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Smooth and discontinuous junctions in the p-system

Rinaldo M. Colombo[a,*], Francesca Marcellini[b]

[a] Dipartimento di Matematica, Università degli Studi di Brescia, Via Branze, 38, 25123 Brescia, Italy
[b] Dip. di Matematica e Applicazioni, Università di Milano – Bicocca, Via Cozzi, 53, 20126 Milano, Italy

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ABSTRACT

Consider the p-system describing the subsonic flow of a fluid in a pipe with section $a = a(x)$. We prove that the resulting Cauchy problem generates a Lipschitz semigroup, provided the total variation of the initial datum and the oscillation of a are small. An explicit estimate on the bound of the total variation of a is provided, showing that at lower fluid speeds, higher total variations of a are acceptable. An example shows that the bound on $\mathrm{TV}(a)$ is mandatory, for otherwise the total variation of the solution may grow arbitrarily.

1. Introduction

Consider a gas pipe with smoothly varying section. In the isentropic or isothermal approximation, the dynamics of the fluid in the pipe is described by the following system of Euler equations:

$$\begin{cases} \partial_t(a\rho) + \partial_x(aq) = 0, \\ \partial_t(aq) + \partial_x\left[a\left(\frac{q^2}{\rho} + p(\rho)\right)\right] = p(\rho)\partial_x a, \end{cases} \tag{1.1}$$

where, as usual, ρ is the fluid density, q is the linear momentum density, $p = p(\rho)$ is the pressure and $a = a(x)$ is *cross-sectional area* of the tube. We provide a basic well-posedness result for (1.1), under the assumptions that the initial data is subsonic, has sufficiently small total variation and the oscillation in the pipe section $a = a(x)$ is also small. We provide an explicit bound on the total variation of a. As it is physically reasonable, as the fluid speed increases this bound decreases and vanishes at sonic speed, see (2.14).

As a tool in the study of (1.1) we use the system recently proposed for the case of a sharp discontinuous change in the pipe's section between the values a^- and a^+, see [3,7,8]. This description is based on the p-system

$$\begin{cases} \partial_t\rho + \partial_x q = 0, \\ \partial_t q + \partial_x\left(\frac{q^2}{\rho} + p(\rho)\right) = 0 \end{cases} \tag{1.2}$$

equipped with a *coupling condition* at the junction of the form

$$\Psi\big(a^-, (\rho, q)(t, 0-); a^+, (\rho, q)(t, 0+)\big) = 0 \tag{1.3}$$

whose role is essentially that of selecting stationary solutions.

* Corresponding author.
E-mail addresses: Rinaldo.Colombo@UniBs.it (R.M. Colombo), F.Marcellini@Campus.UniMiB.it (F. Marcellini).

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Remark that the introduction of condition (1.3) is necessary as soon as the section of the pipe is not smooth. The literature offers different choices for this condition, see [3,7,8]. The construction below does not require any specific choice of Ψ in (1.3), but applies to all conditions satisfying minimal physically reasonable requirements, see **(Σ0)**–**(Σ2)**.

On the contrary, if $a \in \mathbf{W^{1,1}}$ the product in the right-hand side of the second equation in (1.1) is well defined and system (1.1) is equivalent to the 2×2 system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x q = -\dfrac{q}{a}\partial_x a, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho)\right) = -\dfrac{q^2}{a\rho}\partial_x a. \end{cases} \tag{1.4}$$

Systems of this type were considered, for instance, in [5,11,13,15,16,18,21]. In this case the stationary solutions to (1.1) are characterized as solutions to

$$\begin{cases} \partial_x \big(a(x) q\big) = 0, \\ \partial_x \left(a(x)\left(\dfrac{q^2}{\rho} + p(\rho)\right)\right) = p(\rho)\partial_x a \end{cases} \quad \text{or} \quad \begin{cases} \partial_x q = -\dfrac{q}{a}\partial_x a, \\ \partial_x \left(\dfrac{q^2}{\rho} + p(\rho)\right) = -\dfrac{q^2}{a\rho}\partial_x a, \end{cases} \tag{1.5}$$

see Lemma 2.6 for a proof of the equivalence between (1.4) and (1.1).

Thus, the case of a smooth a induces a unique choice for condition (1.3), see (2.3) and (2.19). Even with this choice, in the case of the isothermal pressure law $p(\rho) = c^2\rho$, we show below that a shock entering a pipe can have its strength arbitrarily magnified, provided the total variation of the pipe's section is sufficiently high and the fluid speed is sufficiently near to the sound speed, see Section 2.2. Recall, from the physical point of view, that the present situation neglects friction, viscosity and the conservation of energy. Moreover, this example shows the necessity of a bound on the total variation of the pipe section in any well-posedness theorem for (1.1).

The next section is divided into three parts, the former one deals with a pipe with a single junction, the second with a pipe with a piecewise constant section and the latter with a pipe having a $\mathbf{W^{1,1}}$ section. All proofs are gathered in Section 3.

2. Notation and main results

Throughout this paper, u denotes the pair (ρ, q) so that, for instance, $u^\pm = (\rho^\pm, q^\pm)$, $\bar{u} = (\bar{\rho}, \bar{q})$, ... Correspondingly, we denote by $f(u) = (q, P(\rho, q))$ the flow in (1.2). Introduce also the notation $\mathbb{R}^+ = [0, +\infty[$, whereas $\mathring{\mathbb{R}}^+ =]0, +\infty[$. Besides, we let $a(x\pm) = \lim_{\xi \to x\pm} a(\xi)$. Below, $B(u; \delta)$ denotes the open ball centered in u with radius δ.

The pressure law p is assumed to satisfy the following requirement:

(P) $p \in \mathbf{C^2}(\mathbb{R}^+; \mathbb{R}^+)$ is such that for all $\rho > 0$, $p'(\rho) > 0$ and $p''(\rho) \geqslant 0$.

The classical example is the γ-law, where $p(\rho) = k\rho^\gamma$, for a suitable $\gamma \geqslant 1$.

Recall the expressions of the eigenvalues $\lambda_{1,2}$ and eigenvectors $r_{1,2}$ of the p-system, with c denoting the sound speed,

$$\begin{aligned} &\lambda_1(u) = \frac{q}{\rho} - c(\rho), \qquad c(\rho) = \sqrt{p'(\rho)}, \qquad \lambda_2(u) = \frac{q}{\rho} + c(\rho), \\ &r_1(u) = \begin{bmatrix} -1 \\ -\lambda_1(u) \end{bmatrix}, \qquad r_2(u) = \begin{bmatrix} 1 \\ \lambda_2(u) \end{bmatrix}. \end{aligned} \tag{2.1}$$

The *subsonic* region is given by

$$A_0 = \big\{u \in \mathring{\mathbb{R}}^+ \times \mathbb{R}\colon \lambda_1(u) < 0 < \lambda_2(u)\big\}. \tag{2.2}$$

For later use, we recall the quantities

$$\begin{aligned} \text{flow of the linear momentum:} \quad & P(u) = \frac{q^2}{\rho} + p(\rho), \\ \text{total energy density:} \quad & E(u) = \frac{q^2}{2\rho} + \rho \int_{\rho_*}^{\rho} \frac{p(r)}{r^2}\, dr, \\ \text{flow of the total energy density:} \quad & F(u) = \frac{q}{\rho} \cdot \big(E(u) + p(\rho)\big), \end{aligned}$$

where $\rho_* > 0$ is a suitable fixed constant. As it is well known, see [10, formula (3.3.21)], the pair (E, F) plays the role of the (mathematical) entropy–entropy flux pair.

2.1. A pipe with a single junction

This section is devoted to (1.2)–(1.3). Fix the section $\bar a > \Delta$, with $\Delta > 0$ and the state $\bar u \in A_0$.

First, introduce a function $\Sigma = \Sigma(a^-, a^+; u^-)$ that describes the effects of the junction when the section changes from a^- to a^+ and the state to the left of the junction is u^-. We specify the choice of (1.3) writing

$$\Psi\big(a^-, u^-; a^+, u^+\big) = \begin{bmatrix} a^+q^+ - a^-q^- \\ a^+P(u^+) - a^-P(u^-) \end{bmatrix} - \Sigma\big(a^-, a^+; u^-\big). \tag{2.3}$$

We pose the following assumptions on Σ:

($\boldsymbol{\Sigma}$0) $\Sigma \in \mathbf{C^1}([\bar a - \Delta, \bar a + \Delta] \times B(\bar u; \delta); \mathbb{R}^2)$.
($\boldsymbol{\Sigma}$1) $\Sigma(a, a; u^-) = 0$ for all $a \in [\bar a - \Delta, \bar a + \Delta]$ and all $u^- \in B(\bar u; \delta)$.

Condition **($\boldsymbol{\Sigma}$0)** is a natural regularity condition. Condition **($\boldsymbol{\Sigma}$1)** is aimed to comprehend the standard *"no junction"* situation: if $a^- = a^+$, then the junction has no effects and Σ vanishes.

Conditions **($\boldsymbol{\Sigma}$0)**–**($\boldsymbol{\Sigma}$1)** ensure the existence of stationary solutions to problem (1.2)–(1.3).

Lemma 2.1. *Let* **($\boldsymbol{\Sigma}$0)**–**($\boldsymbol{\Sigma}$1)** *hold. Then, for any $\bar a \in \mathring{\mathbb{R}}^+$, $\bar u \in A_0$, there exist a positive $\bar\delta$ and a Lipschitz map*

$$T :]\bar a - \bar\delta, \bar a + \bar\delta[\times]\bar a - \bar\delta, \bar a + \bar\delta[\times B(\bar u; \bar\delta) \to A_0 \tag{2.4}$$

such that

$$\left.\begin{array}{l} \Psi\big(a^-, u^-; a^+, u^+\big) = 0 \\ a^- \in]\bar a - \bar\delta, \bar a + \bar\delta[\\ a^+ \in]\bar a - \bar\delta, \bar a + \bar\delta[\\ u^-, u^+ \in B(\bar u; \bar\delta) \end{array}\right\} \iff u^+ = T\big(a^-, a^+; u^-\big).$$

In particular, $T(\bar a, \bar a, \bar u) = \bar u$. We may now state a final requirement on Σ:

($\boldsymbol{\Sigma}$2) $\Sigma(a^-, a^0; u^-) + \Sigma(a^0, a^+; T(a^-, a^0; u^-)) = \Sigma(a^-, a^+; u^-)$.

Here T is the map defined in Lemma 2.1. Alternatively, by (2.3), the above condition **($\boldsymbol{\Sigma}$2)** can be restated as

$$\left.\begin{array}{l} \Psi\big(a^-, u^-; a^0, u^0\big) = 0 \\ \Psi\big(a^0, u^0; a^+, u^+\big) = 0 \end{array}\right\} \implies \Psi\big(a^-, u^-; a^+, u^+\big) = 0.$$

Condition **($\boldsymbol{\Sigma}$2)** says that if the two Riemann problems with initial states (a^-, u^-), (a^0, u^0) and (a^0, u^0), (a^+, u^+) both yield the stationary solution, then also the Riemann problem with initial state (a^-, u^-) and (a^+, u^+) is solved by the stationary solution.

Remark that the *"natural"* choice (2.19) implied by a smooth section satisfies **($\boldsymbol{\Sigma}$0)**, **($\boldsymbol{\Sigma}$1)** and **($\boldsymbol{\Sigma}$2)**.

Denote now by $\hat u$ a map satisfying

$$\hat u(x) = \begin{cases} \hat u^- & \text{if } x < 0, \\ \hat u^+ & \text{if } x > 0 \end{cases} \quad \text{with} \quad \begin{array}{l} \Psi(a^-, \hat u^-; a^+, \hat u^+) = 0, \\ \hat u^-, \hat u^+ \in A_0. \end{array} \tag{2.5}$$

The existence of such a map follows from Lemma 2.1. Recall first the definition of weak Ψ-solution, see [7, Definition 2.1] and [8, Definition 2.1].

Definition 2.2. Let Σ satisfy **($\boldsymbol{\Sigma}$0)**–**($\boldsymbol{\Sigma}$2)**. A weak Ψ-solution to (1.2)–(1.3) is a map

$$\begin{aligned} &u \in \mathbf{C^0}\big(\mathbb{R}^+; \hat u + \mathbf{L^1}\big(\mathbb{R}^+; \mathring{\mathbb{R}}^+ \times \mathbb{R}\big)\big), \\ &u(t) \in \mathbf{BV}\big(\mathbb{R}; \mathring{\mathbb{R}}^+ \times \mathbb{R}\big) \quad \text{for a.e. } t \in \mathbb{R}^+, \end{aligned} \tag{2.6}$$

such that

(W) for all $\varphi \in \mathbf{C^1_c}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R})$ whose support does not intersect $x = 0$

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \big(u\,\partial_t\varphi + f(u)\,\partial_x\varphi\big)\,dx\,dt = 0;$$

(Ψ) for a.e. $t \in \mathbb{R}^+$ and with Ψ as in (2.3), the junction condition is met:

$$\Psi\bigl(a^-, u(t,0-); a^+, u(t,0+)\bigr) = 0.$$

It is also an entropy solution if

(E) for all $\varphi \in \mathbf{C_c^1}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R}^+)$ whose support does not intersect $x = 0$

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \bigl(E(u)\partial_t \varphi + F(u)\partial_x \varphi\bigr)\, dx\, dt \geqslant 0.$$

In the particular case of a Riemann Problem, i.e. of (1.1) with initial datum

$$u(0,x) = \begin{cases} u^- & \text{if } x > 0, \\ u^+ & \text{if } x < 0, \end{cases}$$

Definition 2.2 reduces to [8, Definition 2.1].

To state the uniqueness property in the theorems below, we need to introduce the following integral conditions, following [4, Theorem 9.2], see also [14, Theorem 8] and [1]. Given a function $u = u(t,x)$ and a point (τ, ξ), we denote by $U^\sharp_{(u;\tau,\xi)}$ the solution of the homogeneous Riemann Problem consisting of (1.2)–(1.3)–(2.3) with initial datum at time τ

$$w(\tau,x) = \begin{cases} \lim_{x\to\xi-} u(\tau,x) & \text{if } x < \xi, \\ \lim_{x\to\xi+} u(\tau,x) & \text{if } x > \xi \end{cases} \tag{2.7}$$

and with Σ satisfying **(Σ0)**, **(Σ1)** and **(Σ2)**. Moreover, define $U^\flat_{(u;\tau,\xi)}$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$\begin{cases} \partial_t \omega + \partial_x \tilde{A}\omega = 0, & t \geqslant \tau, \\ \omega(\tau,x) = u(\tau,x), \end{cases} \tag{2.8}$$

with $\tilde{A} = Df(u(\tau,\xi))$.

The next theorem applies [8, Theorem 3.2] to (1.2) with the choice (2.3) to construct the semigroup generated by (1.2)–(1.3)–(2.3). The uniqueness part follows from [14, Theorem 2].

Theorem 2.3. *Let p satisfy* **(P)** *and* Σ *satisfy* **(Σ0)**–**(Σ2)**. *Choose any* $\bar{a} > 0$, $\bar{u} \in A_0$. *Then, there exists a positive* Δ *such that for all* a^-, a^+ *with* $|a^- - \bar{a}| < \Delta$ *and* $|a^+ - \bar{a}| < \Delta$, *there exist a map* $\hat{u}$ *as in* (2.5), *positive* δ, L *and a semigroup* $S : \mathbb{R}^+ \times \mathcal{D} \to \mathcal{D}$ *such that*

(1) $\mathcal{D} \supseteq \{u \in \hat{u} + \mathbf{L^1}(\mathbb{R}; A_0)\colon \mathrm{TV}(u - \hat{u}) < \delta\}$.
(2) *For all* $u \in \mathcal{D}$, $S_0 u = u$ *and for all* $t, s \geqslant 0$, $S_t S_s u = S_{s+t} u$.
(3) *For all* $u, u' \in \mathcal{D}$ *and for all* $t, t' \geqslant 0$,

$$\|S_t u - S_{t'} u'\|_{\mathbf{L^1}} \leqslant L \cdot \bigl(\|u - u'\|_{\mathbf{L^1}} + |t - t'|\bigr).$$

(4) *If* $u \in \mathcal{D}$ *is piecewise constant, then for t small,* $S_t u$ *is the gluing of solutions to Riemann problems at the points of jump in u and at the junction at* $x = 0$.
(5) *For all* $u \in \mathcal{D}$, *the orbit* $t \to S_t u$ *is a weak* Ψ*-solution to* (1.2).
(6) *Let* $\hat{\lambda}$ *be an upper bound for the moduli of the characteristic speeds in* $\bar{B}(\hat{u}(\mathbb{R}), \delta)$. *For all* $u \in \mathcal{D}$, *the orbit* $u(t) = S_t u$ *satisfies the integral conditions*:
  (i) *For all* $\tau > 0$ *and* $\xi \in \mathbb{R}$,

$$\lim_{h\to 0} \frac{1}{h} \int_{\xi - h\hat{\lambda}}^{\xi + h\hat{\lambda}} \bigl\|u(\tau + h, x) - U^\sharp_{(u;\tau,\xi)}(\tau + h, x)\bigr\|\, dx = 0. \tag{2.9}$$

  (ii) *There exists a* $C > 0$ *such that for all* $\tau > 0$, $a, b \in \mathbb{R}$ *and* $\xi \in {]a, b[}$,

$$\frac{1}{h} \int_{a + h\hat{\lambda}}^{b - h\hat{\lambda}} \bigl\|u(\tau + h, x) - U^\flat_{(u;\tau,\xi)}(\tau + h, x)\bigr\|\, dx \leqslant C\bigl[\mathrm{TV}\bigl\{u(\tau); {]a, b[}\bigr\}\bigr]^2. \tag{2.10}$$

(7) *If a Lipschitz map* $w : \mathbb{R} \to \mathcal{D}$ *satisfies* (2.9)–(2.10), *then it coincides with the semigroup orbit*: $w(t) = S_t(w(0))$.

The proof is deferred to Section 3.1. Note that, similarly to what happens in the standard case of [4, Theorem 9.2], condition (2.10) is always satisfied at a junction.

2.2. A pipe with piecewise constant section

We consider now a tube with piecewise constant section

$$a = a_0 \chi_{]-\infty,x_1]} + \sum_{j=1}^{n-1} a_j \chi_{[x_j,x_{j+1}[} + a_n \chi_{[x_n,+\infty[}$$

for a suitable $n \in \mathbb{N}$. The fluid in each pipe is modeled by (1.2). At each junction x_j, we require condition (1.3), namely

$$\Psi\left(a_{j-1}, u_j^-; a_j, u_j^+\right) = 0 \quad \text{for all } j = 1, \dots, n, \text{ where } u_j^{\pm} = \lim_{x \to x_j \pm} u_j(x). \tag{2.11}$$

We omit the formal definition of Ψ-solution to (1.2)–(1.3) in the present case, since it is an obvious iteration of Definition 2.2.

Theorem 2.4. *Let p satisfy* **(P)** *and Σ satisfy* **(Σ0)**–**(Σ2)**. *For any $\bar a > 0$ and any $\bar u \in A_0$ there exist positive $M, \Delta, \delta, L, \mathcal{M}$ such that for any profile satisfying*

(A0) $a \in \mathbf{PC}(\mathbb{R};]\bar a - \Delta, \bar a + \Delta[)$ *with* $\mathrm{TV}(a) < M$,

there exist a piecewise constant stationary solution

$$\hat u = \hat u_0 \chi_{]-\infty,x_1[} + \sum_{j=1}^{n-1} \hat u_j \chi_{]x_j,x_{j+1}[} + \hat u_n \chi_{]x_n,+\infty[}$$

to (1.2)–(2.11) *satisfying*

$$\begin{aligned} &\hat u_j \in A_0 \quad \text{with } |\hat u_j - \bar u| < \delta \text{ for } j = 0, \dots n, \\ &\Psi(a_{j-1}, \hat u_{j-1}; a_j, \hat u_j) = 0 \quad \text{for } j = 1, \dots, n, \\ &\mathrm{TV}(\hat u) \leqslant \mathcal{M}\,\mathrm{TV}(a) \end{aligned} \tag{2.12}$$

and a semigroup $S^a : \mathbb{R}^+ \times \mathcal{D}^a \to \mathcal{D}^a$ such that

(1) $\mathcal{D}^a \supseteq \{u \in \hat u + \mathbf{L^1}(\mathbb{R}; A_0)\colon \mathrm{TV}(u - \hat u) < \delta\}$.
(2) *S_0^a is the identity and for all $t, s \geqslant 0$, $S_t^a S_s^a = S_{s+t}^a$.*
(3) *For all $u, u' \in \mathcal{D}^a$ and for all $t, t' \geqslant 0$,*

$$\left\| S_t^a u - S_{t'}^a u' \right\|_{\mathbf{L^1}} \leqslant L \cdot \left(\left\| (u) - u' \right\|_{\mathbf{L^1}} + |t - t'| \right).$$

(4) *If $u \in \mathcal{D}^a$ is piecewise constant, then for t small, $S_t u$ is the gluing of solutions to Riemann problems at the points of jump in u and at each junction x_j.*
(5) *For all $u \in \mathcal{D}^a$, the orbit $t \to S_t^a u$ is a weak Ψ-solution to* (1.2)–(2.11).
(6) *The semigroup satisfies the integral conditions* (2.9)–(2.10) *in* (6) *of Theorem* 2.3.
(7) *If a Lipschitz map $w : \mathbb{R} \to \mathcal{D}$ satisfies* (2.9)–(2.10), *then it coincides with the semigroup orbit*: $w(t) = S_t(w(0))$.

Remark that δ and L depend on a only through $\bar a$ and $\mathrm{TV}(a)$. In particular, all the construction above is independent from the number of points of jump in a. For every $\bar u$, we provide below an estimate of M at the leading order in δ and Δ, see (3.11) and (3.8). In the case of Σ as in (2.19) and with the isothermal pressure law, which obviously satisfies **(P)**,

$$p(\rho) = c^2 \rho, \tag{2.13}$$

the bounds (3.11) and (3.8) reduce to the simpler estimate

$$M = \begin{cases} \frac{\bar a}{4e} & \text{if } \bar v / c \in]0, 1/\sqrt{2}], \\ \frac{\bar a}{4e} \frac{1 - (\bar v/c)^2}{(\bar v/c)^2} & \text{if } \bar v / c \in]1/\sqrt{2}, 1[, \end{cases} \tag{2.14}$$

where $\bar v = \bar q / \bar \rho$. Note that, as it is physically reasonable, M is a weakly decreasing function of $\bar v$, so that at lower fluid speeds, higher values for the total variation of the pipe's section can be accepted.

Furthermore, the estimates proved in Section 3.2 show that the total variation of the solution to (1.2)–(2.11) may grow unboundedly if $\mathrm{TV}(a)$ is large. Consider the case in Fig. 1. A wave σ_2^- hits a junction where the pipe's section increases by

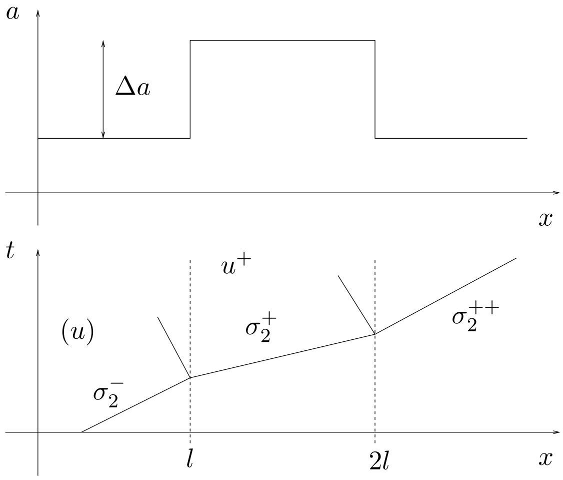


Fig. 1. A wave σ_2^- hits a junction, giving rise to σ_2^+ which hits a second junction.

$\Delta a > 0$. From this interaction, the wave σ_2^+ of the second family arises, which hits the second junction where the section diminishes by Δa. At the leading term in Δa, we have the estimate

$$\left|\sigma_2^{++}\right| \leqslant \left(1 + \mathcal{K}(\bar v/c)\left(\frac{\Delta a}{a}\right)^2\right)\left|\sigma_2^-\right|, \quad \text{where} \tag{2.15}$$

$$\mathcal{K}(\xi) = \frac{-1 + 8\xi^2 - 7\xi^4 + 2\xi^6}{2(1-\xi)^3(1+\xi)^3}, \tag{2.16}$$

see Section 3.2 for the proof. Note that $\mathcal{K}(0) = -1$ whereas $\lim_{\xi \to 1-} \mathcal{K}(\xi) = +\infty$. Therefore, for any fixed Δa, if $\bar v$ is sufficiently near to c, repeating the interactions in Fig. 1 a sufficient number of times makes the 2 shock waves arbitrarily large.

2.3. *A pipe with a* $\mathbf{W^{1,1}}$ *section*

In this section, the pipe's section a is assumed to satisfy

$$\textbf{(A1)} \quad \begin{cases} a \in \mathbf{W^{1,1}}\big(\mathbb{R};]\bar a - \Delta, \bar a + \Delta[\big) & \text{for suitable } \Delta > 0,\ \bar a > \Delta, \\ \mathrm{TV}(a) < M & \text{for a suitable } M > 0, \\ a'(x) = 0 \text{ for a.e. } x \in \mathbb{R} \setminus [-X, X] & \text{for a suitable } X > 0. \end{cases}$$

For smooth solutions, the equivalence of (1.1) and (1.4) is immediate. Note that the latter is in the standard form of a 1D conservation law and the usual definition of weak entropy solution applies, see for instance [19, Definition 3.5.1] or [9, Section 6]. The definition below of weak entropy solution to (1.1) makes the two systems fully equivalent also for non-smooth solutions.

Definition 2.5. A weak solution to (1.1) is a map

$$u \in \mathbf{C^0}\big(\mathbb{R}^+; \hat u + \mathbf{L^1}\big(\mathbb{R}; \mathring{\mathbb{R}}^+ \times \mathbb{R}\big)\big)$$

such that for all $\varphi \in \mathbf{C^1_c}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R})$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a\rho \\ aq \end{bmatrix}\partial_t\varphi + \begin{bmatrix} aq \\ aP(u) \end{bmatrix}\partial_x\varphi + \begin{bmatrix} \\ p(\rho)\partial_x a \end{bmatrix}\varphi\right) dx\,dt = 0. \tag{2.17}$$

u is an entropy weak solution if, for any $\varphi \in \mathbf{C^1_c}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R})$, $\varphi \geqslant 0$,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \big(aE(u)\partial_t\varphi + aF(u)\partial_x\varphi\big)\, dx\,dt \geqslant 0. \tag{2.18}$$

Lemma 2.6. *Let a satisfy* **(A1)***. Then, u is a weak entropy solution to* (1.1) *in the sense of Definition* 2.5*, if and only if it is a weak entropy solution of* (1.4)*.*

The proof is deferred to Section 3.3.

Now, the section a of the pipe is sufficiently regular to select stationary solutions as solutions to either of the systems (1.5), which are equivalent by Lemma 2.6. Hence, the smoothness of a also singles out a specific choice of Σ, see [14, formula (14)].

Proposition 2.7. *Fix $a^-, a^+ \in]\bar a - \Delta, \bar a + \Delta[$ and $u^- \in A_0$. Choose a function a strictly monotone, in $\mathbf{C^1}$, that satisfies* **(A1)** *with $a(-X-) = a^-$ and $a(X+) = a^+$. Call $\rho = R^a(x; u^-)$ the ρ-component of the corresponding solution to either of the Cauchy problems* (1.5) *with initial condition $u(-X) = u^-$. Then,*

(1) *the function*

$$\Sigma\left(a^-, a^+, u^-\right) = \begin{bmatrix} 0 \\ \int_{-X}^{X} p(R^a(x; u^-))a'(x)\,dx \end{bmatrix} \tag{2.19}$$

satisfies **($\boldsymbol{\Sigma}$0)**–**($\boldsymbol{\Sigma}$2)**;

(2) *if $\tilde a$ is a strictly monotone function satisfying the same requirements above for a, the corresponding map $\tilde\Sigma$ coincides with Σ.*

The basic well-posedness theorem in the present $\mathbf{W^{1,1}}$ case is stated similarly to Theorem 2.4.

Theorem 2.8. *Let p satisfy* **(P)**. *For any $\bar a > 0$ and any $\bar u \in A_0$ there exist positive M, Δ, δ, L such that for any profile a satisfying* **(A1)** *there exist a stationary solution $\hat u$ to* (1.1) *satisfying*

$$\hat u \in A_0 \quad \text{with } \left\|\hat u(x) - \bar u\right\| < \delta \text{ for all } x \in \mathbb{R}$$

and a semigroup $S^a : \mathbb{R}^+ \times \mathcal{D}^a \to \mathcal{D}^a$ such that

(1) $\mathcal{D}^a \supseteq \{u \in \hat u + \mathbf{L^1}(\mathbb{R}; A_0)\colon \mathrm{TV}(u - \hat u) < \delta\}$.

(2) *S_0^a is the identity and for all $t, s \geqslant 0$, $S_t^a S_s^a = S_{s+t}^a$.*

(3) *For all $u, u' \in \mathcal{D}^a$ and for all $t, t' \geqslant 0$,*

$$\left\|S_t^a u - S_{t'}^a u'\right\|_{\mathbf{L^1}} \leqslant L \cdot \left(\|u - u'\|_{\mathbf{L^1}} + |t - t'|\right).$$

(4) *For all $u \in \mathcal{D}^a$, the orbit $t \to S_t^a u$ is solution to* (1.2) *in the sense of Definition* 2.5.

(5) *Let $\hat\lambda$ be an upper bound for the moduli of the characteristic speeds in $\bar B(\hat u(\mathbb{R}), \delta)$. For all $u \in \mathcal{D}$, the orbit $u(t) = S_t u$ satisfies the integral conditions*:

 (i) *For all $\tau < 0$ and $\xi \in \mathbb{R}$,*

$$\lim_{h \to 0} \frac{1}{h} \int\limits_{\xi - h\hat\lambda}^{\xi + h\hat\lambda} \left\|u(\tau + h, x) - U^\sharp_{(u;\tau,\xi)}(\tau + h, x)\right\| dx = 0. \tag{2.20}$$

 (ii) *There exists a $C > 0$ such that, for all $\tau > 0$, $a, b \in \mathbb{R}$ and $\xi \in]a, b[$,*

$$\frac{1}{h} \int\limits_{a + h\hat\lambda}^{b - h\hat\lambda} \left\|u(\tau + h, x) - U^\flat_{(u;\tau,\xi)}(\tau + h, x)\right\| dx \leqslant C\left[\mathrm{TV}\left\{u(\tau);]a, b[\right\} + \mathrm{TV}\left\{a;]a, b[\right\}\right]^2. \tag{2.21}$$

(6) *If a Lipschitz map $w : \mathbb{R} \to \mathcal{D}$ solves* (1.1), *then it coincides with the semigroup orbit*: $w(t) = S_t(w(0))$.

Thanks to Theorem 2.4, the proof is obtained approximating a with a piecewise constant function a_n. The corresponding problems (1.2)–(2.11) generate semigroups defined on domains characterized by uniform bounds on the total variation and with a uniformly bounded Lipschitz constants for their time dependence. Then, we pass to the limit (see Section 3.3 for the proof) and we follow the same procedure as in [4, Theorem 9.2] and [14, Theorems 2 and 8] to characterize the solution.

As a byproduct of the proof of Theorem 2.8, we also obtain the following convergence result, relating the construction in Theorem 2.4 to that of Theorem 2.8.

Proposition 2.9. *Under the same assumptions of Theorem* 2.8, *for every $n \in \mathbb{N}$, choose a function β_n such that*:

(i) *β_n is piecewise constant with points of jump $y_n^1, \ldots, y_n^{m_n}$, with $y_n^1 = -X$, $y_n^{m_n} = X$, and $\max_j(y_n^{j+1} - y_n^j) \leqslant 1/n$.*

(ii) *$\beta_n(x) = 0$ for all $x \in \mathbb{R} \setminus [-X, X]$.*

(iii) *$\beta_n \to a'$ in $\mathbf{L^1}(\mathbb{R}; \mathbb{R})$ with $\|\beta_n\|_{\mathbf{L^1}} \leqslant M$, with M as in Theorem* 2.8.

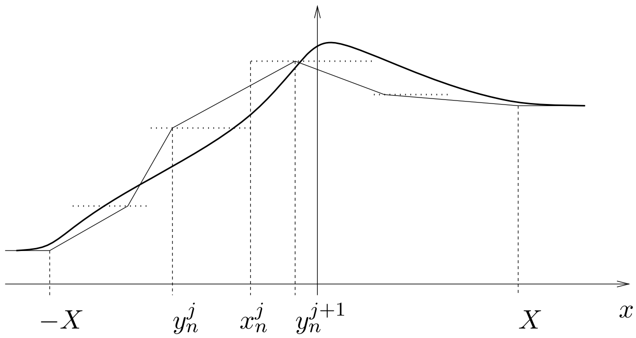


Fig. 2. The thick line is the graph of $a = a(x)$, the dotted line represents a_n while the polygonal line is α_n.

Define $\alpha_n(x) = a(-X-) + \int_{-X}^{x} \beta_n(\xi)\, d\xi$ *and points* $x_n^j \in\,]y_n^j, y_n^{j+1}[$ *for* $j = 1, \ldots, m_n - 1$ *and let*

$$a_n = a(-X-)\chi_{]-\infty, x_j^1[} + \sum_{j=1}^{m_n-1} \alpha_n\bigl(y_n^{j+1}\bigr)\chi_{[x_n^j, x_n^{j+1}[} + a(X+)\chi_{[x_n^{m_n}, +\infty[}$$

(*see Fig.* 2). *Then,* a_n *satisfies* **(A0)** *and the corresponding semigroup* S^n *constructed in Theorem* 2.4 *converges pointwise to the semigroup* S *constructed in Theorem* 2.8.

3. Technical proofs

3.1. Proofs related to Section 2.1

The following equalities will be of use below:

$$\partial_\rho P = -\lambda_1\lambda_2 \quad \text{and} \quad \partial_q P = \lambda_1 + \lambda_2. \tag{3.1}$$

Proof of Lemma 2.1. Apply the Implicit Function Theorem to the equality $\Psi = 0$ in a neighborhood of $(\bar a, \bar u, \bar a, \bar u)$, which satisfies $\Psi = 0$ by **(Σ1)**. Observe that $\partial_u \Sigma(a, a; u^-) = 0$ by **(Σ1)**. Using (3.1), compute

$$\det \partial_{u^+}\Psi(\bar a, \bar u, \bar a, \bar u) = \det\begin{bmatrix} -\partial_{\rho^+}\Sigma_1 & \bar a \\ \bar a\partial_{\rho^+}P & \bar a\partial_{q^+}P \end{bmatrix} = \det\begin{bmatrix} 0 & \bar a \\ \bar a\partial_{\rho^+}P & \bar a\partial_{q^+}P \end{bmatrix} = \bar a^2\lambda_1(\bar u)\lambda_2(\bar u) \neq 0,$$

completing the proof. □

Proof of Theorem 2.3. Let Δ be defined as in Lemma 2.1. Assumption **(F)** in [8, Theorem 3.2] follows from **(P)**, thanks to (2.1) and to the choices (2.2)–(2.5). We now verify condition [8, formula (2.2)]. Recall that $D_{u^-}\Sigma(\bar a, \bar a; \bar u) = 0$ by **(Σ1)**. Hence, using (3.1),

$$\begin{aligned}
&\det\bigl[D_{u^-}\Psi(\bar a, \bar u; \bar a, \bar u)\cdot r_1(\bar u) \quad D_{u^+}\Psi(\bar a, \bar u; \bar a, \bar u)\cdot r_2(\bar u)\bigr] \\
&\quad = \det\begin{bmatrix} \bar a\lambda_1(\bar u) + \partial_{\rho^-}\Sigma_1(\bar a, \bar a; \bar u) + \lambda_1\partial_{q^-}\Sigma_1(\bar a, \bar a; \bar u) & \bar a\lambda_2(\bar u) \\ \bar a(\lambda_1(\bar u))^2 + \partial_{\rho^-}\Sigma_2(\bar a, \bar a; \bar u) + \lambda_1^-\partial_{q^-}\Sigma_2(\bar a, \bar a; \bar u) & \bar a(\lambda_2(\bar u))^2 \end{bmatrix} \\
&\quad = \det\begin{bmatrix} \bar a\lambda_1(\bar u) & \bar a\lambda_2(\bar u) \\ \bar a(\lambda_1(\bar u))^2 & \bar a(\lambda_2(\bar u))^2 \end{bmatrix} \\
&\quad = \bar a^2\lambda_1(\bar u)\lambda_2(\bar u)\bigl(\lambda_2(\bar u) - \lambda_1(\bar u)\bigr) \\
&\quad \neq 0.
\end{aligned}$$

The proof of (1)–(5) is completed applying [8, Theorem 3.2]. The obtained semigroup coincides with that constructed in [14, Theorem 2], where the uniqueness conditions (6) and (7) are proved. □

3.2. Proofs related to Section 2.2

We now work towards the proof of Theorem 2.4. We first use the wave front tracking technique to construct approximate solutions to the Cauchy problem (1.2)–(2.11) adapting the wave front tracking technique introduced in [4, Chapter 7].

Fix an initial datum $u_o \in \hat{u} + \mathbf{L^1}(\mathbb{R}; A_0)$ and an $\varepsilon > 0$. Approximate u_o with a piecewise constant initial datum u_o^ε having a finite number of discontinuities and so that $\lim_{\varepsilon\to 0} \|u_o^\varepsilon - u_o\|_{\mathbf{L^1}} = 0$. Then, at each junction and at each point of jump in u_o^ε along the pipe, we solve the corresponding Riemann Problem according to Definition 2.2. If the total variation of the initial datum is sufficiently small, then Theorem 2.3 ensures the existence and uniqueness of solutions to each Riemann Problem. We approximate each rarefaction wave with a rarefaction fan, i.e. by means of (non-entropic) shock waves traveling at the characteristic speed of the state to the right of the shock and with size at most ε.

This construction can be extended up to the first time $\bar{t}_1$ at which two waves interact in a pipe or a wave hits the junction. At time $\bar{t}_1$ the functions so constructed are piecewise constant with a finite number of discontinuities. At any subsequent interaction or collision with the junction, we repeat the previous construction with the following provisions:

(1) no more than 2 waves interact at the same point or at the junction;
(2) a rarefaction fan of the i-th family produced by the interaction between an i-th rarefaction and any other wave, is *not* split any further;
(3) when the product of the strengths of two interacting waves falls below a threshold $\check{\varepsilon}$, then we let the waves cross each other, their size being unaltered, and introduce a *non-physical* wave with speed $\hat{\lambda}$, with $\hat{\lambda} > \sup_{(u)} \lambda_2(u)$; see [4, Chapter 7] and the refinement [2].

We complete the above algorithm stating how Riemann Problems at the junctions are solved. We use the same rules as in [7, §4.2] and [8, §5]. In particular, at time $t = 0$ and whenever a physical wave with size greater than $\check{\varepsilon}$ hits the junction, the accurate solver is used, i.e. the exact solution is approximated replacing rarefaction waves with rarefaction fans. When a non-physical wave hits the junction, then we let it be refracted into a non-physical wave with the same speed $\hat{\lambda}$ and no other wave is produced.

Repeating recursively this procedure, we construct a wave front tracking sequence of approximate solutions u_ε in the sense of [4, Definition 7.1].

At interactions of waves in a pipe, we have the following classical result.

Lemma 3.1. *Consider interactions in a pipe. Then, there exists a positive K with the properties*:

(1) *An interaction between the wave σ_1^- of the first family and σ_2^- of the second family produces the waves σ_1^+ and σ_2^+ with*

$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| \leqslant K \cdot \left|\sigma_1^- \sigma_2^-\right|. \tag{3.2}$$

(2) *An interaction between σ_i' and σ_i'' both of the same i-th family produces waves of total size σ_1^+ and σ_2^+ with*

$$\begin{aligned} &\left|\sigma_1^+ - (\sigma_1'' + \sigma_1')\right| + \left|\sigma_2^+\right| \leqslant K \cdot \left|\sigma_1' \sigma_1''\right| \quad \text{if } i = 1, \\ &\left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma_2'' + \sigma_2')\right| \leqslant K \cdot \left|\sigma_2' \sigma_2''\right| \quad \text{if } i = 2. \end{aligned}$$

(3) *An interaction between the physical waves σ_1^- and σ_2^- produces a non-physical wave σ_3^+, then*

$$\left|\sigma_3^+\right| \leqslant K \cdot \left|\sigma_1^- \sigma_2^-\right|.$$

(4) *An interaction between a physical wave σ and a non-physical wave σ_3^- produces a physical wave σ and a non-physical wave σ_3^+, then*

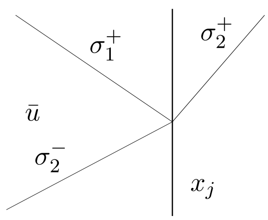
$$\left|\sigma_3^+\right| - \left|\sigma_3^-\right| \leqslant K \cdot \left|\sigma \sigma_3^-\right|.$$

For a proof of this result see [4, Chapter 7]. Differently from the constructions in [7,8], we now cannot avoid the interaction of non-physical waves with junctions. Moreover, the estimates found therein do not allow to pass to the limit $n \to +\infty$, n being the number of junctions.

Lemma 3.2. *Consider interactions at the junction sited at x_j. There exist positive K_1, K_2, K_3 with the following properties*:

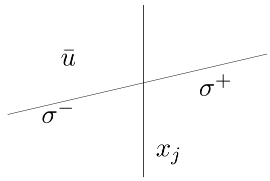
(1) *The wave σ_2^- hits the junction. The resulting waves σ_1^+, σ_2^+ satisfy*

$$\begin{aligned} \left|\sigma_1^+\right| &\leqslant K_1 |a_j - a_{j-1}| \left|\sigma_2^-\right|, \\ \left|\sigma_2^+\right| &\leqslant \left(1 + K_2 |a_j - a_{j-1}|\right) \left|\sigma_2^-\right| \\ &\leqslant e^{K_2 |a_j - a_{j-1}|} \left|\sigma_2^-\right|. \end{aligned}$$



(2) *The non-physical wave σ^- hits the junction. The resulting wave σ^+ satisfies*

$$\begin{aligned}\left|\sigma^+\right| &\leqslant \left(1+K_3|a_j-a_{j-1}|\right)\left|\sigma^-\right| \\ &\leqslant e^{K_3|a_j-a_{j-1}|}\left|\sigma^-\right|.\end{aligned}$$



Proof. Use the notation in the figure above. Recall that σ_1^+ and σ_2^+ are computed through the Implicit Function Theorem applied to a suitable combination of the Lax curves of (1.2), see [7, Proposition 2.4] and [8, Proposition 2.2]. Repeating the proof of Theorem 2.3 one shows that the Implicit Function Theorem can be applied. Therefore, the regularity of the Lax curves and **(P)** ensure that $\sigma_1^+ = \sigma_1^+(\sigma_2^-, a_j - a_{j-1}; \bar u)$ and $\sigma_2^+ = \sigma_2^+(\sigma_2^-, a_j - a_{j-1}; \bar u)$. An application of [4, Lemma 2.5], yields

$$\left.\begin{array}{l}\sigma_1^+(0, a_j - a_{j-1}; \bar u) = 0\\ \sigma_1^+\left(\sigma_2^-, 0; \bar u\right) = 0\end{array}\right\} \implies \left|\sigma_1^+\right| \leqslant K_1|a_j - a_{j-1}|\left|\sigma_2^-\right|,$$

$$\left.\begin{array}{l}\sigma_2^+(0, a_j - a_{j-1}; \bar u) = 0\\ \sigma_2^+\left(\sigma_2^-, 0; \bar u\right) = \sigma_2^-\end{array}\right\} \implies \left|\sigma_2^+ - \sigma_2^-\right| \leqslant K_2|a_j - a_{j-1}|\left|\sigma_2^-\right|$$

$$\implies \left|\sigma_2^+\right| \leqslant \left[1 + K_2|a_j - a_{j-1}|\right]\left|\sigma_2^-\right|, \tag{3.3}$$

completing the proof of (1). The estimate at (2) is proved similarly. □

We now aim at an improvement of (3.3). Solving the Riemann problem at the interaction in case (1) amounts to solve the system

$$\mathcal{L}_2\left(T\left(\mathcal{L}_1\left(\bar u; \sigma_1^+\right)\right); \sigma_2^+\right) = T\left(\mathcal{L}_2\left(\bar u; \sigma_2^-\right)\right). \tag{3.4}$$

By (2.1), the first order expansions in the wave's sizes of the Lax curves exiting u are

$$\mathcal{L}_1(u;\sigma) = \begin{bmatrix}\rho - \sigma + o(\sigma)\\ q - \lambda_1(u)\sigma + o(\sigma)\end{bmatrix} \quad\text{and}\quad \mathcal{L}_2(u;\sigma) = \begin{bmatrix}\rho + \sigma + o(\sigma)\\ q + \lambda_2(u)\sigma + o(\sigma)\end{bmatrix},$$

while the first order expansion in the size's difference $\Delta a = a^+ - a^-$ of the map T defined at (2.4), with $v = q/\rho$, is

$$T(a, a+\Delta a; u) = \begin{bmatrix}(1 + H\frac{\Delta a}{a})\rho + o(\Delta a)\\ (1 - \frac{\Delta a}{a})q + o(\Delta a)\end{bmatrix}, \quad \text{where } H = \frac{v^2 + \frac{\partial_{a^+}\Sigma - p(\rho)}{\rho}}{c^2 - v^2}. \tag{3.5}$$

Inserting these expansions in (3.4), we get the following linear system for σ_1^+, σ_2^+:

$$\begin{cases} -\left(1 + \bar H\dfrac{\Delta a}{\bar a}\right)\sigma_1^+ + \sigma_2^+ = \left(1 + \bar H\dfrac{\Delta a}{\bar a}\right)\sigma_2^-,\\ -\left(1 - \dfrac{\Delta a}{\bar a}\right)\bar\lambda_1\sigma_1^+ + \left(1 + \bar G\dfrac{\Delta a}{\bar a}\right)\bar\lambda_2\sigma_2^+ = \left(1 - \dfrac{\Delta a}{\bar a}\right)\bar\lambda_2\sigma_2^-\end{cases}$$

where

$$\bar H = \frac{\bar v^2 + (\partial_{a^+}\Sigma(\bar a, \bar a, \bar u) - p(\bar\rho))/\bar\rho}{c^2 - \bar v^2} \quad\text{and}\quad \bar G = \frac{(c'(\bar\rho)\bar\rho - \bar v)\bar H - \bar v}{\bar v + c}$$

and all functions are computed in $\bar u$. The solution is

$$\sigma_1^+ = -\frac{\bar\lambda_2}{2c}(1 + \bar G + \bar H)\frac{\Delta a}{a}\sigma_2^-, \tag{3.6}$$

$$\sigma_2^+ = \left(1 - \frac{\bar\lambda_1\bar H + \bar\lambda_2(1 + \bar G)}{2c}\frac{\Delta a}{a}\right)\sigma_2^- \tag{3.7}$$

which implies the following first order estimate for the coefficients in the interaction estimates of Lemma 3.2:

$$K_1 = \frac{1}{2a}\left|\frac{1+\frac{c'\rho}{c}(\frac{v}{c})^2+\frac{1}{c^2}(\frac{c'\rho}{c}+1)\frac{\partial_{a^+}\Sigma-p(\rho)}{\rho}}{1-(\frac{v}{c})^2}\right|,$$
$$K_2 = \frac{1}{2a}\left|\frac{1-2(\frac{v}{c})^2+\frac{c'\rho}{c}(\frac{v}{c})^2+\frac{1}{c^2}(\frac{c'\rho}{c}-1)\frac{\partial_{a^+}\Sigma-p(\rho)}{\rho}}{1-(\frac{v}{c})^2}\right|. \tag{3.8}$$

The estimate (3.7) directly implies the following corollary.

Corollary 3.3. *If $|a_j - a_{j-1}|$ is sufficiently small, then σ_2^+ and σ_2^- are either both rarefactions or both shocks.*

Denote by $\sigma_{i,\alpha}^j$ the wave belonging to the i-th family and sited at the point of jump x^α, with x^α in the j-th pipe I_j, where we set $I_0 =]-\infty, x_1[$, $I_j =]x_j, x_{j+1}[$ for $j = 1,\dots,n-1$ and $I_n =]x_n, +\infty[$. Aiming at a bound on the Total Variation of the approximate solution, we define the Glimm-like functionals, see [4, formulæ (7.53) and (7.54)] or also [10,12,17,20],

$$V = \sum_{j=0}^{n}\sum_{x^\alpha\in I_j}\left(\left|\sigma_{1,\alpha}^j\right|e^{C\sum_{h=1}^{j}|a_h-a_{h-1}|}+\left|\sigma_{2,\alpha}^j\right|e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\right)+\sum_{j=0}^{n}e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\sum_{\sigma\text{ non-physical in }I_j}|\sigma|,$$
$$Q = \sum_{(\sigma_{i,\alpha}^j,\sigma_{i',\alpha'}^{j'})\in\mathcal{A}}\left|\sigma_{i,\alpha}^j\sigma_{i',\alpha'}^{j'}\right|,$$
$$\Upsilon = V + Q, \tag{3.9}$$

where C is a positive constant to be specified below. $\mathcal{A}$ is the set of pairs $(\sigma_{i,\alpha}^j, \sigma_{i',\alpha'}^{j'})$ of approaching waves, see [4, Paragraph 3, Section 7.3]. The i-wave $\sigma_{i,\alpha}^j$ sited at x_α and the i'-wave $\sigma_{i',\alpha'}^{j'}$ sited at $x_{\alpha'}$ are approaching if either $i < i'$ and $x_\alpha > x_{\alpha'}$, or if $i = i' < 3$ and $\min\{\sigma_{i,\alpha}^j, \sigma_{i',\alpha'}^{j'}\} < 0$, independently from j and j'. As usual, non-physical waves are considered as belonging to a fictitious linearly degenerate 3rd family, hence they are approaching to all physical waves to their right.

It is immediate to note that the weights $\exp(C\sum_{h=1}^{j}|a_h - a_{h-1}|)$ and $\exp(C\sum_{h=j}^{n-1}|a_{h+1} - a_h|)$ in the definition of V are uniformly bounded:

$$\forall j\quad \begin{cases} 1 \leqslant \exp\left(C\sum_{h=1}^{j}|a_h-a_{h-1}|\right) \leqslant \exp\big(C\mathrm{TV}(a)\big), \\ 1 \leqslant \exp\left(C\sum_{h=j}^{n-1}|a_{h+1}-a_h|\right) \leqslant \exp\big(C\mathrm{TV}(a)\big). \end{cases} \tag{3.10}$$

Below, the following elementary inequality is of use: if $a < b$, then $e^a - e^b < -(b-a)e^a$.

Lemma 3.4. *There exists a positive δ such that if an ε-approximate wave front tracking solution $u = u(t,x)$ has been defined up to time $\bar t$, $\Upsilon(u(\bar t-)) < \delta$ and an interaction takes place at time $\bar t$, then the ε-solution can be extended beyond time $\bar t$ and $\Upsilon(u(\bar t+)) < \Upsilon(u(\bar t-))$.*

Proof. Thanks to (3.10) and Lemma 3.1, the standard interaction estimates, see [4, Lemma 7.2], ensure that Υ decreases at any interaction taking place in the interior of I_j, for any $j = 0,\dots,n$.

Consider now an interaction at x_j. In the case of (1) in Lemma 3.2,

$$\begin{aligned}
\Delta Q &\leqslant \sum_{(\sigma_1^+,\sigma_{i,\alpha})\in\mathcal{A}}\left|\sigma_1^+\sigma_{i,\alpha}\right| + \sum_{(\sigma_2^+,\sigma_{i,\alpha})\in\mathcal{A}}|\sigma_{i,\alpha}|\left(\left|\sigma_2^+\right|-\left|\sigma_2^-\right|\right)\\
&\leqslant \left(K_1|a_j-a_{j-1}|\sum_{i,\alpha}|\sigma_{i,\alpha}| + \left(e^{K_2|a_j-a_{j-1}|}-1\right)\sum_{i,\alpha}|\sigma_{i,\alpha}|\right)\left|\sigma_2^-\right|\\
&\leqslant (K_1+K_2)\Upsilon(\bar t-)|a_j-a_{j-1}|\left|\sigma_2^-\right|\\
&\leqslant (K_1+K_2)\delta|a_j-a_{j-1}|\left|\sigma_2^-\right|,\\
\Delta V &\leqslant e^{C\sum_{h=1}^{j-1}|a_h-a_{h-1}|}\left|\sigma_1^+\right| + e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\left|\sigma_2^+\right| - e^{C\sum_{h=j-1}^{n-1}|a_{h+1}-a_h|}\left|\sigma_2^-\right|\\
&\leqslant e^{C\sum_{h=1}^{j-1}|a_h-a_{h-1}|}\left(K_1|a_j-a_{j-1}|\left|\sigma_2^-\right|\right) + \left(e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}e^{K_2|a_j-a_{j-1}|} - e^{C\sum_{h=j-1}^{n-1}|a_{h+1}-a_h|}\right)\left|\sigma_2^-\right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \big(K_1|a_j-a_{j-1}|e^{C\sum_{h=1}^{j-1}|a_h-a_{h-1}|}\big)\big|\sigma_2^-\big| + e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\big(e^{K_2|a_j-a_{j-1}|}-e^{C|a_j-a_{j-1}|}\big)\big|\sigma_2^-\big| \\
&\leqslant \big(K_1|a_j-a_{j-1}|e^{C\sum_{h=1}^{j-1}|a_h-a_{h-1}|}\big)\big|\sigma_2^-\big| - (C-K_2)|a_j-a_{j-1}|e^{K_2|a_j-a_{j-1}|}e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\big|\sigma_2^-\big| \\
&\leqslant \big((K_1+K_2)\big(1+e^{K_2|a^+-a^-|}\big)e^{C\mathrm{TV}(a)}-C\big)|a_j-a_{j-1}|\big|\sigma_2^-\big|, \\
\Delta\Upsilon &\leqslant \big((K_1+K_2)\big(1+e^{K_2|a^+-a^-|}\big)e^{C\mathrm{TV}(a)}+(K_1+K_2)\delta-C\big)|a_j-a_{j-1}|\big|\sigma_2^-\big|.
\end{aligned}
$$

Choosing now, for instance,

$$
\delta<1, \qquad C=\frac{1}{\mathrm{TV}(a)}, \qquad \big|a^+-a^-\big| \leqslant \frac{\ln 2}{K_2} \quad \text{and} \quad \mathrm{TV}(a)<\frac{1}{4(K_1+K_2)e} \tag{3.11}
$$

the monotonicity of Υ in this first case is proved.

Consider an interaction as in (2) of Lemma 3.2. Then, similarly,

$$
\begin{aligned}
\Delta Q &\leqslant \sum_{(\sigma^+,\sigma_{i,\alpha})\in\mathcal{A}} |\sigma_{i,\alpha}|\big(\big|\sigma^+\big|-\big|\sigma^-\big|\big) \\
&\leqslant \big(e^{K_3|a_j-a_{j-1}|}-1\big)\sum_{i,\alpha}|\sigma_{i,\alpha}|\big|\sigma^-\big| \\
&\leqslant K_3\Upsilon(\bar t-)|a_j-a_{j-1}|\big|\sigma^-\big| \\
&\leqslant K_3\delta|a_j-a_{j-1}|\big|\sigma^-\big|. \\
\Delta V &\leqslant e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\big|\sigma^+\big| - e^{C\sum_{h=j-1}^{n-1}|a_{h+1}-a_h|}\big|\sigma^-\big| \\
&\leqslant \big(e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}e^{K_3|a_j-a_{j-1}|}-e^{C\sum_{h=j-1}^{n-1}|a_{h+1}-a_h|}\big)\big|\sigma^-\big| \\
&\leqslant e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\big(e^{K_3|a_j-a_{j-1}|}-e^{C|a_j-a_{j-1}|}\big)\big|\sigma^-\big| \\
&\leqslant (K_3-C)|a_j-a_{j-1}|e^{K_3|a_j-a_{j-1}|}e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\big|\sigma^-\big|. \\
\Delta\Upsilon &\leqslant \big(K_3e^{K_3|a^+-a^-|}e^{C\mathrm{TV}(a)}+K_3\delta-C\big)|a_j-a_{j-1}|\big|\sigma^-\big|
\end{aligned}
$$

and the choice $\delta<1$ and $C>2K_3$ ensures that $\Delta\Upsilon<0$. □

Proof of Theorem 2.4. First, observe that the construction of the stationary solution $\hat u$ directly follows from an iterated application of Lemma 2.1. The bound (2.12) follows from the Lipschitz continuity of the map T defined in Lemma 2.1. Define

$$
\tilde{\mathcal{D}} = \big\{u\in\hat u+\mathbf{L}^1(\mathbb{R};A_0)\colon\ u\in\mathbf{PC}\ \text{and}\ \Upsilon(u)\leqslant\delta\big\},
$$

where **PC** denotes the set of piecewise constant functions with finitely many jumps. It is immediate to prove that there exists a suitable $C_1>0$ such that $\frac{1}{C_1}\mathrm{TV}(u)(t)\leqslant V(t)\leqslant C_1\mathrm{TV}(u)(t,\cdot)$ for all $u\in\tilde{\mathcal{D}}$. Any initial data in $\tilde{\mathcal{D}}$ yields an approximate solution to (1.2) attaining values in $\tilde{\mathcal{D}}$ by Lemma 3.4.

We pass now to the $\mathbf{L}^1$-Lipschitz continuous dependence of the approximate solutions from the initial datum. Consider two wave front tracking approximate solutions u_1 and u_2 and define the functional

$$
\Phi(u_1,u_2)=\sum_{j=1}^{n}\sum_{i=1}^{2}\int_0^{+\infty}\big|s_i^j(x)\big|W_i^j(x)\,dx, \tag{3.12}
$$

where $s_i^j(x)$ measures the strengths of the i-th shock wave in the j-th pipe at point x (see [4, Chapter 8]) and the weights W_i^j are defined by

$$
W_i^j(x)=1+\kappa_1A_i^j(x)+\kappa_1\kappa_2\big(\Upsilon(u_1)+\Upsilon(u_2)\big)
$$

for suitable positive constants κ_1,κ_2 chosen as in [4, formula (8.7)]. Here Υ is the functional defined in (3.9), while the A_i^j are defined by

$$
A_i^j(x)=\sum\left\{|\sigma_{k_\alpha,\alpha}^j|\colon\ \begin{matrix} x_\alpha<x,\ i<k_\alpha\leqslant 2,\\ x_\alpha>x,\ 1\leqslant k_\alpha<i\end{matrix}\right\}+\begin{cases}\sum\left\{|\sigma_{i,\alpha}^j|\colon\ \begin{matrix}x_\alpha<x,\ \alpha\in\mathcal{J}_j(u_1),\\ x_\alpha>x,\ \alpha\in\mathcal{J}_j(u_2)\end{matrix}\right\} & \text{if } s_i^j(x)<0,\\[2mm] \sum\left\{|\sigma_{i,\alpha}^j|\colon\ \begin{matrix}x_\alpha<x,\ \alpha\in\mathcal{J}_j(u_2),\\ x_\alpha>x,\ \alpha\in\mathcal{J}_j(u_1)\end{matrix}\right\} & \text{if } s_i^j(x)\geqslant 0;\end{cases}
$$

see [4, Chapter 8]. Here, as above, $\sigma^j_{i,\alpha}$ is the wave belonging to the i-th family, sited at x^α, with $x^\alpha \in I_j$. For fixed κ_1, κ_2 the weights $W^j_i(x)$ are uniformly bounded. Hence the functional Φ is equivalent to $\mathbf{L^1}$ distance:

$$\frac{1}{C_2} \cdot \|u_1 - u_2\|_{\mathbf{L^1}} \leqslant \Phi(u_1, u_2) \leqslant C_2 \cdot \|u_1 - u_2\|_{\mathbf{L^1}}$$

for a positive constant C_2. The same calculations as in [4, Chapter 8] show that, at any time $t > 0$ when an interaction happens neither in u_1 or in u_2,

$$\frac{d}{dt}\Phi\bigl(u_1(t), u_2(t)\bigr) \leqslant C_3\varepsilon$$

where C_3 is a suitable positive constant depending only on a bound on the total variation of the initial data.

If $t > 0$ is an interaction time for u_1 or u_2, then, by Lemma 3.4, $\Delta[\Upsilon(u_1(t)) + \Upsilon(u_2(t))] < 0$ and, choosing κ_2 large enough, we obtain

$$\Delta\Phi\bigl(u_1(t), u_2(t)\bigr) < 0.$$

Thus, $\Phi(u_1(t), u_2(t)) - \Phi(u_1(s), u_2(s)) \leqslant C_2\varepsilon(t - s)$ for every $0 \leqslant s \leqslant t$. The proof is now completed using the standard arguments in [4, Chapter 8].

The proof that in the limit $\varepsilon \to 0$ the semigroup trajectory does indeed yield a Ψ-solution to (1.2) and, in particular, that (2.11) is satisfied on the traces, is exactly as that of [6, Proposition 5.3], completing the proof of (1)–(5).

Due to the *local* nature of the conditions (2.9)–(2.10) and to the finite speed of propagation of (1.2), the uniqueness conditions (6) and (7) are proved exactly as in Theorem 2.3. □

Proof of estimate (2.14). We first compute $\partial_{a^+}\Sigma$, with Σ defined in (2.19). To this aim, by (2) in Proposition 2.7 (in Section 2.3), we may choose

$$a(x) = \begin{cases} a^- & \text{if } x \in]-\infty, -X[, \\ \frac{a^+ - a^-}{2X}(x + X) + a^- & \text{if } x \in [-X, X], \\ a^+ & \text{if } x \in]X, +\infty[, \end{cases}$$

so that we may change variable in the integral in (2.19) to obtain

$$\partial_{a^+}\Sigma = \partial_{a^+}\left(\int_{a^-}^{a^+} p\bigl(R^a(\alpha, u)\bigr)\,d\alpha\right) = p(\rho) + O(\Delta a). \tag{3.13}$$

Now, estimate (2.14) directly follows inserting (2.13) and (3.13) in (3.11) and (3.8). □

Proof of estimates (2.15)–(2.16). Refer to the notation in Fig. 1, where the pipe's section is given by

$$a(x) = \begin{cases} a & \text{if } x \in]-\infty, l[, \\ a + \Delta a & \text{if } x \in [l, 2l], \\ a & \text{if } x \in]2l, +\infty[, \end{cases}$$

where $\Delta a > 0$. The wave σ_2^+ arises from the interaction with the first junction and hence satisfies (3.7). Using the pressure law (2.13) and (3.13), we obtain

$$\sigma_2^+ = \bigl(1 + \psi(a, u)\Delta a\bigr)\sigma_2^-, \quad \text{where } \psi(a, u) = -\frac{1}{a}\left(1 - \frac{1/2}{1 - (v/c)^2}\right).$$

Now we iterate the previous bound to estimate the wave σ_2^{++} which arises from the interaction with the second junction, i.e.

$$\sigma_2^{++} = \bigl(1 - \psi\bigl(a + \Delta a, u^+\bigr)\Delta a\bigr)\sigma_2^+,$$

where, by (3.5),

$$\psi\bigl(a + \Delta a, u^+\bigr) = \psi\left(a + \Delta a, \left(1 + \frac{1}{1 - (\frac{v}{c})^2}\frac{\Delta a}{a}\right)\rho, \left(1 - \frac{\Delta a}{a}\right)q\right).$$

Introduce $\eta = 1/(1-(v/c)^2)$ and $\vartheta = \Delta a/a$ to get the estimate

$$\begin{aligned}\sigma_2^{++} &= \big(1+\big(\psi(a,u)-\psi\big(a+\Delta a,u^+\big)\big)\Delta a\big)\sigma_2^- \\ &= \left(1+\frac{\Delta a}{a}\left(-1+\frac{\eta}{2}\right)+\frac{\Delta a}{a+\Delta a}\left(1-\frac{1/2}{1-(\frac{1-\vartheta}{1+\eta\vartheta}\frac{v}{c})^2}\right)\right)\sigma_2^- \\ &= \left(1+\frac{\Delta a}{a}\left(-1+\frac{\eta}{2}+\frac{1}{1+\vartheta}\left(1-\frac{1/2}{1-(\frac{1-\vartheta}{1+\vartheta\eta}\frac{v}{c})^2}\right)\right)\right)\sigma_2^-,\end{aligned}$$

and a further expansion to the leading term in Δa gives (2.15)–(2.16). □

3.3. Proofs related to Section 2.3

Proof of Lemma 2.6. If $a \in \mathbf{C^1}(\mathbb{R};[a^-,a^+])$ and u is a weak entropy solution of (1.4). Then,

$$\begin{aligned}0 &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(\begin{bmatrix}\rho\\ q\end{bmatrix}\partial_t\varphi+\begin{bmatrix}q\\ P(u)\end{bmatrix}\partial_x\varphi-\begin{bmatrix}\frac{q}{a}\partial_x a\\ \frac{q^2}{a\rho}\partial_x a\end{bmatrix}\varphi\right)dx\,dt \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(\begin{bmatrix}a\rho\\ aq\end{bmatrix}\partial_t\frac{\varphi}{a}+\begin{bmatrix}aq\\ aP(u)\end{bmatrix}\frac{1}{a}\partial_x\varphi-\begin{bmatrix}aq\\ a\frac{q^2}{\rho}\end{bmatrix}\frac{\varphi}{a^2}\partial_x a\right)dx\,dt \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(\begin{bmatrix}a\rho\\ aq\end{bmatrix}\partial_t\frac{\varphi}{a}+\begin{bmatrix}aq\\ aP(u)\end{bmatrix}\partial_x\frac{\varphi}{a}+\begin{bmatrix}0\\ p(\rho)\partial_x a\end{bmatrix}\frac{\varphi}{a}\right)dx\,dt\end{aligned}$$

showing that (2.17) holds. Concerning the entropy inequality, compute preliminarily

$$\begin{aligned}\nabla\big(aE(u)\big)\begin{bmatrix}\frac{q}{a}\partial_x a\\ \frac{q^2}{a\rho}\partial_x a\end{bmatrix} &= a\left[-\frac{q^2}{2\rho^2}+\int_\rho^{\rho_*}\frac{p(r)}{r^2}\,dr+\frac{p(\rho)}{\rho},\ \frac{q}{\rho}\right]\begin{bmatrix}\frac{q}{a}\partial_x a\\ \frac{q^2}{a\rho}\partial_x a\end{bmatrix} \\ &= \left(-\frac{q^3}{2\rho^2}+q\int_\rho^{\rho_*}\frac{p(r)}{r^2}\,dr+\frac{q}{\rho}p(\rho)+\frac{q^3}{\rho^2}\right)\partial_x a \\ &= \frac{q}{\rho}\big(E(u)+p(\rho)\big)\partial_x a \\ &= F(u)\partial_x a.\end{aligned}$$

Consider now the entropy condition for (1.4) and, by the above equality,

$$\begin{aligned}0 &\leqslant \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(E(u)\partial_t\varphi+F(u)\partial_x\varphi-\nabla E(u)\begin{bmatrix}\frac{q}{a}\partial_x a\\ \frac{q^2}{a\rho}\partial_x a\end{bmatrix}\varphi\right)dx\,dt \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(aE(u)\partial_t\frac{\varphi}{a}+aF(u)\partial_x\frac{\varphi}{a}+\left(F(u)\partial_x a-\nabla\big(aE(u)\big)\begin{bmatrix}\frac{q}{a}\partial_x a\\ \frac{q^2}{a\rho}\partial_x a\end{bmatrix}\right)\frac{\varphi}{a}\right)dx\,dt \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(aE(u)\partial_t\frac{\varphi}{a}+aF(u)\partial_x\frac{\varphi}{a}+\big(F(u)\partial_x a-F(u)\partial_x a\big)\frac{\varphi}{a}\right)dx\,dt \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(aE(u)\partial_t\frac{\varphi}{a}+aF(u)\partial_x\frac{\varphi}{a}\right)dx\,dt,\end{aligned}$$

showing that (2.18) holds. The extension to $a \in \mathbf{W^{1,1}}$ is immediate. □

Proof of Proposition 2.7. The regularity condition **($\boldsymbol{\Sigma}$0)** follows from the theory of ordinary differential equations. Condition **($\boldsymbol{\Sigma}$1)** is immediate.

Consider now the item (2). If a_1 and a_2 both satisfy **(A1)**, are strictly monotone, smooth and have the same range, then $a_1 = a_2 \circ \varphi$ for a suitable strictly monotone φ with, say $\varphi' \geqslant 0$, the case $\varphi' \leqslant 0$ is entirely similar. Note that if $u = (R_i(x;u^-), Q_i(x;u^-))$ solves (1.5) with $a = a_i$, then direct computations show that $R_1(x,u^-) = R_2(\varphi(x),u^-)$ and $Q_1(x,u^-) = Q_2(\varphi(x),u^-)$. Hence

$$\begin{aligned}
\Sigma_1\big(a^-,a^+;u^-\big) &= \int_{-X}^{X} p\big(R_1\big(x;u^-\big)\big)a_1'(x)\,dx \\
&= \int_{-X}^{X} p\big(R_2\big(\varphi(x);u^-\big)\big)a_2'\big(\varphi(x)\big)\varphi'(x)\,dx \\
&= \int_{-X}^{X} p\big(R_2\big(\xi;u^-\big)\big)a_2'(\xi)\,d\xi \\
&= \Sigma_2\big(a^-,a^+;u^-\big).
\end{aligned}$$

Having proved **(Σ0)** and **(Σ1)**, we use the map T defined in Lemma 2.1. We first prove that Σ satisfies $\Sigma(a^-,a^+;u^-) + \Sigma(a^+,a^-;T(a^+,a^-;u^-)) = 0$, given a satisfying **(A1)**, strictly monotone and with $a(-X)=a^-$, $a(X)=a^+$, let $\tilde{a}(x)=a^-+a^+-a(x)$. Then, using (2) proved above, and integrating (1.5) backwards, we have

$$\begin{aligned}
\Sigma\big(a^+,a^-;T\big(a^-,a^+;u^-\big)\big) &= \int_{-X}^{X} p\big(\tilde{R}\big(x;T\big(a^-,a^+;u^-\big)\big)\big)\tilde{a}'(x)\,dx \\
&= -\int_{-X}^{X} p\big(R\big(x;a^-,a^+;u^-\big)\big)a'(x)\,dx \\
&= -\Sigma\big(a^-,a^+;u^-\big).
\end{aligned}$$

Finally, condition **(Σ2)** follows from the flow property of R and the additivity of the integral. Indeed, by (2) and (3) we may assume without loss of generality that $a^- < a^0 < a^+$. Then, let $q = Q(x;u^-)$ be the q component in the solution to (1.5) with initial condition $u(0)=u^-$. Then, if T is the map defined in Lemma 2.1, we have

$$T\big(a^-,a^+;u^-\big) = \big(R\big(a^{-1}\big(a^+\big);u^-\big),\ Q\big(a^{-1}\big(a^+\big);u^-\big)\big)$$

so that

$$\begin{aligned}
\Sigma\big(a^-,a^+;u^-\big) &= \int_{-X}^{X} p\big(R\big(x,u^-\big)\big)a'(x)\,dx \\
&= \int_{-X}^{a^{-1}(a^0)} p\big(R\big(x,u^-\big)\big)a'(x)\,dx + \int_{a^{-1}(a^0)}^{X} p\big(R\big(x,u^-\big)\big)a'(x)\,dx \\
&= \int_{-X}^{a^{-1}(a^0)} p\big(R\big(x,u^-\big)\big)a'(x)\,dx \\
&\quad + \int_{a^{-1}(a^0)}^{X} p\big(R\big(x,R\big(a^{-1}\big(a^0\big),u^-\big),\ Q\big(a^{-1}\big(a^0\big),u^-\big)\big)\big)a'(x)\,dx \\
&= \Sigma\big(a^-,a^0;u^-\big) + \Sigma\big(a^0,a^+;T\big(a^-,a^+;u^-\big)\big)
\end{aligned}$$

proving (1). $\square$

Proof of Theorem 2.8. Fix $\bar{a} > 0$, and $\bar{u} \in A_0$. Choose M, Δ, L, δ as in Theorem 2.4. With reference to these quantities, let a satisfy **(A1)**. For $n \in \mathbb{N}$, let a_n, α_n, β_n be as in Proposition 2.9. Note that α_n is piecewise linear and continuous. By (iii), we have that $\alpha_n \to a$ and $a_n \to a$ in $\mathbf{L^1}$. Moreover, $\mathrm{TV}(\alpha_n) \leqslant M$ and $\mathrm{TV}(a_n) \leqslant M$ and, for n sufficiently large, $a_n(\mathbb{R}) \subseteq]\bar{a}-\Delta, \bar{a}+\Delta[$. Hence, for n large, a_n satisfies **(A0)**. Call S^n the semigroup constructed in Theorem 2.4 and denote by $\mathcal{D}^n$ its domain.

Let u_n^0 be a sequence of initial data in $\mathcal{D}^n$. The S^n are uniformly Lipschitz in time and $S_t^n u_n^0$ have total variation in x uniformly bounded in t. Hence, by [4, Theorem 2.4], a subsequence of $u_n(t) = S_t^n u_n^0$ converges pointwise a.e. to a limit, say, u. For any $\varphi \in \mathbf{C_c^1}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R})$ and for any fixed n, let $\varepsilon > 0$ be sufficiently small and introduce a $\mathbf{C_c^\infty}(\mathbb{R}; \mathbb{R})$ function η_ε such that

$$\eta_\varepsilon(x) = 0 \quad \text{for all } x \in \bigcup_{j=1}^{m_n-1} \left[x_n^j - \varepsilon, x_n^j + \varepsilon\right],$$

$$\eta_\varepsilon(x) = 1 \quad \text{for all } x \in \bigcup_{j=1}^{m_n-2} \left[x_n^j + 2\varepsilon, x_n^{j+1} - 2\varepsilon\right].$$

Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a_n\rho_n \\ a_n q_n \end{bmatrix} \partial_t \varphi + \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \partial_x \varphi \right) dx\, dt \\
&\quad = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a_n\rho_n \\ a_n q_n \end{bmatrix} \eta_\varepsilon \partial_t \varphi + \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \eta_\varepsilon \partial_x \varphi \right) dx\, dt \\
&\quad = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a_n\rho_n \\ a_n q_n \end{bmatrix} \partial_t (\eta_\varepsilon \varphi) + \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \partial_x (\eta_\varepsilon \varphi) \right) dx\, dt - \lim_{\varepsilon\to 0} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \varphi \partial_x \eta_\varepsilon \, dx\, dt.
\end{aligned}$$

The first summand in the latter term above vanishes by Definition 2.2 applied in a neighborhood of each x_n^j. The second summand, by the **BV** regularity of u_n, converges as follows:

$$\begin{aligned}
&-\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a_n\rho_n \\ a_n q_n \end{bmatrix} \partial_t \varphi + \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \partial_x \varphi \right) dx\, dt \\
&\quad = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \varphi \partial_x \eta_\varepsilon \, dx\, dt \\
&\quad = \sum_{j=1}^{m_n-1} \int_{\mathbb{R}^+} \begin{bmatrix} a_n(x_n^j+) q_n(x_n^j+) - a_n(x_n^j-) q_n(x_n^j-) \\ a_n(x_n^j+) P_n(x_n^j+) - a_n(x_n^j-) P_n(x_n^j-) \end{bmatrix} \varphi\big(t, x_n^j\big)\, dt \\
&\quad = \sum_{j=1}^{m_n-1} \int_{\mathbb{R}^+} \begin{bmatrix} 0 \\ \Sigma(a_n(x_n^j-), a_n(x_n^j+), u(t, x_n^j-)) \end{bmatrix} \varphi\big(t, x_n^j\big)\, dt.
\end{aligned}$$

We proceed now considering only the second component. Using the map

$$\varphi_n(t,x) = \varphi(t,x)\chi_{]-\infty, y_n^1[}(x) + \sum_{j=1}^{m_n-1} \varphi\big(t, x_n^j\big)\chi_{[y_n^j, y_n^{j+1}[}(x) + \varphi(t,x)\chi_{]y_n^{m_n}, +\infty[}(x),$$

we obtain

$$\begin{aligned}
&\sum_{j=1}^{m_n-1} \int_{\mathbb{R}^+} \Sigma\big(a_n\big(x_n^j-\big), a_n\big(x_n^j+\big), u\big(t, x_n^j-\big)\big)\varphi\big(t, x_n^j\big)\, dt \\
&\quad = \sum_{j=1}^{m_n-1} \int_{\mathbb{R}^+} \Sigma\big(a_n\big(y_n^j\big), a_n\big(y_n^{j+1}\big), u\big(t, x_n^j-\big)\big)\varphi\big(t, x_n^j\big)\, dt \\
&\quad = \sum_{j=1}^{m_n-1} \int_{\mathbb{R}^+} \Sigma\big(\alpha_n\big(y_n^j\big), \alpha_n\big(y_n^{j+1}\big), u\big(t, x_n^j-\big)\big)\varphi\big(t, x_n^j\big)\, dt \\
&\quad = \sum_{j=1}^{m_n-1} \int_{\mathbb{R}^+} \int_{y_n^j}^{y_n^{j+1}} p\big(R^{\alpha_n}\big(x; u_n\big(t, x_n^j-\big)\big)\big)\alpha_n'(x)\, dx\, \varphi\big(t, x_n^j\big)\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^+} \sum_{j=1}^{m_n-1} \int_{y_n^j}^{y_n^{j+1}} p\big(R^{\alpha_n}\big(x; u_n\big(t, x_n^j-\big)\big)\big)\alpha_n'(x)\,dx\,\varphi\big(t, x_n^j\big)\,dt \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}} \sum_{j=1}^{m_n-1} p\big(R^{\alpha_n}\big(x; u_n\big(t, x_n^j-\big)\big)\big)\alpha_n'(x)\varphi\big(t, x_n^j\big)\chi_{[y_n^j, y_n^{j+1}[}(x)\,dx\,dt \\
&\to \int_{\mathbb{R}^+} \int_{\mathbb{R}} p\big(\rho(x)\big)\partial_x a(x)\varphi\big(t, x_n^j\big)\,dx\,dt \quad \text{as } n \to +\infty,
\end{aligned}
$$

where we used (i) in the choice of the approximation α_n.

We thus constructed a solution to (1.1), for any initial datum in $\mathcal{D}$. Note that this solution satisfies (2.20)–(2.21), as can be proved using exactly the techniques in [14, Theorem 8]. Therefore, the whole sequence u_n converges to a unique limit u, which is Lipschitz with respect to time. This uniqueness implies the semigroup property (2) in Theorem 2.8. The Lipschitz continuity with respect to the initial datum follows from the uniform Lipschitz regularity of the approximate solutions u_n, completing the proof of (3). Finally, (6) is proved exactly as in [14, Theorem 8]. □

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