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On the kth Laplacian eigenvalues of trees with perfect matchings ☆

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ABSTRACT

Let $\mathcal{T}_n^+$ be the set of all trees of order n with perfect matchings. In this paper, we prove that for any tree $T \in \mathcal{T}_n^+$, its kth largest Laplacian eigenvalue $\mu_k(T)$ satisfies $\mu_k(T) = 2$ when $n = 2k$, and $\mu_k(T) \leqslant \frac{\lceil \frac{n}{2k} \rceil + 2 + \sqrt{(\lceil \frac{n}{2k} \rceil)^2 + 4}}{2}$ when $n \neq 2k$. Moreover, this upper bound is sharp when $n = 0(mod\ 2k)$.

1. Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Denote by $d_i = d(v_i)$ the degree of vertex $v_i \in V(G)$. Let $A(G)$ and $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. Then the matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the graph G. Moreover, the eigenvalues of $L(G)$ are called the Laplacian eigenvalues of G. It is well known that $L(G)$ is positive semi-definite symmetric and singular. Moreover, since G is connected, $L(G)$ is irreducible. Denote its eigenvalues by

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$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0,$$

which are always enumerated in non-increasing order and repeated according to their multiplicity. We shall use the notation $\mu_k(G)$ to denote the kth Laplacian eigenvalue of the graph G.

Two distinct edges in a graph G incident to the same vertex will be called adjacent edges. A matching of G is a set of edges in G such that no two of which are adjacent. The largest matching is called a maximum matching. The cardinality of a maximum matching of G is commonly known as its matching number, denoted by $\alpha'(G)$. Let M be a matching of G. M is called the s-matching of G if M contains exactly s edges of G. A vertex $v \in V(G)$ is said to be M-saturated if it incident with an edge of M, otherwise v is called an M-unsaturated vertex. The matching M of G is called a perfect matching if all vertices of G are M-saturated.

Throughout this paper, we always denote $\mathcal{T}_n^+$ the set of all trees of order n with perfect matchings. We use $G - x$ and $G - xy$ to denote the graphs obtained by deleting the vertex $x \in V(G)$ and the edge $xy \in E(G)$ from G, respectively. Similarly, $G + xy$ is the graph obtained by adding an edge $xy \notin E(G)$ to G, where $x, y \in V(G)$. Other undefined notations may be referred to [1].

For any graph G of order n, it is well known that $\mu_1(G) \leqslant n$, and the equality holds if and only if the complement of G is disconnected. Thus, if T is a tree of order n, then $\mu_1(T) \leqslant n$, and the equality holds if and only if $T = K_{1,n-1}$, the star on $n \geqslant 2$ vertices. For a real number x, let $\lceil x \rceil$ denote the least integer not less than x. Guo [2] studied the kth Laplacian eigenvalue of a tree, and got the following result.

Theorem 1.1. *Let T be a tree of order n. Then $\mu_k(T) \leqslant \lceil \frac{n}{k} \rceil$, and the equality holds if and only if $k < n$, $n = 0 (mod\ k)$ and T is spanned by k vertex disjoint copies of $K_{1,\frac{n}{k}-1}$.*

In this paper, we investigate the kth Laplacian eigenvalue of a tree with a perfect matching, and obtain the following result:

Let $T \in \mathcal{T}_n^+$. Then $\mu_k(T) = 2$ when $n = 2k$, and $\mu_k(T) \leqslant \frac{\lceil \frac{n}{2k} \rceil + 2 + \sqrt{(\lceil \frac{n}{2k} \rceil)^2 + 4}}{2}$ when $n \neq 2k$. Moreover, this upper bound is sharp for $n = 0 (mod\ 2k)$.

2. Preliminaries

Let G be a graph of order n and let $G' = G + e$ be the graph obtained from G by adding a new edge e into G. Then $L(G') = L(G) + zz^T$, where z is a column n-vector with two non-zero entries $1, -1$ in suitable places. The next Lemma follows from the well-known Courant–Weyl inequalities and the fact that $\mu_n(zz^T) = 0$, where $\mu_n(zz^T)$ is the least eigenvalue of zz^T.

Lemma 2.1 [3]. *The Laplacian eigenvalues of G and G' interlace, that is,*

$$\mu_1(G') \geqslant \mu_1(G) \geqslant \mu_2(G') \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G') = \mu_n(G) = 0.$$

Lemma 2.2 [3]. *Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and B be a principal submatrix of A. Let B has eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$. Then the inequalities $\lambda_{n-m+i} \leqslant \mu_i \leqslant \lambda_i$ $(i = 1, 2, \ldots, m)$ hold.*

Lemma 2.3 [4]. *Let T be a tree of order n. Then for any positive integer a, there exists a vertex $v \in V(T)$ such that there is one component of $T - v$ with order not exceeding* $\max\{n - 1 - a, a\}$ *and all the other components of $T - v$ have order not exceeding a.*

Lemma 2.4 [5]. *Let T be a tree on $n = 2k$ vertices. Then $\mu_k(T) \leqslant 2$, and the equality holds if and only if T has a perfect matching.*

Lemma 2.5 [6]. *Let T be a tree on n $(n \geqslant 3)$ vertices with $\alpha'(T) = m$. Then $\mu_1(T) \leqslant \mu_1(T(n, m))$, where $\mu_1(T(n, m))$ is the largest root of the equation*

$$x^3 - (n - m + 4)x^2 + (3n - 3m + 4)x - n = 0.$$

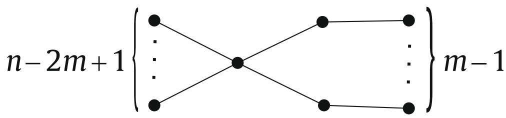

Fig. 1. $T(n, m)$.

The equality holds if and only if $T \cong T(n, m)$ *(see Fig. 1).*

When T is a tree on two vertices, i.e., $T = K_2$, it is easy to see that $\mu_1(K_2) = 2$. When T is a tree of order $2t$ ($t \geqslant 2$) and has a perfect matching, then $\alpha'(T) = t$. Hence Lemma 2.5 implies that $\mu_1(T) \leqslant \mu_1(T(2t, t)) = \frac{t+2+\sqrt{t^2+4}}{2}$, and the equality holds if and only if $T \cong T(2t, t)$.

3. Main results

In this section, we give a sharp upper bound for the kth Laplacian eigenvalue of a tree with a perfect matching. Our results is as follows.

Lemma 3.1. *Let* $T \in \mathcal{T}_n^+$ *and* $n = k(2t)$. *Then* $\mu_k(T) = 2$ *when* $t = 1$ *and* $\mu_k(T) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}$ *when* $t > 1$.

Proof. When $t = 1$, then $n = 2k$. Lemma 2.4 implies that $\mu_k(T) = 2$. When $t > 1$, fix t. We proceed by induction on k. By Lemma 2.5, the result is obvious for $k = 1$. Assume that for any tree T' on $(k-1)(2t)$ vertices, we have $\mu_{k-1}(T') \leqslant \frac{t+2+\sqrt{t^2+4}}{2}$, $k \geqslant 2$. Now let T be a tree on $n = k(2t)$ vertices. We only need to show that $\mu_k(T) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}$.

Let $a = 2t - 1$ in Lemma 2.3. Then $a \leqslant n - a - 1$, and so there exists a vertex $v \in V(T)$ such that there is one component T_0 of $T - v$ with order not exceeding

$$n - a - 1 = n - 2t = (k-1)(2t),$$

and all the other components of $T - v$, say T_j ($j = 1, 2, \ldots, s$), have order not exceeding $2t - 1$. Suppose that $v_0, v_1, \ldots, v_s$ are the vertices of T and $v_j \in V(T_j)$, $vv_j \in E(T)$ ($j = 0, 1, \ldots, s$). For each j ($0 \leqslant j \leqslant s$), let T'_j be the tree obtained from T_j by attaching its vertex v_j to the vertex v. Then $|T'_j| = |T_j| + 1$. Let $L_v(T'_j)$ be the matrix obtained from $L(T'_j)$ by deleting the row and column corresponding to the vertex v. Note that $L_v(T'_j)$ is a principal submatrix of $L(T'_j)$. Let $L_v(T)$ be the matrix obtained from $L(T)$ by deleting the row and column corresponding to the vertex v. Without loss of generality, we may assume that

$$L_v(T) = \begin{bmatrix} L_v(T'_0) & 0 & \cdots & 0 \\ 0 & L_v(T'_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_v(T'_s) \end{bmatrix}_{(n-1)\times(n-1)}$$

Since T is a tree with a perfect matching, it is obvious that only one of the components of $T - v$ has not a perfect matching. Now we consider the following two cases.

Case 1. T_0 has a perfect matching. Then only one of T_j ($j = 1, \ldots, s$) has not a perfect matching. Without loss of generality, we assume that T_1 has not a perfect matching. Then $\alpha'(T_1) = \frac{1}{2}(|T_1| - 1)$ and

$$|T_1| \leqslant 2t - 1 \quad \text{and} \quad |T_j| \leqslant 2t - 2 \ (j = 2, 3, \ldots, s).$$

We consider the following two subcases.

Subcase 1.1. $|T_0| \leqslant (k-1)(2t) - 2$. Let $\widetilde{T_0}$ be the tree obtained from T'_0 by attaching a new vertex and $\frac{(k-1)(2t)-|T'_0|-1}{2}$ paths of length two to its vertex v. Then $\widetilde{T_0}$ has a perfect matching, and $|\widetilde{T_0}| = (k-1)(2t)$. Hence by Lemmas 2.1 and 2.2, and induction, we have

$$\mu_{k-1}(L_v(T_0')) \leqslant \mu_{k-1}(L(T_0')) \leqslant \mu_{k-1}(L(\widetilde{T}_0)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}.$$

Recall that T_1' has a perfect matching and T_j' $(j = 2, 3, \ldots, s)$ has not a perfect matching, and

$$|T_1'| \leqslant 2t \quad \text{and} \quad |T_j'| \leqslant 2t-1 \ \ (j = 2, 3, \ldots, s).$$

Let $\widetilde{T}_1$ be the tree obtained from T_1' by attaching $\frac{2t-|T_1'|}{2}$ paths of length two to its vertex v, $\widetilde{T}_j$ be the tree obtained from T_j' by attaching a new vertex and $\frac{2t-|T_j'|-1}{2}$ paths of length two to its vertex v $(j = 2, 3, \ldots, s)$. Then $\widetilde{T}_j$ has a perfect matching, and $|\widetilde{T}_j| = 2t$ $(j = 1, 2, \ldots, s)$. By Lemmas 2.1, 2.2 and 2.5, we have

$$\mu_1(L_v(T_j')) \leqslant \mu_1(L(T_j')) \leqslant \mu_1(L(\widetilde{T}_j)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2} \quad (j = 1, 2, \ldots, s).$$

Then $\mu_{k-1}(L_v(T)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}$. Hence by Lemma 2.2, we have

$$\mu_k(T) \leqslant \mu_{k-1}(L_v(T)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}.$$

Subcase 1.2. $|T_0| = (k-1)(2t)$. Then the edge $vv_0 \in E(T)$ gives

$$T - vv_0 = T_v \bigcup T_0, \quad |T_v| = 2t \quad \text{and} \quad |T_0| = (k-1)(2t),$$

where T_v and T_0 are the two connected components of $T - vv_0$.

Hence by Lemma 2.5 we have $\mu_1(T_v) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}$; and by induction, we have $\mu_{k-1}(T_0) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}$. Therefore, by Lemma 2.1 we have

$$\mu_k(T) \leqslant \mu_{k-1}(T - vv_0) \leqslant \max\{\mu_{k-1}(T_0), \mu_1(T_v)\} \leqslant \frac{t+2+\sqrt{t^2+4}}{2}.$$

Case 2. T_0 has not a perfect matching. Then $\alpha'(T_0) = \frac{1}{2}(|T_0|-1)$, and all of T_j $(j = 1, 2, \ldots, s)$ have a perfect matching. We consider the following two subcases.

Subcase 2.1. $|T_0| \leqslant (k-1)(2t) - 3$. Since T_0' has a perfect matching, let $\widetilde{T}_0$ be the tree obtained from T_0' by attaching $\frac{(k-1)(2t)-|T_0'|}{2}$ paths of length two to its vertex v, then $\widetilde{T}_0$ has a perfect matching, and $|\widetilde{T}_0| = (k-1)(2t)$. By Lemmas 2.1 and 2.2, and induction, we have

$$\mu_{k-1}(L_v(T_0')) \leqslant \mu_{k-1}(L(T_0')) \leqslant \mu_{k-1}(L(\widetilde{T}_0)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}.$$

Recall that T_j' $(j = 1, 2, \ldots, s)$ has not a perfect matching and

$$|T_j'| \leqslant 2t-1 \quad (j = 1, 2, \ldots, s).$$

Let $\widetilde{T}_j$ be the tree obtained from T_j' by attaching a new vertex and $\frac{2t-|T_j'|-1}{2}$ paths of length two to its vertex v $(j = 1, 2, \ldots, s)$. Then $\widetilde{T}_j$ has a perfect matching, and $|\widetilde{T}_j| = 2t$ $(j = 1, 2, \ldots, s)$. By Lemmas 2.1, 2.2 and 2.5, we have

$$\mu_1(L_v(T_j')) \leqslant \mu_1(L(T_j')) \leqslant \mu_1(L(\widetilde{T}_j)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2} \quad (j = 1, 2, \ldots, s).$$

Then $\mu_{k-1}(L_v(T)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}$. Hence by Lemma 2.2, we have

$$\mu_k(T) \leqslant \mu_{k-1}(L_v(T)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}.$$

Subcase 2.2. $|T_0| = (k-1)(2t) - 1$. Then T_0' has a perfect matching, and $|T_0'| = (k-1)(2t)$. By Lemma 2.2 and induction, we have

$$\mu_{k-1}(L_v(T_0')) \leqslant \mu_{k-1}(L(T_0')) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}.$$

Similarly, for each T_j', we can construct $\widetilde{T}_j$ such that $\widetilde{T}_j$ has a perfect matching, and $|\widetilde{T}_j| = 2t$ $(j = 1, 2, \ldots, s)$. Hence

$$\mu_1(L_v(T_j')) \leqslant \mu_1(L(T_j')) \leqslant \mu_1(L(\widetilde{T}_j)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2} \quad (j = 1, 2, \ldots, s).$$

Then $\mu_{k-1}(L_v(T)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}$. Hence by Lemma 2.2, we have

$$\mu_k(T) \leqslant \mu_{k-1}(L_v(T)) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}.$$

From above discussion, the proof is completed. □

Lemma 3.2. *Let $T \in \mathcal{T}_n^+$ and $n = k(2t) + r$, where r is even and $1 \leqslant r \leqslant 2k-1$. Then $\mu_k(T) \leqslant \frac{(t+1)+2+\sqrt{(t+1)^2+4}}{2}$.*

Proof. Let $\widehat{T}$ be a tree on $n + 2k - r = k(2t+2)$ vertices that contains T and let T' be a subtree on $\widehat{T}$ on $n+1$ vertices that contains T. By Lemmas 2.1, 2.2 and 3.1, we have

$$\mu_k(T) \leqslant \mu_k(T') \leqslant \mu_k(\widehat{T}) \leqslant \frac{(t+1)+2+\sqrt{(t+1)^2+4}}{2}. \quad \square$$

By Lemmas 3.1 and 3.2, it is easy to get our main result.

Theorem 3.1. *Let $T \in \mathcal{T}_n^+$. Then $\mu_k(T) = 2$ when $n = 2k$ and $\mu_k(T) \leqslant \frac{\lceil \frac{n}{2k} \rceil + 2 + \sqrt{(\lceil \frac{n}{2k} \rceil)^2+4}}{2}$ when $n \neq 2k$.*

Remark. Let $T \in \mathcal{T}_n^+$ and $n = k(2t)$. If T contains $kT(2t,t)$ as a spanning subgraph, then when $t = 1$,

$$\mu_1(kK_2) = \mu_2(kK_2) = \cdots = \mu_k(kK_2) = 2;$$

when $t > 1$,

$$\mu_1(kT(2t,t)) = \mu_2(kT(2t,t)) = \cdots = \mu_k(kT(2t,t)) = \frac{t+2+\sqrt{t^2+4}}{2};$$

and

$$|E(T)| - |E(kT_{2t,t})| = k-1.$$

From Lemmas 2.1 and 3.1, we have when $t = 1$,

$$2 \leqslant \mu_k(kK_2) \leqslant \mu_k(T) = 2;$$

when $t > 1$,

$$\frac{t+2+\sqrt{t^2+4}}{2} = \mu_k(kT(2t,t)) \leqslant \mu_k(T) \leqslant \frac{t+2+\sqrt{t^2+4}}{2}.$$

Thus the upper bound in Theorem 3.1 is sharp.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \bigcap V_2 = \emptyset$. A connected sum of G_1 and G_2 is a graph $G = (V, E)$, where $V = V_1 \bigcup V_2$, and E differs from $E_1 \bigcup E_2$ by the addition of a single edge joining some (arbitrary) vertex of V_1 to some vertex of V_2. At the end of this paper, we propose the following problem.

Problem. Let $T \in \mathcal{T}_n^+$ and $n = k(2t)$. Then $\mu_k(T) = 2$ when $t = 1$ and $\mu_k(T) = \frac{t+2+\sqrt{t^2+4}}{2}$ when $t > 1$ if and only if T is connected sum of $kT(2t, t)$, i.e., T contains $kT(2t, t)$ as a spanning subgraph.

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