On Sequences with Zero Autocorrelation and Orthogonal Designs

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We propose some methods for multiplying the length and type of sequences with elements on a set of commuting variables which have zero non-periodic autocorrelation function. We use base sequences of lengths $n+1$, $n+1$, $n$, $n$ in order to construct four directed sequences of lengths $n+1$, $n+1$, $n$, $n$ and type $(2n+1, 2n+1)$ with zero NPAF as well as normal sequences of length $n$ in order to construct four directed sequences of length $2n$ and type $(2n, 2n, 2n, 2n)$ with zero NPAF. We construct two and four directed sequences with zero PAF of length $34$ and type $(34, 34)$, and $(34, 34, 34, 34)$, respectively, as well as four directed sequences with zero NPAF of lengths $34$, $34$, $33$, $33$ and type $(67, 67)$. We also indicate that from $m$ directed sequences of lengths $n_1, n_2, ..., n_m$ which consist of $t$ variables, we obtain $k \cdot m$ directed sequences of lengths $n_1, n_2, ..., n_m$ ($k$ sequences will be of lengths $n_i$, $i=1, 2, ..., m$) which consist of $k \cdot t$ variables for $k=1, 2, ...$. The above methods lead to the construction of many new orthogonal designs.

Key Words: directed sequences; multiplication; autocorrelation; orthogonal design.

1. INTRODUCTION

An orthogonal design of order $n$ and type $(s_1, s_2, ..., s_u)$ ($s_i > 0$), denoted $OD(n; s_1, s_2, ..., s_u)$, on the commuting variables $x_1, x_2, ..., x_u$ is an $n \times n$ matrix $A$ with entries from $\{0, \pm x_1, \pm x_2, ..., \pm x_u\}$ such that

$$AA^T = \left( \sum_{i=1}^{u} s_i x_i^2 \right) I_n$$
Alternatively, the rows of $A$ are formally orthogonal and each row has precisely $s_i$ entries of the type $\pm x_i$. In [1], where this was first defined, it was mentioned that

$$A^T A = \left( \sum_{i=1}^{n} x_i^2 \right) I_n$$

and so our alternative description of $A$ applies equally well to the columns of $A$. It was also shown in [1] that $u \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, $b$ odd, $a = 4c + d$, $0 \leq d < 4$.

A weighing matrix $W = W(n, k)$ is a square matrix with entries $0, \pm 1$ having $k$ non-zero entries per row and column and inner product of distinct rows zero. Hence $W$ satisfies $WW^T = kI_n$, and $W$ is equivalent to an orthogonal design $OD(n; k)$. The number $k$ is called the weight of $W$. If $k = n$, that is, all the entries of $W$ are $\pm 1$ and $WW^T = nI_n$, then $W$ is called an Hadamard matrix of order $n$. In this case $n = 1, 2$ or $n \equiv 0(\text{mod } 4)$.

Given the sequence $A = [a_1, a_2, ..., a_n]$ of length $n$ the non-periodic autocorrelation function $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, ..., n - 1.$$ (1)

Given $A$ as above of length $n$ the periodic autocorrelation function $P_A(s)$ is defined, reducing $i + s$ modulo $n$, as

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, ..., n - 1.$$ (2)

The following theorem which uses four circulant matrices in the Goethals-Seidel array is very useful in our construction for orthogonal designs.

**Theorem 1** [2, Theorem 4.49]. Suppose there exist four circulant matrices $A$, $B$, $C$, $D$ of order $n$ satisfying

$$AA^T + BB^T + CC^T + DD^T = fI_n.$$

Let $R$ be the back diagonal matrix. Then

is a $W(n, f)$ when $A, B, C, D$ are $(0, 1, -1)$ matrices, and an orthogonal design $OD(4n; s_1, s_2, ..., s_u)$ on $x_1, x_2, ..., x_n$ when $A, B, C, D$ have entries from $\{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$ and $f = \sum_{j=1}^{u} (s_j x_j^2)$.

**Corollary 1.** If there are four sequences $A, B, C, D$ of length $n$ with entries from the set $\{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$ with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals–Seidel array to form an $OD(4n; s_1, s_2, s_3, s_4)$. If there are sequences of length $n$ with zero non-periodic autocorrelation function, then there are sequences with zero non-periodic autocorrelation function and the same list of weight as the sequences of length $n$, but of length $n+m$ for all $m \geq 0$.

In this paper we propose some methods for multiplying the length and type of sequences with elements on a set of commuting variables which have zero non periodic autocorrelation function. In Section 2 we give some preliminary results and in Section 3 we use normal sequences of length $n$ in order to construct four new directed sequences of length $2n$ and type $(2n, 2n, 2n, 2n)$ with zero NPAF. We construct two and four directed sequences with zero PAF of length 34 and type $(34, 34)$, and $(34, 34, 34, 34)$ respectively, as well as four directed sequences of lengths 34, 34, 33, 33 and type $(67, 67)$ with zero NPAF. Furthermore, we indicate that from $m$ directed sequences of lengths $n_1, n_2, ..., n_m$ which consist of $t$ variables, we obtain $k \cdot m$ directed sequences of lengths $n_1, n_2, ..., n_m$ ($k$ sequences will be of lengths $n_i, i=1, 2, ..., m$) which consist of $k \cdot t$ variables for $k=1, 2, ...$, As a consequence we state the construction for many new orthogonal designs.

### 2. PRELIMINARY RESULTS

**Notation.** We use the following notation throughout this paper.

1. We use $\bar{a}$ to denote $-a$.
2. We use $0_n$ to denote a sequence of length $n$ with every element zero.
3. If $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$ then $X/Y$ means the interleaved sequence $x_1, y_1, x_2, y_2, ..., x_n, y_n$. 


and $Y_0$ means the interleaved sequence

$$y_1, 0_m, y_2, 0_m, \ldots, y_n, 0_m.$$  

(4) We will say that two sequences of variables are \textit{directed} if the sequences have zero autocorrelation function independently from the properties of the variables; i.e., they do not rely on commutativity to ensure zero autocorrelation. For example $\{a, b\}$ and $\{a, -b\}$ are two directed sequences while $\{a, b\}$ and $\{b, -a\}$ are not directed. Also $\{a, b\}$, $\{a, -b\}$, $\{c, d\}$ and $\{c, -d\}$ are four directed sequences while $\{a, b\}$, $\{b, -a\}$, $\{c, d\}$ and $\{c, -d\}$ are not directed.

(5) If we have $m$ sequences $A_j$, $j = 1, \ldots, m$ we will use the symbol $u'_j$ to denote the number of times that the elements $x_i$ appear in the sequence $A_j$, $i = 1, \ldots, t$, $j = 1, \ldots, m$. Thus for example $u'_3$ denotes the number of times that the elements $x_2$ appear in the sequence $A_3$.

(6) Let $A_j$, $j = 1, 2, \ldots, m$ be $m$ sequences of length $n$. We will say that these sequences are of type $(u_1, \ldots, u_t)$ if the sequences are composed of $t$ variables, say $x_i$, $i = 1, \ldots, t$ and $x_i$ and $-x_i$ occur a total of $u_i$ times, $i = 1, \ldots, t$. Thus $u_j = \sum_{i=1}^t u'_i$.

(7) We define the $\text{NPAF (PAF)}$ of a set of sequences the sum of the corresponding $\text{NPAF (PAF)}$ of the individual sequences.

Let $M = \{n$ is a positive integer: $n = 2^n \cdot 6^n \cdot 10^n \cdot 14^n \cdot 24^n \cdot 26^n \cdot 30^n \cdot 40^n\}$ and $W = \{w$ is a positive integer: $w = 2^n \cdot 5^n \cdot 10^n \cdot 13^n \cdot 17^n \cdot 26^n \cdot 25^n \cdot 34^n\}$.

\textbf{Theorem 2.} Let $a_1, a_2, \ldots, a_8$ be positive integers or zero. Then there exist four directed sequences with $\text{NPAF} = 0$ of lengths $m_1, m_2, m_3$ and type $(w_1, w_2, w_2)$ where $m_i \in M$ and the corresponding $w_i \in W$, $i = 1, 2$.

\textit{Proof.} Use two pairs of directed sequences given in Table 13 of [4]. Even though the derived sequences are directed, their variables can not be replaced by four sequences with zero NPAF but only by two pairs of sequences where each pair has zero NPAF.

\textbf{Example 1.} Suppose we wish to construct four directed sequences of lengths $24, 24, 14, 14$ and type $(20, 20, 13, 13)$. From Table 13 of [4] we have that $\{a, b\}$ and $\{a, b\}$ are two directed sequences of length 2 and type $(2, 2)$. If we replace $a$ by the sequence $\{a, b\}$ and $b$ by the sequence $\{a, b\}$ then we get the sequences $\{a, b, a, b\}$ and $\{a, b, a, b\}$. These are two directed sequences of length 4 and type $(4, 4)$. From Table 13 of [4] we have that $\{a, a, a, b, 0, b\}$ and $\{a, 0, a, b, b, b\}$ are two directed sequences of length 6.
and type (5, 5). If we replace $a$ by the sequence $\{a, b, a, \tilde{b}\}$, $b$ by the sequence $\{a, b, \tilde{a}, b\}$ and 0 by 0, then we get the sequences

$$A = \{a, b, a, \tilde{b}, a, b, a, b, a, b, \tilde{a}, b, a, b, a, b, \tilde{a}, b, 0, 0, 0, a, b, \tilde{a}, b\}$$

$$B = \{a, b, a, \tilde{b}, 0, 0, 0, a, b, a, b, \tilde{a}, b, a, b, a, b, \tilde{a}, b, a, b\}.$$ 

These are two directed sequences of length 24 and type (20, 20). Finally, from Table 13 of [4] we have the following directed sequences of length 14 and type (13, 13),

$$C = \{c, c, c, d, \tilde{c}, c, c, d, \tilde{c}, c, \tilde{c}, d, 0, d\}$$

$$D = \{c, 0, c, d, \tilde{d}, d, \tilde{c}, d, d, \tilde{c}, \tilde{d}, d, \tilde{d}\}.$$ 

Hence the sequences $A$, $B$, $C$, $D$ are four directed sequences of lengths 24, 24, 14, 14 and type (20, 20, 13, 13) with NPAF = 0.

**Theorem 3.** Let $a_1, a_2, \ldots, a_n$ be positive integers or zero. Then there exist four directed sequences with NPAF = 0 of length $m$ and type $(w, w, w, w)$ where $m \in M$ and the corresponding $w \in W$. As the derived sequences are directed of type $(w_1, w_1, w_1, w_1)$ their variables can be replaced by sequences with zero NPAF.

**Proof.** Use twice the same pair of sequences given in Table 13 of [4]. For the first pair we use the variables named $a, b$ whereas for the second pair we use the variables named $c, d$. 

**Example 2.** Suppose we wish to construct four directed sequences of length 6 and type (5, 5, 5, 5). From Table 13 of [4] we have that $A = \{a, a, \tilde{a}, b, 0, b\}$ and $B = \{a, 0, a, b, \tilde{b}, \tilde{b}\}$ are two directed sequences of length 6 and type (5, 5). Then we can get the following four directed sequences of length 6 and type (5, 5, 5, 5) with zero NPAF. Those are $A = \{a, a, \tilde{a}, b, 0, b\}$, $B = \{a, 0, a, b, \tilde{b}, \tilde{b}\}$, $C = \{c, c, \tilde{c}, d, 0, d\}$ and $D = \{c, 0, c, d, \tilde{d}, \tilde{d}\}$.

**Example 3.** Suppose we wish to construct an $OD(72; 5, 10, 15, 30)$. If we replace $a$ by $\{d, a, \tilde{d}\}$, $b$ by $\{c, \tilde{d}, b\}$, $c$ by $\{\tilde{c}, d, b\}$, $d$ by $\{d, c, d\}$ and 0 by 0, in the four directed sequences of length 6 and type (5, 5, 5, 5) which are constructed in Example 2, we get the sequences

$$X = \{d, a, \tilde{d}, d, a, \tilde{d}, \tilde{a}, d, c, \tilde{d}, b, 0, 0, 0, c, \tilde{d}, b\}$$

$$Y = \{d, a, \tilde{d}, 0, 0, 0, a, \tilde{d}, d, c, \tilde{d}, b, \tilde{c}, d, \tilde{b}\}$$

$$Z = \{\tilde{c}, d, b, \tilde{c}, d, b, c, \tilde{d}, b, \tilde{c}, d, \tilde{b}\}$$

$$W = \{\tilde{c}, d, b, 0, 0, 0, \tilde{c}, d, b, d, c, d, \tilde{d}, \tilde{c}, d, \tilde{d}, \tilde{c}, d\}.$$
which are 4 sequences of length 18 and type (5, 10, 15, 30) with zero NPAF and thus can be used to construct an \(OD(72; 5, 10, 15, 30)\).

**Corollary 2.** Let \(D_1\) be an orthogonal design \(OD(4n; u_1, u_2, ..., u_t)\) constructed by four sequences of length \(n\), with NPAF = 0. Then there exists an orthogonal design \(OD(4nm; u_1w, u_2w, ..., u_tw)\), \(\forall m \in M\) and the corresponding \(w \in W\).

**Proof.** Let \(A_i, i = 1, 2, 3, 4\) be the four sequences of length \(n\) with NPAF = 0 from which the orthogonal design \(D_1\) is constructed. Thus \(\forall m \in M\) and the corresponding \(w \in W\) applying Theorem 3, we obtain four directed sequences of length \(m\) and type \((w, w, w, w)\) with NPAF = 0. As the derived sequences are directed of type \((w, w, w, w)\) their variables can be replaced by sequences with zero NPAF. If we replace \(a\) by \(A_1\), \(b\) by \(A_2\), \(c\) by \(A_3\), \(d\) by \(A_4\) then we get four sequences of length \(l = nm\) and type \((s_1, s_2, ..., s_t)\) where \(s_i = wa_i, i = 1, 2, ..., t\) and NPAF = 0. Hence using these sequences we obtain an orthogonal design \(OD(4nm; u_1w, u_2w, ..., u_tw)\).

**Example 4.** Suppose we wish to construct an orthogonal design \(OD(168; 13, 26, 39, 78)\). From Table 13 of [4] we have two directed sequences of length 14 and type \((13, 13)\),

\[
A = \{a, a, a, b, a, a, a, b, a, a, b, 0, b\}
\]

\[
B = \{a, 0, a, b, b, b, b, a, b, b, b, b\}.
\]

Replace \(a\) by \(c\) and \(b\) by \(d\) and set

\[
C = \{c, c, c, c, c, c, c, d, 0, d\}
\]

\[
D = \{c, 0, c, d, d, d, d, c, d, d, d\}.
\]

Then \(A, B, C, D\) are four directed sequences of length 14 and type \((13, 13, 13, 13)\) with NPAF = 0. As the derived sequences are directed of type \((s, s, s, s)\) their variables can be replaced by sequences. Set \(X = \{d, a, d\}\), \(Y = \{c, d, b\}\), \(Z = \{c, d, b\}\), \(W = \{d, c, d\}\). Then \(X, Y, Z, W\) have NPAF = 0 and can be used to construct an \(OD(12; 1, 2, 3, 6)\). Now in the four directed sequences replace \(a\) by \(X\) sequence, \(b\) by \(Y\) sequence, \(c\) by \(Z\) sequence, \(d\) by \(W\) sequence and 0 by 0 sequence to obtain four sequences of length 14-3 and type \((1 \cdot 13, 2 \cdot 13, 3 \cdot 13, 6 \cdot 13)\). Thus we have the orthogonal design \(OD(168; 13, 26, 39, 78)\).

Set

\[
(M_2, W_2) = \{(10, 9), (11, 9), (13, 9), (15, 9), (19, 9), (21, 9), (23, 9),
(19, 17), (21, 18), (23, 18), (34, 34), (37, 36)\}.
\]
The sequences

\[ X = \{ a, a, a, \bar{a}, \bar{b}, b, \bar{b}, b, a, b, b, a, b, a, b \} \]

\[ Y = \{ b, a, a, b, b, \bar{a}, \bar{b}, b, \bar{b}, b, a, b, \bar{a}, \bar{a}, a, b, b, \bar{b}, b, a, b, \bar{a}, \bar{a}, a, b \} \]

Copy sequences \( X, Y \) in \( Z, W \) and replace their variables by \( c \) and \( d \). Then the desirable sequences are \( A = X/0_n \); \( B = Y/0_n \); \( C = Z/0_n \); \( D = W/0_n \).

**Example 5.** For \( n = 0, m = 34, w = 34 \) we obtain four directed sequences of length 34 and type (34, 34, 34, 34) with zero PAF.

**Corollary 3.** Let \( D_4 \) be an orthogonal design \( OD(4s; u_1, u_2, ..., u_s) \) constructed by four sequences of length \( n \), with \( NPAF = 0 \) and \( n \) a non negative integer. Then there exists an orthogonal design \( OD(4ms(n + 1); u_1w, u_2w, ..., u_tw) \), \( \forall (m, w) \in (M_2, W_2) \).

**Proof.** Let \( A_i, i = 1, 2, 3, 4, \) be the four sequences of length \( s \) with \( NPAF = 0 \) from which the orthogonal design \( D_4 \) is constructed. Thus from Theorem 4 we have four directed sequences of length \( mn \) and type \( (w, w, w, w) \) with zero PAF. If we replace \( a \) by \( A_1 \), \( b \) by \( A_2 \), \( c \) by \( A_3 \), \( d \) by \( A_4 \) and 0 by 0, then we get four sequences of length \( \ell = ms(n + 1) \), and type \( (s_1, s_2, ..., s_4) \) where \( s_i = wu_i, i = 1, 2, ..., 4 \) and \( PAF = 0 \). Hence using these sequences we obtain an orthogonal design \( OD(4ms(n + 1); u_1w, u_2w, ..., u_tw) \).

**Example 6.** Suppose we wish to construct an \( OD(132; 9, 18, 27, 54) \). Using Theorem 4 with \( m = 11, n = 0, w = 9 \) we get four directed sequences of length \( m(n + 1) \) of type \( (w, w, w, w) \) with zero PAF. These are

\[ A = \{ a, b, 0, b, 0, a, b, a, b, a \} \]

\[ B = \{ b, a, b, a, b, 0, a, b, a, b \} \]

\[ C = \{ c, \bar{d}, 0, \bar{d}, 0, c, \bar{d}, \bar{d}, \bar{c} \} \]

\[ D = \{ \bar{c}, \bar{d}, c, \bar{d}, 0, c, \bar{c}, 0, \bar{d} \} \]

The sequences

\[ X = \{ d, a, d \} \]

\[ Y = \{ e, d, b \} \]

\[ Z = \{ \bar{e}, d, b \} \]

\[ W = \{ d, c, d \} \]
have zero NPAF so in the directed sequences $A, B, C, D$ we can replace their variables $a, b, c, d$ by the sequences $X, Y, Z, W$ respectively and 0 by $0_3$ and get the following sequences

$$A_1 = \{d, a, -d, c, -d, b, 0, 0, c, -d, b, c, -d, b, 0, 0, 0, -d, c, -d, b, c, -d, b, 0, 0, 0, -d, c, -d, b, c, -d, b, 0, 0, 0, 0, c, -d, b, 0, 0, 0, -d, c, -d, b, c, -d, b, 0, 0, 0, 0, c, -d, b, 0, 0, 0, 0, c, -d, b, -d, -a, d, -b, -d, -a, d\}$$

$$A_2 = \{c, -d, b, -d, -a, d, c, -d, b, d, a, -d, c, -d, b, 0, 0, 0, d, a, -d, c, -d, b\}$$

$$A_3 = \{-c, d, b, c, d, a, -d, d, a, -d, 0, 0, 0, d, a, -d, c, -d, b, -d, c, -d, b\}$$

$$A_4 = \{d, c, d, c, -d, -b, d, c, d, -c, d, b, -d, -c, -d, 0, 0, 0, d, b, -d, c, -d, d\}$$

of length 33 and type $(9, 18, 27, 54)$ with zero PAF which can be used to construct an $OD(132; 9, 18, 27, 54)$.

### 3. MAIN RESULTS

In this section we use base sequences of lengths $n + 1, n + 1, n, n$, i.e., $BS(n + 1, n)$ in order to construct four directed sequences of lengths $n + 1, n + 1, n, n$ and type $(2n + 1, 2n + 1)$ with zero NPAF. We also use normal sequences of length $n$ in order to construct four directed sequences of length $2n$ and type $(2n, 2n, 2n, 2n)$ with zero NPAF. We first give the definition of base and normal sequences, see also [3, 5].

**Definition 1.** Four $(1, -1)$ sequences $X, Y, Z, W$ of lengths $n + p, n + p, n, n$ are base sequences, (abbreviated as $BS(n + p, n)$) if

$$N_X(s) + N_Y(s) + N_Z(s) + N_W(s) = \begin{cases} 0, & s = 1, \ldots, n - 1 \\ 4n + 2p, & s = 0 \end{cases}$$

$$N_X(s) + N_Y(s) = 0, \quad s = n, \ldots, n + p - 1.$$
Proof. (\("\Rightarrow\) ) Suppose \(A, B, C, D\) are four directed sequences of lengths \(n+1, n+1, n, n\) and type \((2n+1, 2n+1)\) with zero NPAF. Then we replace both variables by 1 and thus we obtain the required base sequences \(BS(n+1, n)\).

\((\Rightarrow\) ) Let \(X = \{x_1, ..., x_n, x_{n+1}\}, Y = \{y_1, ..., y_n, y_{n+1}\}, Z = \{z_1, ..., z_n\}, W = \{w_1, ..., w_n\}\) be base sequences of lengths \(n+1, n+1, n, n\). Set

\[
A = \left\{ a \frac{x_1 + y_1}{2} + b \frac{x_1 - y_1}{2}, ..., a \frac{x_n + y_n}{2} + b \frac{x_n - y_n}{2}, \right.
\]

\[
B = \left\{ b \frac{x_{n+1} + y_{n+1}}{2} - a \frac{x_{n+1} - y_{n+1}}{2}, b \frac{x_n + y_n}{2} - a \frac{x_n - y_n}{2}, \right. 
\]

\[
C = \left\{ a \frac{z_1 + w_1}{2} + b \frac{z_1 - w_1}{2}, ..., a \frac{z_{n-1} + w_{n-1}}{2} + b \frac{z_{n-1} - w_{n-1}}{2}, \right. 
\]

\[
D = \left\{ b \frac{z_{n+1} + w_{n+1}}{2} - a \frac{z_{n+1} - w_{n+1}}{2}, b \frac{z_n + w_n}{2} - a \frac{z_n - w_n}{2}, \right. 
\]

Then we have

\[N_A(s) = a^2 \sum_{i=1}^{n+1-s} \left( \frac{x_i + y_i}{2} \right) \left( \frac{x_{i+s} + y_{i+s}}{2} \right) \]

\[+ b^2 \sum_{i=1}^{n+1-s} \left( \frac{x_i - y_i}{2} \right) \left( \frac{x_{i+s} - y_{i+s}}{2} \right) \]

\[+ ab \sum_{i=1}^{n+1-s} \left( \frac{x_i + y_i}{2} \right) \left( \frac{x_{i+s} - y_{i+s}}{2} \right) \]

\[+ ba \sum_{i=1}^{n+1-s} \left( \frac{x_i - y_i}{2} \right) \left( \frac{x_{i+s} + y_{i+s}}{2} \right) \]

\[N_B(s) = b^2 \sum_{i=1}^{n+1-s} \left( \frac{x_i + y_i}{2} \right) \left( \frac{x_{i+s} + y_{i+s}}{2} \right) \]

\[+ a^2 \sum_{i=1}^{n+1-s} \left( \frac{x_i - y_i}{2} \right) \left( \frac{x_{i+s} - y_{i+s}}{2} \right) \]

\[+ ab \sum_{i=1}^{n+1-s} \left( \frac{x_i + y_i}{2} \right) \left( \frac{x_{i+s} - y_{i+s}}{2} \right) \]

\[+ ba \sum_{i=1}^{n+1-s} \left( \frac{x_i - y_i}{2} \right) \left( \frac{x_{i+s} + y_{i+s}}{2} \right) \]
\[ N_A(s) = a^2 \left( \sum_{i=1}^{n+1-s} \left( \frac{X_{i+s} - Y_{i+s}}{2} \right) \left( \frac{X_i - Y_i}{2} \right) \right) \]
\[ + b^2 \left( \sum_{i=1}^{n+1-s} \left( \frac{X_{i+s} + Y_{i+s}}{2} \right) \left( \frac{X_i + Y_i}{2} \right) \right) \]
\[ - ab \left( \sum_{i=1}^{n+1-s} \left( \frac{X_{i+s} - Y_{i+s}}{2} \right) \left( \frac{X_i + Y_i}{2} \right) \right) \]
\[ - ba \left( \sum_{i=1}^{n+1-s} \left( \frac{X_{i+s} + Y_{i+s}}{2} \right) \left( \frac{X_i - Y_i}{2} \right) \right) \]
\[ N_C(s) = a^2 \left( \sum_{i=1}^{n+1-s} \left( \frac{z_i + W_i}{2} \right) \left( \frac{z_{i+s} + W_{i+s}}{2} \right) \right) \]
\[ + b^2 \left( \sum_{i=1}^{n+1-s} \left( \frac{z_i - W_i}{2} \right) \left( \frac{z_{i+s} - W_{i+s}}{2} \right) \right) \]
\[ + ab \left( \sum_{i=1}^{n+1-s} \left( \frac{z_i + W_i}{2} \right) \left( \frac{z_{i+s} - W_{i+s}}{2} \right) \right) \]
\[ + ba \left( \sum_{i=1}^{n+1-s} \left( \frac{z_i - W_i}{2} \right) \left( \frac{z_{i+s} + W_{i+s}}{2} \right) \right) \]
\[ N_D(s) = a^2 \left( \sum_{i=1}^{n+1-s} \left( \frac{z_{i+s} - W_{i+s}}{2} \right) \left( \frac{z_i - W_i}{2} \right) \right) \]
\[ + b^2 \left( \sum_{i=1}^{n+1-s} \left( \frac{z_{i+s} + W_{i+s}}{2} \right) \left( \frac{z_i + W_i}{2} \right) \right) \]
\[ - ab \left( \sum_{i=1}^{n+1-s} \left( \frac{z_{i+s} - W_{i+s}}{2} \right) \left( \frac{z_i + W_i}{2} \right) \right) \]
\[ - ba \left( \sum_{i=1}^{n+1-s} \left( \frac{z_{i+s} + W_{i+s}}{2} \right) \left( \frac{z_i - W_i}{2} \right) \right) \].

Thus

\[ N_A(s) + N_B(s) = (a^2 + b^2) \left( \frac{N_X(s) + N_Y(s)}{2} \right) = 0, \quad \text{for} \quad s = n \]
and

\[ N_4(s) + N_6(s) + N_C(s) + N_D(s) \]
\[ = (a^2 + b^2) \left( \frac{N_X(s) + N_Y(s) + N_Z(s) + N_W(s)}{2} \right) = 0, \]

for \( s = 1, \ldots, n - 1 \). The coefficients of \( a^2, b^2, ab, ba \) are zero. So, \( N_4(s) + N_6(s) + N_C(s) + N_D(s) = 0, \ s = 1, 2, \ldots, 2n - 1 \) without relying on commutativity of the variables. Thus, \( A, B, C, D \) are four directed sequences of lengths \( n + 1, n + 1, n, n \) and type \((2n + 1, 2n + 1)\) with NPAF = 0.

**Example 7.** The following sequences are four directed sequences of lengths 34, 34, 33, 33 and type \((67, 67)\) with zero NPAF,

\[
X = \{ b, a, b, b, a, b, b, a, a, b, b, a, b, a, b, a, b, b, a \}
\]
\[
Y = \{ \bar{b}, \bar{a}, b, a, b, a, b, a, b, a, b, a, b, a, b, a, b, a \}
\]
\[
Z = \{ \bar{a}, \bar{a}, a, b, a, a, b, a, a, b, a, a, b, a, a, b, a, a \}
\]
\[
W = \{ \bar{\bar{b}}, \bar{b}, a, b, b, a, \bar{\bar{b}}, a, b, a, a, b, a, b, a, b, a \}
\]

**Corollary 4.** There are four directed sequences of lengths \( n + 1, n + 1, n, n \) and type \((2n + 1, 2n + 1)\) with zero NPAF for all \( n = 0, 1, \ldots, 33 \) and \( n = g \) where \( g \) is a Golay number.

**Corollary 5.** If there are four directed sequences of lengths \( n + 1, n + 1, n, n \) and type \((2n + 1, 2n + 1)\) with zero NPAF then there are four directed sequences of lengths \( n + 1, n + 1, n + 1, n + 1 \) and type \((2n + 1, 2n + 1)\) with zero NPAF.

**Corollary 6.** If there are four directed sequences of lengths \( n + 1, n + 1, n, n \) and type \((2n + 1, 2n + 1)\) with zero NPAF then there are four directed sequences of lengths \( 2n + 1, 2n + 1, 2n + 1, 2n + 1 \) and type \((4n + 2, 4n + 2)\) with zero NPAF.
Proof. We use the directed sequences $AC; A\bar{C}; BD; B\bar{D}$.

Corollary 7. If there are base sequences BS($n+1, n$) and normal sequences of length $m$ then there are four directed sequences of length $m(2n+1)$ and type $(2m(2n+1), 2m(2n+1))$ with zero NPAF.

Proof. We combine Theorems 5 and 6.

Corollary 8. If there are base sequences BS$(n+1, n)$, normal sequences of length $m$ and an orthogonal design OD$(2^k; x_1, x_2)$ constructed from two sequences with zero NPAF then there exist an orthogonal design OD$(4km(2n+1); 2x_1m(2n+1), 2x_2m(2n+1))$.

Definition 2. A triple $(F, G, H)$ of sequences is said to be a set of normal sequences for length $n$ (abbreviated as NS$(n)$) if the following conditions are satisfied.

(i) $F = (f_k)$ is a $(1,1)$ sequence of length $n$.
(ii) $G = (g_k)$ and $H = (h_k)$ are sequences of length $n$ with entries 0, 1, &1, such that $G + H = (g_k + h_k)$ is a $(1,1)$ sequence of length $n$.
(iii) $N_F(s) + N_G(s) + N_H(s) = 0$, $s = 1, ..., n$. 

Theorem 6. Let $F, G, H$ be normal sequences of length $n$ (NS$(n)$). Then there are four sequences $X, Y, Z, W$ of length $2n$ and type $(2n, 2n, 2n, 2n)$ with NPAF $= 0$ which are directed.

Proof. We define

$X = \{af_n, cg_1 + dh_1, af_{n-1}, cg_2 + dh_2, ..., af_1, cg_n + dh_n\}$

$Y = \{ag_n - bh_1, -cf_1, ag_{n-1} - bh_2, -cf_2, ..., ag_1 - bh_n, -cf_n\}$

$Z = \{bf_n, dg_n - ch_n, bf_{n-1}, dg_{n-1} - ch_{n-1}, ..., bf_1, dg_1 - ch_1\}$

$W = \{bg_1 + ah_n, -df_1, bg_2 + ah_{n-1}, -df_2, ..., bg_n + ah_1, -df_n\}$

We will prove that:

(i) $N_X(2n-2k) + N_Y(2n-2k) + N_Z(2n-2k) + N_W(2n-2k) = 0$, $k = 1, 2, ..., n - 1$

(ii) $N_X(2n-2k+1) + N_Y(2n-2k+1) + N_Z(2n-2k+1) + N_W(2n-2k+1) = 0$, $k = 1, 2, ..., n$

and that this result do not rely on commutativity of the variables.
\( N_x(2n-2k) + N_y(2n-2k) + N_x(2n-2k) + N_w(2n-2k) \)

\[ = a^2(N_x(n-k) + N_v(n-k) + N_w(n-k)) \]

\[ + b^2((N_x(n-k) + N_v(n-k) + N_w(n-k)) \]

\[ + c^2(N_x(n-k) + N_v(n-k) + N_w(n-k)) \]

\[ + d^2(N_x(n-k) + N_v(n-k) + N_w(n-k)) \]

\[ + cd \left( \sum_{i=1}^{k} g_i h_{n-k+i} - \sum_{i=1}^{k} h_{n-k+i} g_i \right) \]

\[ + dc \left( \sum_{i=1}^{k} h_i g_{n-k+i} - \sum_{i=1}^{k} g_{n-k+i} h_i \right) \]

\[ + ab \left( \sum_{i=1}^{k} h_{n-k+i} g_{n-i+1} - \sum_{i=1}^{k} g_{n-i+1} h_{n-k+i} \right) \]

\[ + ba \left( \sum_{i=1}^{k} h_{i} g_{k-i+1} - \sum_{i=1}^{k} g_{k-i+1} h_i \right) = 0. \]

The coefficients of \( a^2, b^2, c^2, d^2, cd, dc, ab, ba \) are zero. So,

\( N_x(2n-2k) + N_y(2n-2k) + N_x(2n-2k) + N_w(2n-2k) = 0 \)

without relying on commutativity of the variables.

(ii) \( N_x(2n-2k+1) + N_y(2n-2k+1) \)

\[ + N_x(2n-2k+1) + N_w(2n-2k+1) \]

\[ = ac \left( \sum_{i=1}^{k} f_{n-i+1} g_{n-k+i} - \sum_{i=1}^{k} g_{n-k+i} f_{n-i+1} \right) \]

\[ + ca \left( \sum_{i=1}^{k} g_i f_{k-i} - \sum_{i=1}^{k} f_{k-i} g_i \right) \]

\[ + ad \left( \sum_{i=1}^{k} f_{n-i+1} h_{n-k+i} - \sum_{i=1}^{k} h_{n-k+i} f_{n-i+1} \right) \]

\[ + da \left( \sum_{i=1}^{k-1} h_i f_{k-i} - \sum_{i=1}^{k-1} f_{k-i} h_i \right) \]

\[ + bc \left( \sum_{i=1}^{k} h_i f_{n-k+i} - \sum_{i=1}^{k} f_{n-k+i} h_i \right) \]
The coefficients of $ac$, $ca$, $ad$, $da$, $bc$, $cb$, $bd$, $db$, are zero. So,

$$N_X(2n-2k+1) + N_Y(2n-2k+1) + N_Z(2n-2k+1) + N_W(2n-2k+1) = 0$$

without relying on commutativity of the variables.

Then $N_X(s) + N_Y(s) + N_Z(s) + N_W(s) = 0, \quad s = 1, 2, ..., 2n-1$ without relying on commutativity of the variables. So, $X, Y, Z, W$ are four directed sequences of length $2n$ and type $(2n, 2n, 2n)$ with $NPAF = 0$.  

**Example 8.** Let $F = \{-1, 1, 1\}$, $G = \{1, 0, 1\}$, $H = \{0, 1, 0\}$ be normal sequences of length 3 (NS(3)). Then we have that $X = \{a, c, a, d, a, c\}$, $Y = \{a, c, b, c, a, c\}$, $Z = \{b, d, a, d, b, d\}$, $W = \{b, d, a, d, b, d\}$ are four directed sequences of length 6 and type (6, 6, 6).

**Remark 1.** If we have directed sequences with $NPAF = 0$ ($PAF = 0$) and of type $(w, w, w, w)$ then their variables can be replaced by sequences with zero NPAF. Furthermore if the sequences we plug in are also directed with $NPAF = 0$ then the derived sequences are directed with $NPAF = 0$ ($PAF = 0$) as well.

In Table I, we give the directed sequences of length $2n$ and type $(2n, 2n, 2n, 2n)$ with $NPAF = 0$, which are constructed from normal sequences of length $n$ (see [5] and [3]), for $2n = 2, 6, 10, 14, 18, 22, 26, 30, 38, 50, 58$.

Let $M_1 = \{n \text{ is a positive integer: } n = 2^a \cdot 6^b \cdot 10^c \cdot 14^d \cdot 18^e \cdot 22^f \cdot 26^g \cdot 30^h \cdot 38^i \cdot 50^j \cdot 58^k\}$.

**Corollary 9.** Let $a_1, a_2, ..., a_1$ be positive integers or zero. Then there exist four directed sequences of length $m$ and type $(m, m, m, m)$ for all $m \in M_1$.

**Proof.** Use the sequences given in Table I with Remark 1.  

<table>
<thead>
<tr>
<th>$2n$</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$ae$</td>
<td>$a\bar{e}$</td>
<td>$bd$</td>
<td>$bd$</td>
</tr>
<tr>
<td>6</td>
<td>$ac\bar{e}d\bar{b}c\bar{d}$</td>
<td>$bd\bar{b}\bar{d}c\bar{d}$</td>
<td>$\bar{a}\bar{e}c\bar{e}$</td>
<td>$bd\bar{a}d\bar{b}$</td>
</tr>
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<td>$ac\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{c}d\bar{a}b\bar{e}$</td>
<td>$\bar{a}e\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{a}d\bar{b}$</td>
</tr>
<tr>
<td>14</td>
<td>$ac\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{c}d\bar{a}b\bar{e}$</td>
<td>$\bar{a}e\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{a}d\bar{b}d\bar{a}$</td>
</tr>
<tr>
<td>18</td>
<td>$ac\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{c}d\bar{a}b\bar{e}$</td>
<td>$\bar{a}e\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{a}d\bar{b}d\bar{a}$</td>
</tr>
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<td>$ac\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{c}d\bar{a}b\bar{e}$</td>
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<td>$bd\bar{a}d\bar{b}d\bar{a}$</td>
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<tr>
<td>30</td>
<td>$ac\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{c}d\bar{a}b\bar{e}$</td>
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<td>$bd\bar{a}d\bar{b}d\bar{a}$</td>
</tr>
<tr>
<td>34</td>
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<td>$bd\bar{c}d\bar{a}b\bar{e}$</td>
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<td>$bd\bar{a}d\bar{b}d\bar{a}$</td>
</tr>
<tr>
<td>54</td>
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<td>$bd\bar{c}d\bar{a}b\bar{e}$</td>
<td>$\bar{a}e\bar{b}d\bar{c}a\bar{e}$</td>
<td>$bd\bar{a}d\bar{b}d\bar{a}$</td>
</tr>
</tbody>
</table>
Corollary 10. Let $D_1$ be an orthogonal design $OD(4m; u_1, u_2, ..., u_t)$ constructed by four sequences of length $n$, with $NPAF=0$. Then there exists an orthogonal design $OD(4nm; u_1m, u_2m, ..., u_tm), \forall m \in M_1$.

Proof. Let $A_i, i = 1, 2, 3, 4$ be the four sequences of length $n$ from which the orthogonal design $D_1$ is constructed and $m \in M_1$. We know that these sequences have $NPAF=0$. Using the previous corollary we obtain four directed sequences of length $m$ and type $(m, m, m, m)$ with $NPAF=0$. As the derived sequences are directed of type $(m, m, m, m)$ their variables can be replaced by sequences with zero $NPAF$. If we replace $a$ by $A_1$, $b$ by $A_2$, $c$ by $A_3$, $d$ by $A_4$ and 0 by 0, then we get four sequences of length $\ell = nm$, and type $(s_1, s_2, ..., s_t)$ where $s_i = mu_i, i = 1, 2, ..., t$ with $NPAF=0$. Hence using these sequences we obtain an orthogonal design $OD(4nm; u_1m, u_2m, ..., u_tm)$.

Example 9. Suppose we wish to construct an orthogonal design $OD(72; 6, 12, 18, 36)$ and an $OD(72; 18, 18, 18, 18)$. From Table I we have four directed sequences $A, B, C, D$ of length 6 and type $(6, 6, 6, 6)$.

Set $X = \{d, a, d\}$, $Y = \{c, d, b\}$, $Z = \{c, d, b\}$, $W = \{d, c, d\}$. Then $X, Y, Z, W$ have $NPAF=0$ and can be used to construct an $OD(12; 1, 2, 3, 6)$. Now, if in the four directed sequences $A, B, C, D$ we replace $a$ by sequence $X$, $b$ by $Y$, $c$ by $Z$, and $d$ by sequence $W$, then we obtain four sequences of length 6 - 3 and type $(1, 6, 2, 6, 3, 6, 6, 6)$. Thus, we have the orthogonal design $OD(72; 6, 12, 18, 36)$.

Set $X = \{d, a, b\}$, $Y = \{a, b, d\}$, $Z = \{a, c, d\}$, $W = \{c, d, b\}$. Then $X, Y, Z, W$ have $NPAF=0$ and can be used to construct an $OD(12; 3, 3, 3, 3)$. Now, if in the four directed sequences $A, B, C, D$ we replace $a$ by sequence $X$, $b$ by $Y$, $c$ by $Z$, and $d$ by sequence $W$, then we obtain four sequences of length 6 - 3 and type $(3, 6, 3, 6, 3, 6)$ Thus, we have the orthogonal design $OD(72; 18, 18, 18, 18)$.

Example 10. Suppose we wish to construct an orthogonal design $OD(144; 12, 24, 36, 72)$. From Table I we have four directed sequences $A, B, C, D$ of length 6 and type $(6, 6, 6, 6)$. Also from Table I, $\{a, c\}, \{a, c\}, \{b, d\}, \{b, d\}$ are four directed sequences of length 2 and type $(2, 2, 2, 2)$. So, in these sequences if we replace $a$ by $X$, $b$ by $Y$, $c$ by $Z$, and $d$ by $W$ then we get four directed sequences $A, B, C, D$ of length 12 and type $(12, 12, 12, 12)$. As the derived sequences are directed of type $(w, w, w, w)$ their variables can be replaced by sequences.

Set $X = \{d, a, d\}$, $Y = \{c, d, b\}$, $Z = \{c, d, b\}$, $W = \{d, c, d\}$. Then $X, Y, Z, W$ have $NPAF=0$ and can be used to construct an $OD(12; 1, 2, 3, 6)$. Now, if in the four directed sequences $A, B, C, D$ we replace $a$ by sequence
\[X, b \text{ by } Y, c \text{ by } Z, \text{ and } d \text{ by sequence } W, \text{ then we obtain four sequences of length } 12 \cdot 3 \text{ and type } (1 \cdot 12, 2 \cdot 12, 3 \cdot 12, 6 \cdot 12). \text{ Thus, we have the orthogonal design } OD(144; 12, 24, 36, 72).\]

**Theorem 7.** Let \( a_1, a_2, ..., a_4, n \) be positive integers or zero. There exist four directed sequences of length \((n + 1)m_m m_2\) and type \((w_2 w_1, w_2 m_1, w_2 m_1, w_2 m_1)\) with zero PAF where \( m \in M \) and the corresponding \( w \in W, m_1 \in M_1 \) and \((m_2, w_2) \in (M_2, W_2).\)

**Proof.** Using Theorem 3 we can construct four directed sequences \( A_1, A_2, A_3, A_4 \) of length \( m \) and type \((w, w, w, w).\) Using Corollary 9 we can construct four directed sequences \( B_1, B_2, B_3, B_4 \) of length \( m_1 \) and type \((m_1, m_1, m_1, m_1).\) From Remark 1 using sequences \( A_1, A_2, A_3, A_4 \) and \( B_1, B_2, B_3, B_4 \) we get four directed sequences \( C_1, C_2, C_3, C_4 \) of length \( m m_1 \) and type \((m m_1, m m_1, m m_1, m m_1)\) with zero NPAF.

From Theorem 4 we can construct four directed sequences \( D_1, D_2, D_3, D_4 \) of length \((n + 1)m_2\) and type \((w_2, w_2, w_2, w_2)\) with zero PAF.

Now if we replace the variables \( a, b, c, d \) and zeros in \( D_1, D_2, D_3, D_4 \) by sequences \( C_1, C_2, C_3, C_4, 0_{mm}, \) respectively we obtain the result. \( \blacksquare \)

**Corollary 11.** Let \( D_1 \) be an orthogonal design \( OD(4s; u_1, u_2, ..., u_s) \) constructed by four sequences of length \( n, \) with NPAF = 0. Then there exists an orthogonal design \( OD(4s(n + 1)mm_1 m_2; u_1, w_2 m_1, u_2 w_2 m_1, ..., u_s w_2 m_1) \)

where \( m \in M \) and the corresponding \( w \in W, m_1 \in M_1 \) and \((m_2, w_2) \in (M_2, W_2).\)

**Proof.** Let \( A, B, C, D \) be the four sequences of length \( s \) and type \((u_1, u_2, \ldots, u_s)\) from which the orthogonal design is constructed. If we replace the variables \( a, b, c, d \) of the directed sequences of length \((n + 1)mm_1 m_2\) and type \((w_2 m_1, w_2 m_1, w_2 m_1, w_2 m_1)\) constructed by Theorem 7 by the sequences \( A, B, C, D \) respectively, then we obtain four sequences of length \( s(n + 1)mm_1 m_2\) and type \((u_1 w_2 m_1, u_2 w_2 m_1, \ldots, u_s w_2 m_1)\) with zero PAF. These sequences can be used in the Goethals-Seidel array to obtain the desirable orthogonal design. \( \blacksquare \)

**Remark 2.** If \( X_1, X_2, ..., X_m \) are \( m \) directed sequences of lengths \( n_1, n_2, ..., n_m \) which are consist of \( t_1 \) variables and \( Y_1, Y_2, ..., Y_k \) are \( k \) directed sequences of lengths \( p_1, p_2, ..., p_k \) which are consist of \( t_2 \) variables then \( \{X_i, Y_j, i = 1, 2, ..., m, j = 1, 2, ..., k\} \) are \( m + k \) directed sequences of lengths \( n_1, n_2, ..., n_m, p_1, p_2, ..., p_k \) which are consist of \( t_1 + t_2 \) variables.
Corollary 12. If we have \( m \) directed sequences of lengths \( n_1, n_2, \ldots, n_m \) which consist of \( t \) variables, then we can obtain \( k \cdot m \) directed sequences of lengths \( n_1, n_2, \ldots, n_m \) \((k \text{ sequences will be of lengths } n_i, i = 1, 2, \ldots, m)\) which consist of \( k \cdot t \) variables for \( k = 1, 2, \ldots \).

Proof. Suppose \( Y_{11}, Y_{12}, \ldots, Y_{1m} \) are \( m \) directed sequences of lengths \( n_1, n_2, \ldots, n_m \) which consist of \( t \) variables say \( a_{11}, a_{12}, \ldots, a_{1t} \). Choose a \( k \in \{1, 2, 3, \ldots\} \) and write these sequences \( k \) times renaming the variables and sequences each time. Finally we will have

\[
Y_{11}, Y_{12}, \ldots, Y_{1m} \text{ are } m \text{ directed sequences of lengths } n_1, n_2, \ldots, n_m \text{ which consist of } t \text{ variables say } a_{11}, a_{12}, \ldots, a_{1t}.
\]

\[
Y_{11}, Y_{12}, \ldots, Y_{1m} \text{ are } m \text{ directed sequences of lengths } n_1, n_2, \ldots, n_m \text{ which consist of } t \text{ variables say } a_{21}, a_{22}, \ldots, a_{2t}.
\]

\[
Y_{11}, Y_{12}, \ldots, Y_{1m} \text{ are } m \text{ directed sequences of lengths } n_1, n_2, \ldots, n_m \text{ which consist of } t \text{ variables say } a_{31}, a_{32}, \ldots, a_{3t}.
\]

\[
\ldots
\]

\[
Y_{11}, Y_{12}, \ldots, Y_{1m} \text{ are } m \text{ directed sequences of lengths } n_1, n_2, \ldots, n_m \text{ which consist of } t \text{ variables say } a_{k1}, a_{k2}, \ldots, a_{kt}.
\]

Totally the above are \( k \cdot m \) directed sequences of lengths \( n_1, n_2, \ldots, n_m \) \((k \text{ sequences will be of lengths } n_i, i = 1, 2, \ldots, m)\) which consist of \( k \cdot t \) variables, for \( k = 1, 2, \ldots \).

Example 11. From Table 13 of [4] we have that \( \{a_1, a_1, a_2, a_2\} \) and \( \{a_1, 0, a_1, a_2, a_2\} \) are two directed sequences of length \( n = 6 \) and type \((5, 5)\). Let \( k = 3 \), then the following sequences

\[
\{a_1, a_1, a_1, a_2, 0, a_2\}, \quad \{a_1, 0, a_1, a_2, a_2, a_2\},
\]

\[
\{a_3, a_3, a_4, a_4, 0, a_4\}, \quad \{a_3, 0, a_3, a_4, a_4, a_4\},
\]

\[
\{a_5, a_5, a_6, a_6, 0, a_6\}, \quad \{a_5, 0, a_5, a_6, a_6, a_6\}
\]

are \( k \cdot m = 3 \cdot 2 = 6 \) directed sequences of length \( n = 6 \) which consist of \( k \cdot t = 3 \cdot 2 = 6 \) variables.

It is obvious, that the above multiplication can be applied on one, two or three variable orthogonal designs as well. These multiplication methods give many new orthogonal designs, particularly for higher orders.
REFERENCES