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# Local duality for equitable partitions of a Hamming space

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ABSTRACT

The Hamming space $Q_n$ is the set of binary words of length $n$. A partition $(C_1, C_2, \ldots, C_r)$ of $Q_n$ with quotient matrix $B = [b_{ij}]_{r\times r}$ is equitable if for all $i$ and $j$, any word in the cell $C_i$ has exactly $b_{ij}$ neighbors in the cell $C_j$. In this paper, we provide an explicit formula relating the local spectrum of cells in the face to that in the orthogonal face.



## 1. Introduction

The equitable partition of a Hamming space has been extensively studied due of its importance in coding theory [2,3,6,7] and algebraic graph theory [4]. A completely regular code of covering radius $\rho$ is part of the cells for an equitable $(\rho + 1)$-partition of a Hamming space.

In [10], A.Y. Vasil'eva proved that if $C$ is 1-perfect (one-error-correcting perfect), then there is a connection between the local spectrum of $C$ with respect to a $u$-face and its orthogonal one. An explicit formula describing the connection is given as well. We refer to her result as the local duality for 1-perfect codes.

The 1-perfect codes generate an equitable bipartition of a Hamming space. It is natural to extend the local duality for 1-perfect codes to that for arbitrary equitable partitions of a Hamming space. By using the relations on neighboring component of the complete local spectrum, Vasil'eva also settled the local duality for arbitrary equitable partitions of a Hamming space [11].

The purpose of the paper is to provide a more explicit formula relating the local spectrum of an equitable partition to its orthogonal local spectrum. This permits the easy derivation of a formula relating the local spectrum of eigenfunctions and its orthogonal local spectrum, first obtained in [12].

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According to [12], the reconstruction problem for eigenfunctions is closely related to the local duality for eigenfunctions.

In the next section we state our main result, Theorem 2.2, solving the local duality problem for any equitable partition of a Hamming space in an explicit fashion. In Section 3 this result is proved, after which some simple consequences are derived, see Corollaries 3.4, 3.6 and 3.7.

## 2. Main result

The Hamming space $Q_n$ is the set of binary word of length $n$. A subset $C$ of $Q_n$ is called a (binary) code of length $n$ and the elements of $C$ are called codewords. Let $u$ be a word in $Q_n$. The support of $u$ is the set of nonzero coordinate positions of $u$. Throughout this paper, we identify $u$ with its support, and by $\bar{u}$ we denote the complement of $u$. The weight of $u$ is the cardinality of its support, and it is denoted by $|u|$. The Hamming distance $d(u, v)$ between $u$ and $v$ equals the number of coordinate positions in which $u$ and $v$ differ.

**Definition 2.1.** An $r$-partition $(C_1, C_2, \ldots, C_r)$ of $Q_n$ is equitable with quotient matrix $B = [b_{ij}]$ if for any word $v$ in the cell $C_i$, the number of neighbors in the cell $C_j$ is $b_{ij}$ for all $i, j = 1, 2, \ldots, r$.

In the literature [2,3], the concept of equitable partitions of $Q_n$ is also referred to as "perfect $r$-colorings" or "$r$-partition designs".

There are known examples of equitable bipartition of $Q_n$. A binary code $C$ of $Q_n$ is 1-perfect if for any word $v \in Q_n$ there exists a unique codeword $u$ in $C$ such that the number of coordinate positions in which $u$ and $v$ differ is at most one. It is well known [9] that an 1-perfect code of length $n$ exists if and only if $n = 2^m - 1$, $m > 1$. One easily verifies that any 1-perfect code of length $n$ generates an equitable bipartition of $Q_n$ with quotient matrix $\left[\begin{smallmatrix} 0 & n \\ 1 & n-1 \end{smallmatrix}\right]$. For $u, w \in Q_n$, define the $u$-face $B_u^w$ (centered at $w$) to be the set $\{z + w\colon z \subseteq u\}$. If $w$ is the zero word, then we simply denote $B_u^w$ by $B_u$. By convention, if $u$ is the zero word $\mathbf{0}$, then we set $B_{\mathbf{0}}^w = \{w\}$. It should be mentioned that we do not assume the restriction $u \cap w = \emptyset$ for $B_u^w$ as in [5,6].

We introduce the local weight enumerator of $C$ with respect to a $u$-face as follows:

$$W_C^{u,w}(x) = \sum_{c \in C \cap B_u^w} x^{|c+w|} = \sum_{c \in (C+w) \cap B_u} x^{|c|}. \tag{2.1}$$

If $u$ is the all-one word $\mathbf{1}$ in (2.1), then $W_C^{\mathbf{1},w}(x)$ becomes the weight enumerator of $C$ with respect to $w$. More generally, we define the local weight enumerator of a real valued function $f$ of $Q_n$. The local weight enumerator of $f$ with respect to a $u$-face is defined by

$$W_f^{u,w}(x) = \sum_{v \in B_u^w} f(v) x^{|v+w|}.$$

If $f$ is a characteristic function of a code $C$, then the local weight enumerator of $f$ with respect to a $u$-face is equal to that of $C$ with the same $u$-face.

We are now ready to state our main result concerning the local duality for equitable partitions of $Q_n$.

**Theorem 2.2.** *Let $(C_1, C_2, \ldots, C_r)$ be an equitable partition of $Q_n$ with quotient matrix $B$. For $j = 1, 2, \ldots, r$, let $(\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jr})^T$ be an eigenvector of $B$ with eigenvalue $\lambda_j$. Then*

$$\begin{bmatrix} W_{C_1}^{\bar{u},w}(x) \\ W_{C_2}^{\bar{u},w}(x) \\ \vdots \\ W_{C_r}^{\bar{u},w}(x) \end{bmatrix} = [\alpha_{ij}]^{-1} \operatorname{diag}\bigl[G_1^u(-x), G_2^u(-x), \ldots, G_r^u(-x)\bigr][\alpha_{ij}] \begin{bmatrix} W_{C_1}^{u,w}(-x) \\ W_{C_2}^{u,w}(-x) \\ \vdots \\ W_{C_r}^{u,w}(-x) \end{bmatrix},$$

*where $G_t^u(x) = (1-x)^{\frac{n+\lambda_t}{2} - |u|}(1+x)^{\frac{n-\lambda_t}{2} - |u|}$.*

## 3. Proof of the main theorem

Let $\mathcal{V}_n$ be the $2^n$-dimensional vector space of real-valued functions on $Q_n$ equipped with the inner product $\langle f, g\rangle = \sum_{x\in Q_n} f(x)g(x)$. For any $u \in Q_n$, define a function $f^u$ as

$$f^u(v) = (-1)^{|u\cap v|}.$$

Then the set $\{f^u\colon u \in Q_n\}$ of characters is an orthogonal basis of $\mathcal{V}_n$. If $f$ is in $\mathcal{V}_n$, then

$$f = \sum_{u\in Q_n} a_u f^u,$$

where $a_u$ is the Fourier coefficient at $u$, i.e., $a_u = \langle f, f^u\rangle/\langle f^u, f^u\rangle = 2^{-n}\langle f, f^u\rangle$.

Let $S(u)$ denote the sphere of radius one centered at $u$, i.e.,

$$S(u) = \bigl\{v \in Q_n\colon d(u, v) = 1\bigr\}.$$

The adjacent matrix $D$ of $Q_n$ is defined by

$$D_{uv} = \begin{cases} 1 & \text{if } d(u, v) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in \mathcal{V}_n$, we define

$$(Df)(u) = \sum_{v\in S(u)} f(v).$$

Let $n$ be an integer and let

$$(1 - z)^x(1 + z)^{n-x} = \sum_{k\geqslant 0} P_k(x; n)z^k.$$

If $n$ is a positive integer and $x$ is an integer with $0 \leqslant x \leqslant n$, then

$$P_k(x; n) = \sum_{j=0}^{k}(-1)^j \binom{x}{j}\binom{n-x}{k-j}$$

which is called the Krawtchouk polynomial in $x$ of degree $k$.

It is well known [9] that $f^u$ is an eigenfunction of $D$ with eigenvalue $P_1(|u|; n)$, that is,

$$Df^u = P_1\bigl(|u|; n\bigr) f^u = \bigl(n - 2|u|\bigr) f^u.$$

The character sum of a code $C$ with respect to $v \in Q_n$ is defined as

$$\chi_v(C) = \sum_{c\in C}(-1)^{|c\cap v|}.$$

The following three lemmas play an important role to prove our main result.

**Lemma 3.1.** *(See [5].) Let $C$ be a subset of $Q_n$. Then*

$$\sum_{c\in C}\prod_{i=1}^{n} x_i^{c_i} = \frac{1}{2^n}\sum_{v\in Q_n}\chi_v(C)\prod_{i=1}^{n}(1 - x_i)^{v_i}(1 + x_i)^{1-v_i}.$$

**Lemma 3.2.** *(See [6].) Let $f$ be an eigenfunction of $D$ with eigenvalue $\lambda$. Then the following statements are true.*

(a) *$(n - \lambda)/2$ is a nonnegative integer, and hence $\lambda$ is an integer.*
(b) *$\langle f, f^u\rangle = 0$ if $|u| \neq (n - \lambda)/2$.*

We denote $1_C$ the characteristic function of a code $C$.

**Lemma 3.3.** *(See [6].) Let $(C_1, C_2, \ldots, C_r)$ be an equitable partition with quotient matrix $B$ and let $(\alpha_1, \alpha_2, \ldots, \alpha_r)^T$ be an eigenvector of $B$ with eigenvalue $\lambda$. Then $f = \sum_{i=1}^{r} \alpha_i 1_{C_i}$ is an eigenfunction of $D$ with eigenvalue $\lambda$.*

We are ready to prove our main result concerning the local duality for equitable partitions of $Q_n$. We denote $E_i$ the set of words of weight $i$.

**Proof of Theorem 2.2.** Recall from [2] that the quotient matrix $B$ is diagonalizable. For $j = 1, 2, \ldots, r$, let $f_j = \sum_{i=1}^{r} \alpha_{ji} 1_{C_i}$, where $(\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jr})^T$ is an eigenvector of $B$ with eigenvalue $\lambda_j$. Note that

$$\langle f_j, f^v \rangle = \sum_{i=1}^{r} \alpha_{ji} \langle 1_{C_i}, f^v \rangle = \sum_{i=1}^{r} \alpha_{ji} \chi_v(C_i). \tag{3.1}$$

We claim that the following identity holds for $t = 1, 2, \ldots, r$,

$$\sum_{j=1}^{r} \alpha_{tj} \sum_{c \in C_j} \prod_{i=1}^{n} x_i^{c_i} = \frac{1}{2^n} \sum_{v \in E_{\frac{1}{2}(n-\lambda_t)}} \langle f_t, f^v \rangle \prod_{i=1}^{n} (1 - x_i)^{v_i} (1 + x_i)^{1 - v_i}. \tag{3.2}$$

It follows from Lemma 3.1 and (3.1) that

$$\sum_{j=1}^{r} \alpha_{tj} \sum_{c \in C_j} \prod_{i=1}^{n} x_i^{c_i} = \frac{1}{2^n} \sum_{v \in Q_n} \langle f_t, f^v \rangle \prod_{i=1}^{n} (1 - x_i)^{v_i} (1 + x_i)^{1 - v_i}. \tag{3.3}$$

We now apply Lemma 3.2 and Lemma 3.3 to obtain our claim (3.2).

Let $u$ be a word in $Q_n$. Setting $x_i = 0$ for all $i \in \bar{u}$ in (3.2), we have

$$\sum_{j=1}^{r} \alpha_{tj} \sum_{c \in C_j \cap B_u} \prod_{i=1}^{n} x_i^{c_i} = \frac{1}{2^n} \sum_{v \in E_{\frac{1}{2}(n-\lambda_t)}} \langle f_t, f^v \rangle \prod_{i=1}^{n} (1 - x_i)^{v_i} (1 + x_i)^{1 - v_i} \Big|_{x_i = 0, i \in \bar{u}}. \tag{3.4}$$

Setting $x = x_i$ for all $i \in u$ in (3.4), we have

$$\sum_{j=1}^{r} \alpha_{tj} \sum_{c \in C_j \cap B_u} x^{|c|} = \frac{1}{2^n} \sum_{v \in E_{\frac{1}{2}(n-\lambda_t)}} \langle f_t, f^v \rangle (1 - x)^{|u \cap v|} (1 + x)^{|u| - |u \cap v|}, \tag{3.5}$$

for all $t = 1, 2, \ldots, r$. Put

$$F_t^u(x) = \frac{1}{2^n} \sum_{v \in E_{\frac{1}{2}(n-\lambda_t)}} \langle f_t, f^v \rangle (1 - x)^{|u \cap v|} (1 + x)^{|u| - |u \cap v|} \quad (t = 1, 2, \ldots, r).$$

Then the system of $r$ equations in (3.5) becomes

$$[\alpha_{ij}] \begin{bmatrix} W_{C_1}^{u,\mathbf{0}}(x) \\ W_{C_2}^{u,\mathbf{0}}(x) \\ \vdots \\ W_{C_r}^{u,\mathbf{0}}(x) \end{bmatrix} = \begin{bmatrix} F_1^u(x) \\ F_2^u(x) \\ \vdots \\ F_r^u(x) \end{bmatrix}. \tag{3.6}$$

Notice that

$$F_t^{\bar{u}}(-x) = \frac{1}{2^n} \sum_{v \in E_{\frac{1}{2}(n-\lambda_t)}} \langle f_t, f^v \rangle (1+x)^{|\bar{u} \cap v|} (1-x)^{|\bar{u}| - |\bar{u} \cap v|}$$
$$= (1-x)^{\frac{1}{2}(n+\lambda_t)-|u|} (1+x)^{\frac{1}{2}(n-\lambda_t)-|u|} F_t^u(x). \tag{3.7}$$

Put

$$G_t^u(x) = (1-x)^{\frac{1}{2}(n+\lambda_t)-|u|} (1+x)^{\frac{1}{2}(n-\lambda_t)-|u|}.$$

After replacing $x$ (resp. $u$) by $-x$ (resp. $\bar{u}$) in (3.6), apply (3.7) to obtain

$$[\alpha_{ij}] \begin{bmatrix} W_{C_1}^{\bar{u},\mathbf{0}}(-x) \\ W_{C_2}^{\bar{u},\mathbf{0}}(-x) \\ \vdots \\ W_{C_r}^{\bar{u},\mathbf{0}}(-x) \end{bmatrix} = \operatorname{diag}\left[G_1^u(x), G_2^u(x), \ldots, G_r^u(x)\right] \begin{bmatrix} F_1^u(x) \\ F_2^u(x) \\ \vdots \\ F_r^u(x) \end{bmatrix}. \tag{3.8}$$

Using the fact that $B$ is diagonalizable and combining (3.6) and (3.8), we have

$$\begin{bmatrix} W_{C_1}^{\bar{u},\mathbf{0}}(x) \\ W_{C_2}^{\bar{u},\mathbf{0}}(x) \\ \vdots \\ W_{C_r}^{\bar{u},\mathbf{0}}(x) \end{bmatrix} = [\alpha_{ij}]^{-1} \operatorname{diag}\left[G_1^u(-x), G_2^u(-x), \ldots, G_r^u(-x)\right][\alpha_{ij}] \begin{bmatrix} W_{C_1}^{u,\mathbf{0}}(-x) \\ W_{C_2}^{u,\mathbf{0}}(-x) \\ \vdots \\ W_{C_r}^{u,\mathbf{0}}(-x) \end{bmatrix}. \tag{3.9}$$

We remark that if $(C_1, C_2, \ldots, C_r)$ is an equitable partition of $Q_n$ with quotient matrix $B$, then $(C_1 + w, C_2 + w, \ldots, C_r + w)$ is also the equitable partition of $Q_n$ with the same quotient matrix. It then follows from (2.1) and (3.9) that

$$\begin{bmatrix} W_{C_1}^{\bar{u},w}(x) \\ W_{C_2}^{\bar{u},w}(x) \\ \vdots \\ W_{C_r}^{\bar{u},w}(x) \end{bmatrix} = [\alpha_{ij}]^{-1} \operatorname{diag}\left[G_1^u(-x), G_2^u(-x), \ldots, G_r^u(-x)\right][\alpha_{ij}] \begin{bmatrix} W_{C_1}^{u,w}(-x) \\ W_{C_2}^{u,w}(-x) \\ \vdots \\ W_{C_r}^{u,w}(-x) \end{bmatrix}.$$

This proves Theorem 2.2. □

Setting $u = \mathbf{0}$ in Theorem 2.2, leads to the following corollary.

**Corollary 3.4.** *Let $(C_1, C_2, \ldots, C_r)$ be an equitable partition of $Q_n$ with quotient matrix $B$. For $j = 1, 2, \ldots, r$, let $(\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jr})^T$ be an eigenvector of $B$ with eigenvalue $\lambda_j$. If $w \in C_l$, then for $t = 1, 2, \ldots, r$, we have*

$$W_{C_t}^{\mathbf{1},w}(x) = \sum_{i=1}^{r} \alpha_{il} \beta_{ti} (1-x)^{\frac{1}{2}(n-\lambda_i)} (1+x)^{\frac{1}{2}(n+\lambda_i)},$$

*where $\beta_{ij}$ is an $(i, j)$ entry of $[\alpha_{ij}]^{-1}$.*

We remark that the formula obtained from Corollary 3.4 is more effective computationally than the matrix formula given in [8].

**Example 3.5.** Let $n = 2^m - 3, m > 2$. D.S. Krotov proved in [7] that an $(n, 2^{n-m}, 3)$ code $C_1$ generates an equitable partition $(C_1, C_2, C_3, C_4)$ of $Q_n$ with quotient matrix

$$B=\begin{bmatrix} 0 & 1 & n-1 & 0 \\ 1 & 0 & n-1 & 0 \\ 1 & 1 & n-4 & 2 \\ 0 & 0 & n-1 & 1 \end{bmatrix}.$$

The set of eigenvalues of $B$ is $\{-3,-1,1,n\}$. Let $(D_1,D_2,D_3,D_4)$ be any equitable partition of $Q_n$ with the same quotient matrix $B$. It follows from Corollary 3.4 that if $w$ is in $D_2$, then $W_{D_2}^{\mathbf{1},w}(x)$ becomes

$$\frac{n-1}{4n+12}(1-x)^{\frac{n+3}{2}}(1+x)^{\frac{n-3}{2}}+\frac{1}{2}(1-x)^{\frac{n+1}{2}}(1+x)^{\frac{n-1}{2}} \\ +\frac{1}{4}(1-x)^{\frac{n-1}{2}}(1+x)^{\frac{n+1}{2}}+\tfrac{1}{n+3}(1+x)^{n}.$$

In [1], M.R. Best and A.E. Brouwer used the weight enumerator of $D_2$ in another fashion to show that the maximum cardinality of length $n=2^m-3$ and minimum distance 3 equals $2^{n-m}$.

An equitable bipartition of $Q_n$ is equivalent to a completely regular code of covering radius one, and the quotient matrix of an equitable bipartition has two distinct eigenvalues, see [2] for details. The following corollary is a generalization of the local duality for 1-perfect codes established in [10].

**Corollary 3.6.** *Let $(C_1,C_2)$ be an equitable bipartition of $Q_n$ with quotient matrix $\left[\begin{smallmatrix} i & j \\ k & l\end{smallmatrix}\right]$. Then*

$$W_{C_1}^{\bar{u},w}(x)=\frac{k}{j+k}(1+x)^{|\bar{u}|} \\ +(1-x)^{\frac{1}{2}(j+k)-|u|}(1+x)^{\frac{1}{2}(i+l)-|u|}\left(-\frac{k}{j+k}(1-x)^{|u|}+W_{C_1}^{u,w}(-x)\right).$$

*In particular, if $w\in C_1$, then*

$$W_{C_1}^{\mathbf{1},w}(x)=\frac{k}{j+k}(1+x)^{n}+\frac{j}{j+k}(1-x)^{\frac{1}{2}(j+k)}(1+x)^{\frac{1}{2}(i+l)}.$$

**Proof.** It is a straightforward consequence of the fact that $(-j,k)$ (resp. $(1,1)$) is an eigenvector of the quotient matrix with eigenvalue $i-k$ (resp. $n$). □

Next, we derive directly, without the use of induction, the local duality for eigenfunctions of $D$ established in [12]. Corollary 3.7 plays an important role in settling the reconstruction problem for eigenfunctions of $D$ [12].

**Corollary 3.7.** *(See [12].) Let $f$ be an eigenfunction of $D$ with eigenvalue $\lambda$. Then*

$$W_{f}^{\bar{u},w}(x)=(1+x)^{\frac{1}{2}(n+\lambda)-|u|}(1-x)^{\frac{1}{2}(n-\lambda)-|u|}W_{f}^{u,w}(-x).$$

*In particular,*

$$W_{f}^{\mathbf{1},w}(x)=f(w)(1+x)^{\frac{1}{2}(n+\lambda)}(1-x)^{\frac{1}{2}(n-\lambda)}.$$

**Proof.** The partition $(\{v\})_{v\in Q_n}$ is equitable with quotient matrix $D$. We expand $f$ in the standard basis $\{1_{\{v\}}\colon\ v\in Q_n\}$ as follows:

$$f=\sum_{v\in Q_n}f(v)1_{\{v\}}.$$

Since $f$ is an eigenfunction of $D$ with eigenvalue $\lambda$, we can apply (3.5) to obtain

$$\sum_{v\in Q_n} f(v) \sum_{c\in\{v\}\cap B_u} x^{|c|} = \frac{1}{2^n} \sum_{v\in E_{\frac{1}{2}(n-\lambda)}} \langle f, f^v \rangle (1-x)^{|u\cap v|}(1+x)^{|u|-|u\cap v|}. \tag{3.10}$$

The left-hand side of (3.10) becomes the local weight enumerator of $f$ with respect to the zero word. It then follows from (3.7) that

$$W_f^{\bar{u},\mathbf{0}}(x) = (1+x)^{\frac{1}{2}(n+\lambda)-|u|}(1-x)^{\frac{1}{2}(n-\lambda)-|u|} W_f^{u,\mathbf{0}}(-x). \tag{3.11}$$

We define a function $g$ by $g(u) = f(u+w)$ that is also an eigenfunction of $D$ with eigenvalue $\lambda$. To obtain the result, we apply $g$ to (3.11). $\square$

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