$-factor contains exactly three edges joining vertices from $V' \cap U$. Among the vertices of $V' \cap U$ there must be two vertices b_x, b_y such that $|x-y| \equiv 1 \pmod{2n-1}$. Without loss of generality, let $x=1$, $y=2$. Then the definition of the set $\mathscr{B}$ implies $b_3, b_5, b_6, b_7 \in V' \cap U$ which in turn implies $V' \cap U = U$ which is a contradiction.

Case 3b. $|V' \cap X| = 1$. Then $|V' \cap U| \geqslant n-1$. Since edges joining vertices from U to vertices in X are contained only in $\mathscr{A}$-factors it follows that $\mathscr{F}'$ contains at least $n-1$ 1-factors which are subfactors of $\mathscr{A}$-factors. On the other hand, $(V \cap V', \mathscr{F} \cap \mathscr{F}')$ is a sub-OF of $(S, \mathscr{H})$ and $|V \cap V'| \leqslant n$; therefore, $\mathscr{F}'$ contains at most $n-1$ 1-factors which are subfactors of $\mathscr{A}$-factors; thus, it contains exactly $n-1$ such 1-factors. It follows that $|V \cap V'| = n$ and $|V' \cap U| = n-1$. The remaining n 1-factors of $\mathscr{F}'$ must be subfactors of $\mathscr{B}$-factors and $\mathscr{C}$-factors. Since the latter do not contain any edges joining vertices from V, it must be that in all these 1-factors of $\mathscr{F}'$ edges join vertices from V to vertices in $V' \cap U$ except for one edge which joins the unique vertex of $V' \cap X$ to a vertex of V. Among the $n-1$ vertices of $V' \cap U$ there must be two vertices b_x, b_y such that $|x-y| \equiv 1, 2$, or $3 \pmod{2n-1}$. But the edge b_x, b_y does not occur in any 1-factor of $\mathscr{F}'$ which contradicts the fact the $(V', \mathscr{F}')$ is an $OF(K_{2n})$.

Case 3c. $V' \cap X = \varnothing$. Then $|V' \cap U| \geqslant |V' \cap V|$. If $|V' \cap U| = |V' \cap V| = n$ then no 1-factor of $\mathscr{F}'$ can contain an edge joining two vertices $b_x, b_y \in V' \cap U$ such that $|x-y| \equiv 1, 2$, or $3 \pmod{2n-1}$. Since $V' \cap U$ must contain two such vertices, this case cannot occur. If $|V' \cap U| > |V' \cap V|$, then no 1-factor of $\mathscr{F}'$ can be a subfactor of a $\mathscr{C}$-factor. If $|V' \cap U| = n+1$, $|V' \cap V| = n-1$ then there must be $n-2$ 1-factors of $\mathscr{F}'$ which are subfactors of $\mathscr{A}$-factors and $n+1$ 1-factors of $\mathscr{F}'$ which are subfactors of $\mathscr{B}$-factors. It follows that each vertex from $V' \cap U$ must be joined by an edge to another vertex from $V' \cap U$ in exactly two 1-factors of $\mathscr{F}'$ which are subfactors of $\mathscr{B}$-factors. That is, for every vertex b_x in $V' \cap U$ there must be exactly two other vertices b_y, b_z in $V' \cap U$ such that $|x-y| \equiv 1$, 2, or 3 $\pmod{2n-1}$ and $|x-z| \equiv 1$, 2, or 3 $\pmod{2n-1}$. As $|V' \cap U| = n+1$ this is evidently impossible. The impossibility of the case $|V' \cap U| = n+2$,

$|V' \cap V| = n - 2$ is shown in a similar fashion, while the assumption $|V' \cap U| = n + 3$, $|V' \cap V| = n - 3$ implies $V' \cap U = U$, a contradiction. Obviously, $|V' \cap U|$ cannot exceed $|V' \cap V|$ by more than 6 which completes this case.

Thus, $(V, \mathscr{F})$ is the unique sub-*OF* of $(S, \mathscr{H})$ of index $2 + 3/n$. Assume now that α is a nontrivial automorphism of $(S, \mathscr{H})$. Then α must map $(V, \mathscr{F})$ onto itself and as $(V, \mathscr{F})$ is automorphism-free, we have $a_i\alpha = a_i$, for all $i = 1, 2,..., 2n - 1$, and $a^*\alpha = a^*$. Therefore each of the 1-factors A_k, $k = 1, 2,..., 2n - 1$, must be fixed under α which is obviously possible only if $b_i\alpha = b_i$ for all $i = 1, 2,..., 2n - 1$ and $\infty_i\alpha = \infty_i$ for $i = 1, 2,..., 7$. This completes the proof of Theorem 15.

COROLLARY 16. $A(4n + 6) \geqslant (2n - 2)!\, A(2n)/42$.

Proof. To obtain $(S, \mathscr{H})$ from $(V, \mathscr{F})$ by the construction in Theorem 15 we have $(2n - 1)!$ choices for the bijection β. Thus we obtain $(2n - 1)!$ distinct AF $OF(K_{4n+6})$ corresponding to a fixed AF $OF(K_{2n})$ (and to a fixed $(A, n - 1)$ or $(B, n - 1)$-system). Further we observe that any automorphism α of $(S, \mathscr{H})$ has to map the set of subfactors $\{B_k' \mid k = 1, 2,..., 2n - 1\}$ onto itself and so we must have $b_i\alpha = b_{i+x}$ for each $i = 1, 2,..., 2n - 1$ and for some $x \in \{1, 2,..., 2n - 1\}$. The proof of the corollary is then completed by taking into account that the order of the automorphism group of $GF(8)$ is 42.

LEMMA 17. *There exists an* AF $OF(K_{2n})$ *for* $n = 6, 7,$ *and* 9.

Proof. Here they are! (For the sake of brevity all brackets are omitted.)

AF *OF*(12):	1,6	2,5	3,4	7,12	8,11	9,10
	2,6	1,3	4,5	9,12	8,10	7,11
	3,6	2,4	1,5	8,12	7,9	10,11
	4,6	3,5	1,2	10,12	9,11	7,8
	5,6	1,4	2,3	11,12	7,10	8,9
	1,7	2,9	3,12	4,8	5,10	6,11
	1,8	2,7	3,10	4,9	5,11	6,12
	1,9	2,11	3,7	4,12	5,8	6,10
	1,10	2,8	3,11	4,7	5,12	6,9
	1,11	2,12	3,9	4,10	5,7	6,8
	1,12	2,10	3,8	4,11	5,9	6,7

AF *OF*(14):	1,2	3,5	4,7	6,13	8,11	9,10	12,14
	2,3	4,6	1,5	7,14	9,12	10,11	8,13
	3,4	5,7	2,6	1,8	10,13	11,12	9,14
	4,5	1,6	3,7	2,9	11,14	12,13	8,10

AF *OF*(14):	5,6	2,7	1,4	3,10	8,12	13,14	9,11
	6,7	1,3	2,5	4,11	9,13	8,14	10,12
	1,7	2,4	3,6	5,12	10,14	8,9	11,13
	1,9	2,8	3,14	4,12	5,10	6,11	7,13
	1,10	2,13	3,8	4,14	5,11	6,9	7,12
	1,11	2,14	3,9	4,8	5,13	6,12	7,10
	1,12	2,10	3,13	4,9	5,8	6,14	7,11
	1,13	2,12	3,11	4,10	5,14	6,8	7,9
	1,14	2,11	3,12	4,13	5,9	6,10	7,8

AF *OF*(18):	1,10	2,6	3,5	4,7	8,9	11,18	12,17	13,16	14,15
	2,11	3,7	4,6	5,8	1,9	10,12	13,18	14,17	15,16
	3,12	4,8	5,7	6,9	1,2	11,13	10,14	15,18	16,17
	4,13	5,9	6,8	1,7	2,3	12,14	11,15	10,16	17,18
	5,14	1,6	7,9	2,8	3,4	13,15	12,16	11,17	10,18
	6,15	2,7	1,8	3,9	4,5	14,16	13,17	12,18	10,11
	7,16	3,8	2,9	1,4	5,6	15,17	14,18	10,13	11,12
	8,17	4,9	1,3	2,5	6,7	16,18	10,15	11,14	12,13
	9,18	1,5	2,4	3,6	7,8	10,17	11,16	12,15	13,14
	1,11	2,10	3,18	4,14	5,17	6,16	7,15	8,12	9,13
	1,18	2,12	3,11	4,15	5,16	6,14	7,17	8,13	9,10
	1,12	2,13	3,10	4,17	5,15	6,18	7,11	8,16	9,14
	1,13	2,14	3,17	4,10	5,18	6,11	7,12	8,15	9,16
	1,14	2,15	3,13	4,16	5,10	6,17	7,18	8,11	9,12
	1,15	2,16	3,14	4,12	5,11	6,10	7,13	8,18	9,17
	1,16	2,17	3,15	4,18	5,12	6,13	7,10	8,14	9,11
	1,17	2,18	3,16	4,11	5,13	6,12	7,14	8,10	9,15

In order to verify that our $OF(K_{2n})$ are automorphism-free we proceed as follows:

Given an $OF(K_{2n})$, the union of any two of its 1-factors is a 2-factor each component of which is an even circuit of length at least four. Therefore to any pair of 1-factors of $OF(K_{2n})$ corresponds a partition of $2n$ into even parts not smaller than 4. If $T_1, T_2, \ldots, T_q$ are all such partitions we may assign to each 1-factor F_i of $OF(K_{2n})$ a t-vector $(t_1{}^i, t_2{}^i, \ldots, t_q{}^i)$ where $t_j{}^i$ is the number of 1-factors G such that to the 2-factor $F_i \cup G$, the partition T_j corresponds, and $\sum_{j=1}^{q} t_j{}^i = 2n - 2$. For instance, for our $OF(K_{12})$ we get the t-vectors in Table I.

Since the types of the 1-factors are invariant under isomorphism it follows that any automorphism of our $OF(K_{12})$ must map each factor F_i onto itself except possibly F_2 and F_3 which could be interchanged. It is then verified rather easily that this can be achieved only by the identity mapping of the vertices.

TABLE I

Factors	Partitions: 12	8 + 4	6 + 6	4 + 4 + 4	Σ
F_1	0	3	4	3	10
F_2	4	2	4	0	10
F_3	4	2	4	0	10
F_4	2	3	4	1	10
F_5	3	2	4	1	10
F_6	4	5	1	0	10
F_7	2	6	1	1	10
F_8	7	1	0	2	10
F_9	7	3	0	0	10
F_{10}	7	2	0	1	10
F_{11}	3	5	1	1	10

TABLE II

Partitions: 14	10 + 4	8 + 6	6 + 4 + 4	Σ
4	7	0	1	12
4	6	1	1	12
0	10	2	0	12
4	7	1	0	12
4	8	0	0	12
4	7	1	0	12
3	8	1	0	12
6	3	2	1	12
7	4	1	0	12
3	6	1	2	12
6	2	3	1	12
6	5	1	0	12
5	5	2	0	12

Similarly, for our $OF(K_{14})$ and $OF(K_{18})$ we obtain the t-vectors in Tables II and III. From these tables it is again easily seen that both 1-factorizations are automorphism-free.

THEOREM 18. *An* AF $OF(K_{2n})$ *exists if and only if* $n \geqslant 5$.

Proof. As already mentioned, there is no AF $OF(K_{2n})$ for $n \leqslant 4$. By [9], there exists an AF $OF(K_{10})$ (actually, there are exactly 298 noniso-

TABLE III

Partitions									
18	14 + 4	12 + 6	10 + 8	10 + 4 + 4	8 + 6 + 4	6 + 6 + 6	6 + 4 + 4 + 4	Σ
11	2	0	1	0	0	2	0	16
8	4	0	2	0	0	2	0	16
8	2	0	2	2	0	2	0	16
8	3	1	1	1	0	2	0	16
11	2	0	0	0	1	2	0	16
9	0	0	3	1	1	2	0	16
9	1	1	1	1	1	2	0	16
11	2	0	1	0	0	2	0	16
7	3	0	3	1	0	2	0	16
6	4	1	1	0	3	0	1	16
4	3	0	4	2	3	0	0	16
7	4	1	2	1	1	0	0	16
5	4	1	3	1	0	1	1	16
7	2	2	5	0	2	0	0	16
5	4	3	0	3	0	1	0	16
3	3	1	7	1	1	0	0	16

morphic AF $OF(K_{10})$), and by Theorem 13, there exists an AF $OF(K_{20})$. Further, an AF $OF(K_{2n})$ for $n = 6$, 7 and 9 exists by Lemma 17 while the existence of an AF $OF(K_{2n})$ for $n = 8$ and 11 follows from Theorem 12. Assume therefore $n \geqslant 12$, and assume that for all $m < n$ ($m \geqslant 5$) there exists an AF $OF(K_{2m})$. If $n \equiv 0 \pmod 2$ then there exists an AF $OF(K_n)$ and by Theorem 13 there exists an AF $OF(K_{2n})$. If $n \equiv 1 \pmod 2$ then there exists an AF $OF(K_{n-3})$ where $n \geqslant 10$ therefore by Theorem 15 there is an AF $OF(K_{2n})$.

THEOREM 19. $\lim_{n\to\infty} A(2n) = \infty$.

Proof. The statement follows from Corollary 14 and Corollary 16.

6. EMBEDDINGS INTO AF STS AND AF OF

In order to prove the two theorems of this section, we need a lemma concerning the following (well-known) construction.

CONSTRUCTION A. Let $(S, \mathscr{B})$ be a Steiner triple system of order v where $S = \{a_1, a_2, ..., a_v\}$ Put $v + 1 = 2n$ and let $(T, \mathscr{F})$, $\mathscr{F} = \{F_i\}_{i\in I_{\mathscr{F}}}$, be an $OF(K_{2n})$ and $S \cap T = \varnothing$. Put $S^* = S \cup T$ and $\mathscr{B}^* = \mathscr{B} \cup \mathscr{C}$

where $\mathscr{C} = \{\{a_i, x, y\} \mid [x, y] \in F_i, i \in I_{\mathscr{F}}\}$. Then $(S^*, \mathscr{B}^*)$ is an STS of order $2v + 1$.

LEMMA 20. *If* $(T, \mathscr{F})$ *is an* AF $OF(K_{2n})$ *which does not contain a sub-*OF *of index* 2 *then the Steiner triple system* $(S^*, \mathscr{B}^*)$ *is also automorphism-free.*

Proof. Assume first that there is a nontrivial automorphism α of $(S^*, \mathscr{B}^*)$ which maps $(S, \mathscr{B})$ onto itself. Then the set of triples $\mathscr{C}$ must be also mapped onto itself by α. Since $(T, \mathscr{F})$ is automorphism-free it follows that $x\alpha = x$ for all $x \in T$. However, any automorphism of $(S^*, \mathscr{B}^*)$ fixing at least half the number of elements of S^* necessarily fixes all elements of S^*.

II. Assume now that there is a nontrivial automorphism α of $(S^*, \mathscr{B}^*)$ which maps $(S, \mathscr{B})$ onto $(S', \mathscr{B}')$ where $(S', \mathscr{B}')$ is another STS of order v. Then $(S \cap S', \mathscr{B} \cap \mathscr{B}')$ must be an STS of order $\frac{1}{2}(v - 1)$ and $(S' \backslash S, \mathscr{H})$ (where $\mathscr{H} = \{H_i\}_{i \in I_{\mathscr{H}}}$, $H_i = \{[x, y] \mid \{a_i, x, y\} \in B' \backslash B$, $a_i \in S \cap S'\})$ is a sub-OF of $(T, \mathscr{F})$ of index 2 (cf. [15, Theorem 3.1]). This contradiction completes the proof.

A *partial* Steiner triple system is a pair $(P, \mathscr{Q})$ where P is a finite set and $\mathscr{Q}$ is a collection of 3-subsets of P such that each pair of distinct elements of P belongs to *at most* one triple of $\mathscr{Q}$.

THEOREM 21. *Every partial Steiner triple system can be embedded into an automorphism-free Steiner triple system.*

Proof. Let $(P, \mathscr{Q})$ be a partial STS. Then $(P, \mathscr{Q})$ can be completed to a finite STS $(S', \mathscr{B}')$ [17]. Let $|S'| = u$. Put

$$\begin{aligned} v &= u && \text{if } u \equiv 1 \text{ or } 9 \pmod{12} \\ &= 2u + 3 && \text{if } u \equiv 3 \pmod{12} \\ &= 2u + 7 && \text{if } u \equiv 7 \pmod{12}. \end{aligned}$$

By [8], $(S', \mathscr{B}')$ can be embedded into an STS of order v, say $(S, \mathscr{B})$. Let $(T, \mathscr{F})$ be an AF $OF(K_{2n})$ where $2n = v + 1$ and $T \cap S = \varnothing$. Since $v \equiv 1$ or $9 \pmod{12}$ we have $n \equiv 1$ or $5 \pmod 6$. But no $OF(K_{2n})$ with n odd can contain a sub-OF of index 2. Therefore using Construction A and Lemma 20, we obtain an STS or order $2v + 1$ $(S^*, \mathscr{B}^*)$ which is automorphism-free with $(P, \mathscr{Q})$ embedded into $(S^*, \mathscr{B}^*)$.

COROLLARY 22. *For every* STS *of order* v $(S, \mathscr{B})$ *there exists an* AF STS$(S^*, \mathscr{B}^*)$ *or order not exceeding* $4v + 15$ *such that* $(S, \mathscr{B})$ *can be embedded into* $(S^*, \mathscr{B}^*)$.

THEOREM 23. *Any* $OF(K_{2s})$ *can be embedded into an* AF $OF(K_{2n})$ *for some* n.

Proof. Let $(V, \mathscr{G})$ be an $OF(K_{2s})$. Put

$$\begin{aligned} t &= s && \text{if } s \equiv 1 \text{ or } 2 \pmod 3 \\ &= 2s + 2 && \text{if } s \equiv 0 \pmod 3). \end{aligned}$$

If $t > s$, then by Theorem 8 $(V, \mathscr{G})$ can be embedded into an $OF(K_{2t})$, say $(W, \mathscr{F})$. (If $t = 2$ we just put $W = V$, and $\mathscr{F} = \mathscr{G}$). Taking $(W, \mathscr{F})$ and any STS of order $2t - 1$, we can use Construction A to obtain an STS of order $4t - 1$, which, in turn, can be embedded by Corollary 22 into an AF STS or order v, say, $(S, \mathscr{B})$. Then the Steiner AF $OF(K_{2n})$ (where $2n = v + 1$) obtained from $(S, \mathscr{B})$ obviously contains $(V, \mathscr{G})$ as a sub OF.

REFERENCES

1. B. A. ANDERSON, Finite topologies and Hamiltonian paths, *J. Combinatorial Theory* **14** (1973), 87–93.
2. B. A. ANDERSON, Symmetry groups of perfect 1-factorizations on some K_{2n}, to appear.
3. R. H. BRUCK, "A Survey of Binary Systems," Springer, Berlin, 1958.
4. R. H. BRUCK, What is a loop? *in* "Studies in Modern Algebra" (Ed. A. A. Albert), pp. 59–99, Prentice-Hall, Englewood Cliffs, N.J., 1963.
5. A. B. CRUSE, On embedding incomplete symmetric Latin squares, *J. Combinatorial Theory Ser. A* **16** (1974), 18–22.
6. J. DÉNES AND A. D. KEEDWELL, "Latin Squares and Their Applications," Academic Press, New York, 1974.
7. L. E. DICKSON AND F. H. SAFFORD, Solution to problem 8 (group theory), *Amer. Math. Monthly* **13** (1906), 150–151.
8. J. DOYEN AND R. M. WILSON, Embeddings of Steiner triple systems, *Discrete Math.* **5** (1973), 229–239.
9. E. N. GELLING, "On 1-Factorizations of the Complete Graph and the Relationship to Round Robin Schedules," M.A. Thesis, University of Victoria, 1973.
10. E. N. GELLING AND R. E. ODEH, On 1-factorizations of the complete graph and the relationship to round robin schedules, *in* "Proceedings of the 3rd Manitoba Conference on Numerical Mathematics," October 1973, pp. 213–221, Utilitas Math. Publ. Inc., Winnipeg, 1974.
11. M. HALL, JR., "Combinatorial Theory," Ginn-Blaisdall, Waltham, Mass., 1967.
12. F. HARARY, "Graph Theory," Addison-Wesley, Reading, Mass., 1969.
13. A. J. W. HILTON, Embedding incomplete latin rectangles and extending the edge colourings of graphs, *J. London Math. Soc.* (2) **12** (1975), 123–128.
14. A. KOTZIG, Groupoids and partitions of complete graphs, *in* "Combinatorial Structures and their Applications," Proc. Conf. Calgary 1969, pp. 215–221, Gordon and Breach, New York, 1970.

15. C. C. LINDNER AND A. ROSA, On the existence of automorphism-free Steiner triple systems, *J. Algebra* **34** (1975), 440–443.
16. A. ROSA, A remark on cyclic Steiner triple systems (in Slovak), *Mat.-Fyz. Časopis* **16** (1966), 238–239.
17. C. A. TREASH, The completion of finite incomplete Steiner triple systems with applications to loop theory, *J. Combinatorial Theory Ser. A* **10** (1971), 259–265.
18. W. D. WALLIS, A. P. STREET, AND J. S. WALLIS, "Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices," Lecture Notes in Math., Vol. 292, Springer, New York, 1972.
19. W. D. WALLIS, On one-factorizations of complete graphs, *J. Austral. Math. Soc.* **16** (1973), 167–171.
20. R. M. WILSON, Nonisomorphic Steiner triple systems, *Math. Zeitschr.* **135** (1974), 303–313.