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Abelian Groups Projective over Their Endomorphism Rings

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INTRODUCTION

As the title indicates, we are concerned in this paper with abelian groups that are projective as modules over their endomorphism rings. We will in fact limit our attention to those groups that are finitely generated over their endomorphism rings. The abelian groups that are cyclic as modules over their endomorphism rings play a central role in this study. Some general results concerning these are derived in Section 2 and a basic invariant of such groups, here called K, is introduced and studied. This object K is a subgroup of the group G in question which admits a ring structure in a natural way and G becomes a K-module. K becomes the "natural" ring of operators for the groups in question. In Section 3 we characterize the abelian groups that are cyclic projective over their endomorphism rings. For such groups, K is an *E-ring* in the sense of Schultz [6] (definition in Section 2). The results of Section 3 establish an intimate connection between the theory of such rings and the groups cyclic projective over their endomorphism rings. Their theory is reduced to the cyclic case by use of

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an appropriate Morita equivalence. Finally in Section 5 we give some fairly general examples and make a few remarks connecting our work with other results in the literature.

Our terminology is, we think, standard, but we might refer to Fuchs [3] as a basic reference for abelian groups and Jacobson [4] for module theory. The unadorned symbols Hom and $\otimes$ are understood to be taken over Z, which of course denotes the ring of integers. All groups mentioned are understood to be abelian with G the generic symbol. We denote by E the ring of endomorphisms of G. When necessary, we write $E(G)$ for E. The center of E is denoted by $z(E)$.

2. E-Cyclic Groups

Let A be a ring with identity 1. Then A, viewed as a module over itself, is a cyclic module with 1 as generator; hence the additive group of A is cyclic, again with 1 as generator, when viewed as a module over its endomorphism ring E. More generally, if A is any ring and if M is a cyclic A-module, then the additive group of M is cyclic as a module over its endomorphism ring. We consider in this section the general situation.

Definition. Let G be an abelian group with endomorphism ring E. We say that G is E-cyclic if G is generated as module over E by a single element.

Let G be E-cyclic and choose a generator e for G. Then we have the exact sequence of E-modules

$$0 \longrightarrow L \longrightarrow E \xrightarrow{\varphi} G \longrightarrow 0, \tag{1}$$

where φ is defined by $\varphi(\alpha) = \alpha e$ for $\alpha \in E$, and

$$L = \{\beta \in E \mid \beta e = 0\}.$$

Having chosen e, we also have

$$K_e = K = \{x \in G \mid L_x = 0\}.$$

Properties of K will play an important role in our discussion. We derive some of them here. First observe that in the exact sequence (1), L is a left ideal of E. We recall that there is a largest subring, W, of E containing L as two-sidel ideal, the *idealizer* of L in E,

$$W = \{\alpha \in E \mid L\alpha \subseteq L\}.$$

PROPOSITION 1. *Let* φ *be the map defined in the exact sequence* (1). *Then* $W=\varphi^{-1}K$ *and* $K=\varphi W$.

Proof. Let $\alpha\in W$ and $\beta\in L$. Then $\varphi(\alpha)=\alpha e$ and we have $\beta(\alpha e)=(\beta\alpha e)=0$ since $\beta\alpha\in L$. Hence, $\alpha e\in K$ so $\alpha\in\varphi^{-1}K$. Conversely if $\alpha\in\varphi^{-1}K$ and $\beta\in L$ then $0=\beta(\alpha e)=(\beta\alpha)e$ so $L\alpha\subseteq L$ as required. Clearly now $K=\varphi W$.

COROLLARY. *K admits a ring structure in which multiplication is given by*

$$xy=\alpha\beta e \qquad \text{if} \quad x=\alpha e, \qquad y=\beta e. \tag{2}$$

Proof. By definition of W, W/L is a ring, whose additive group is isomorphic to K, via φ, by the result above. Transferring the multiplication on W/L to K results, as is easy to see, in (2).

A consequence of this corollary is the fact that the operation defined by (2) is in fact well defined. We note a second relation between W and K.

PROPOSITION 2. $W=\{\alpha\in E\mid\alpha K\subseteq K\}$.

Proof. If $\alpha K\subseteq K$ for some $\alpha\in E$ then in particular $\alpha e\in K$ so that $\beta\alpha e=0$ for all $\beta\in L$. This yields $L\alpha\subseteq L$ so that $\alpha\in W$. Conversely if $\alpha\in W$, $x\in K$ and $\beta\in L$, then $\beta(\alpha x)=(\beta\alpha)x=0$ since $\beta\alpha\in L$. Thus $L\alpha x=0$ so $\alpha x\in K$.

The operation defined in (2) does not extend to G. However, we have

PROPOSITION 3. *The group G is a right K-module under the action defined by*

$$yx=\beta\alpha e, \qquad \text{where} \quad y=\beta e\in G, \quad x=\alpha e\in K.$$

Proof. The only question is whether this operation is well defined. Let $\alpha e=\alpha' e\in K$, $\beta e=\beta' e\in G$ for α, α', β, $\beta'\in E$. Then $\alpha\in W$ by Proposition 1 and $\beta-\beta'\in L$ so that $(\beta-\beta')\alpha\in L$. Similarly $\alpha-\alpha'\in L$ so $\beta'(\alpha-\alpha')\in L$. Hence

$$\beta\alpha-\beta'\alpha'=(\beta-\beta')\alpha+\beta'(\alpha-\alpha')\in L$$

as required.

COROLLARY. *Denote the K-module G defined above by* G_K. *Then* $E=\mathrm{Hom}_K(G_K, G_K)$.

Proof. Let $\gamma \in E$, $y = \beta e \in G$, and $x = \alpha e \in K$. Then

$$\begin{aligned} \gamma(yx) &= \gamma(\beta\alpha e) \\ &= (\gamma\beta)(\alpha e) \\ &= [(\gamma\beta)\, e](\alpha e) \\ &= [\gamma(\beta e)](\alpha e) \\ &= (\gamma y)\, x. \end{aligned}$$

PROPOSITION 4. *The ring K is isomorphic to the center, $z(E)$, of E. In fact,*

$$W = L \oplus z(E), \qquad K = z(E)\, e.$$

Proof. Clearly $z(E) \subseteq W$ and $L \cap z(E) = 0$, since G is cyclic with e as generator. Let $x \in K$. Then right multiplication by x in G determines a central endomorphism, say $\gamma \in z(E)$, by Proposition 3 and its corollary. Let $x = \alpha e$. Then $\alpha e = (1 \cdot e)(\alpha e) = ex = \gamma e$. Thus $x = \alpha e \in z(E)\, e$ and it follows that φ takes $z(E)$ onto K. This completes the proof.

A certain special kind of ring will play an important role in what follows. We follow Schultz [6] in

DEFINITION. An associative ring R with identity is an *E-ring* if every endomorphism of the additive group, R^+, of R is given by left multiplication by some element of R.

Examples of E-rings are the subrings of Q and their quotients modulo ideals, and pure subgroups of the p-adic integers. It is easy to see that E-rings are commutative and that a ring R is an E-ring if and only if the only endomorphism of R^+ that annihilates 1 is 0 (cf. [2, 6]). We use this criterion in

PROPOSITION 5. *Let G be E-cyclic. Then $z(E)$ is an E-ring if and only if every endomorphism of K extends to an endomorphism of G.*

Proof. Suppose $z(E)$ is an E-ring. Then K is an E-ring and if $\alpha \in E(K)$, $\alpha e = 0$, then $\alpha = 0$. Given $\alpha \in E(K)$, define $\bar{\alpha}$ on G by $\bar{\alpha}(y) = y\alpha(e)$. This makes sense by Proposition 3 and we have $\bar{\alpha}K \subseteq K$. Hence $(\bar{\alpha} - \alpha)|_K \in E(K)$ and

$$(\bar{\alpha} - \alpha)\, e = \bar{\alpha}(e) - \alpha(e) = e\alpha(e) - \alpha(e) = 0$$

since e is the identity for K. Since K is an E-ring we obtain $\bar{\alpha} = \alpha$ on K, so α extends to G.

Conversely if every $\alpha \in E(K)$ extends to an endomorphism $\bar{\alpha}$ of G, then $\alpha e = 0$ implies that $\bar{\alpha}K = 0$ by definition of K. Then $\alpha = 0$ as well. Hence K, and with it $z(E)$ is an E-ring.

COROLLARY. *If K is a direct summand of G then $z(E)$ is an E-ring.*

3. CYCLIC *E*-PROJECTIVE GROUPS

We retain the hypothesis that G is cyclic over its endomorphism ring E, with generator e, and with K defined as above.

THEOREM 1. *Let G be cyclic over E. Then G is projective over E if and only if K is a summand of G.*

Proof. Suppose that G is E-projective. We have the exact sequence of E-modules

$$0 \longrightarrow L \longrightarrow E \xrightarrow{\varphi} G \longrightarrow 0$$

as in the previous section, and this splits now, so L is a summand of E as E-module. Thus there exists an idempotent $\lambda \in E$ such that $L = E\lambda$. Then $L(1-\lambda) = 0$ so that $(1-\lambda)\, G \subseteq K$ by definition of K. On the other hand, since $\lambda \in L$, we have $\lambda K = 0$ and $(1-\lambda)|_K = 1_K$. In particular, $K = (1-\lambda)\, K \subseteq (1-\lambda)\, G \subseteq K$. Therefore, $K = (1-\lambda)\, G$, hence is a summand of G.

Conversely, suppose that $G = K \oplus H$ for some subgroup H of G and let ε be the projection of G onto K with kernel H. Define a map ρ: $G \to E$ as

$$\rho(g) = \alpha\varepsilon \qquad \text{if} \quad g = \alpha e.$$

If $\alpha e = g = \beta e$ we have $\alpha - \beta \in L$ so $(\alpha - \beta)\, \varepsilon = 0$ since $\varepsilon G = K$. Thus ρ is an E-map of G into E and we have, with φ as in the sequence (1), $\varphi\rho(g) = \varphi(\alpha\varepsilon) = \alpha\varepsilon e = \alpha e = g$, where $g = \alpha e$. Thus the sequence (1) splits and G is projective since it is isomorphic to the summand $\rho(G)$ of E.

COROLLARY. *If G is cyclic projective over its endomorphism ring E then the center of E is an E-ring.*

Proof. Theorem 1 and Corollary to Proposition 5.

Any E-ring, being commutative and equal to its own endomorphism ring, is the center of its endomorphism ring on the one hand, and is cyclic projective over that ring on the other. Hene, by the corollary above, the E-rings are precisely the centers of the endomorphism rings of those abelian groups that are cyclic projective over their endomorphism rings. This would not be very illuminating from either point of view if the E-cyclic projectives were just the additive groups of the E-rings. The following theorem adds some insight into the situation. In this regard as well as in providing examples, it is helpful to have at hand the following

LEMMA. *Let G be E-cyclic with generator e. Let H be a subgroups of G containing e. Suppose that*

(1) $\mathrm{Hom}(H/Ze, G) = 0$ *and*

(2) $H = \bigcap \ker \beta$ *for some set of endomorphism* $\{\beta\}$ *of* G.

Then $H = K_e$.

Proof. By (1), H is annihilated by every endomorphism of G that annihilates e, so $H \subseteq K$. By (2), should $x \in K$ and $x \notin H$ then there would exist an endomorphism β such that $\beta H = 0$ and $\beta x \neq 0$. Since $e \in H$, this would contradict the definition of K. Hence $K \subseteq H$ and we are done.

COROLLARY. *If H is summand of G containing e and if* $\mathrm{Hom}(H/Ze, G) = 0$ *then* $H = K$.

We can now construct all groups that are cyclic projective over their endomorphism rings—provided that we know all E-rings.

THEOREM 2. *Let R be any E-ring and let M be an R-module such that* $\mathrm{Hom}(R/Z \cdot 1, M) = 0$, *where 1 is the identity of R. Then $G = R \oplus M$ is cyclic projective over its endomorphism ring with generator $e = (1, 0)$ and with $K \approx R$. Conversely every group cyclic projective over its endomorphism ring has this form for a suitable E-ring R and R-module M.*

Proof. Clearly $G = R \oplus M$ is cyclic over its endomorphism ring with $e = (1, 0)$ as generator. By the corollary to the lemma above, $R = K$ here (more precisely $K = \{(r, 0) \mid r \in R\}$). Since K is summand of G, G is projective over its endomorphism ring by Theorem 1. The converse follows from Theorem 1, its corollary, and the corollary to Proposition 3.

4. FINITELY GENERATED E-PROJECTIVE GROUPS

We consider now abelian groups G that are finitely generated over their endomorphism rings E; we will reduce the study of those that are projective over their endomorphism rings to the cyclic case. Thus let G be generated over E by $\{g_1, ..., g_n\}$ and let $X = G^n$, the product of n copies of G. Then the endomorphism ring, $E(X)$, if X is the ring, E_n, of $n \times n$

matrices over E. Let $e=(g_1, ..., g_n)$ be the element of X given by the generators of G. Then it is clear that X is cyclic over $E(X)$ with e as a generator. In this way we obtain an E-cyclic group X from the group G. The fact that E and the matrix ring E_n are Morita equivalent allows us to reduce the finitely generated case to the cyclic case. We provide some details.

As above let G be finitely generated over its endomorphism ring E, with a set $\{g_1, ..., g_n\}$ of generators, and let $X=G^n$ so X is cyclic over $E(X)=E_n$. Let P be the free left E-module of rank n viewed as row vectors so P is an $E-E_n$ bimodule. Let Q be the free right E-module viewed as column vectors over E so Q is an E_n-E bimodule. Let $\varepsilon \in E_n$ be the matrix with 1 in the upper left and 0 elsewhere. Then it is clear that we may identify P with εE_n and that $E_n\varepsilon E_n=E_n$; i.e., P is a cyclic progenerator for E_n (cf. [4], for example, for details concerning this, and Morita theory in general). Now it is easy to see that

$$Q\otimes_E G\approx X \qquad \text{as left } E_n\text{-modules,}$$

$$P\otimes_{E_n} X\approx G \qquad \text{as left } E\text{-modules.}$$

In fact we have

LEMMA. *With the notation above,* $G\approx \varepsilon X\approx P\otimes_{E_n} X$.

Proof. The first isomorphism is clear since

$$\varepsilon X=\{(x, 0, ..., 0) \mid x\varepsilon G\}.$$

We establish the second isomorphism assuming only that $P=\varepsilon E_n$ is a cyclic progenerator for the endomorphism ring E_n of X. We have a bilinear, E_n-balanced map of $P\times X$ to εX given by $(\varepsilon\alpha, x)\to \varepsilon\alpha x$, thus a map $P\otimes X\to \varepsilon X$ of abelian groups. On the other hand, if $\varepsilon x=\varepsilon y$ for $x, y\in X$ then $\varepsilon\otimes x=\varepsilon\otimes y$ so we have the obvious map of εX into $P\otimes X$. Composition in both directions is the identity so $\varepsilon X\approx P\otimes_{E_n} X$ as abelian groups. Now if we identify E with the subring $\varepsilon E_n\varepsilon$ of E_n, then εX becomes an E-module and it is clear that the maps above are E-maps.

We can now prove

THEOREM 3. *If G is finitely generated over its endomorphism ring E, with $\{g_1, ..., g_n\}$ as a set of generators, then $X=G^n$ is cyclic over $E(X)$. Moreover G is E-projective if and only if X is $E(X)$ projective.*

Conversely, if X is cyclic projective over its endomorphism ring $E(X)$ and E is any ring Morita equivalent to $E(X)$ via a cyclic progenerator P of $E(X)$, then $E(P\otimes_{E(X)} X)=E$ and $P\otimes X$ is finitely generated projective over E.

Proof. We have seen that $X = G^n$ is cyclic over $E(X)$ if G admits n generators over $E = E(G)$ and that $E(X) = E_n$. Moreover, G and X correspond under the standard Morita equivalence of E with E_n described above and since Morita equivalence preserves projectivity, X is $E(X) = E_n$ projective if and only if G is E-projective.

Conversely suppose X is cyclic projective over its endomorphism ring $E(X)$ and let P be a cyclic progenerator, so $P = \varepsilon E(X)$ for some idempotent ε of $E(X)$ with $E(X)\,\varepsilon E(X) = E(X)$. The ring equivalent to $E(X)$ under the equivalence induced by P is $\mathrm{Hom}_{E(X)}(P, P) = \varepsilon E(X)\,\varepsilon$. We denote this by E here. We must show that if G is the abelian group $P \otimes_{E(X)} X$, then G is finitely generated over its endomorphism ring $E(G)$, and G is projective over $E(G)$. Since Morita equivalence preserves both projectivity and the property of being finitely generated, it suffices to show that $E(G) = E$.

By the lemma above (cf. the remarks in its proof) we may identify $G = \varepsilon X$, so G is a summand of X, say $X = G \oplus H$. Let α be any Z-endomorphism of G. We may extend α to an endomorphism of X, denoted by $\bar{\alpha}$, such that $\bar{\alpha}H = 0$. Then $\bar{\alpha} \in E(X)$ so

$$\bar{\alpha} = \varepsilon\bar{\alpha}\varepsilon + \varepsilon\bar{\alpha}(1-\varepsilon) + (1-\varepsilon)\,\bar{\alpha}\varepsilon + (1-\varepsilon)\,\bar{\alpha}(1-\varepsilon).$$

Now $(1-\varepsilon)\,X = H$ so $\varepsilon\bar{\alpha}(1-\varepsilon) = (1-\varepsilon)\,\bar{\alpha}(1-\varepsilon) = 0$. Similarly, since $\varepsilon X = G$ and $\bar{\alpha}G = G$ we have $(1-\varepsilon)\,\bar{\alpha}\varepsilon = 0$. Thus $\bar{\alpha} = \varepsilon\bar{\alpha}\varepsilon \in E$ as desired. Conversely any element of E yields, by restriction, an endomorphism of $G = \varepsilon X$. Since $X = \varepsilon X \oplus (1-\varepsilon)\,X$ and $E(1-\varepsilon) = 0$, restriction of E to act in G is a monomorphism. Thus we may identify E with $E(G)$. This proves the theorem.

COROLLARY. *If G is finitely generated and projective over its endomorphism ring E then the center, $z(E)$, is an E-ring and* $\mathrm{Hom}(z(E)/Z \cdot 1, G) = 0$.

Proof. Morita equivalent rings have isomorphic centers, so the first statement follows from Theorem 3 and Corollary to Theorem 1. The second statement is clear.

5. EXAMPLES AND REMARKS

One of the early homological results on abelian groups as modules over their endomorphism rings is the characterization by Richman and Walker [5] of the E-projective p-groups as precisely the bounded p-groups. It is interesting to see how this fits into our discussion. If G is a p-group with $p^nG = 0$ for some n, which we choose to be minimal, then G has a cyclic summand K with generator e of order p^n. It is clear that K generates G over E so e does too and it is well known that $z(E)$ here is $Z/(p^n)$. This is an

E-ring isomorphic to K and we have the situation of Theorem 1. (It is not hard to see that conversely a p-group that is *E*-projective must be bounded [5].)

As a second example, let K be the ring of p-adic integers. Then K is an *E*-ring and K/Z is a divisible abelian group. Hence any abelian group G of the form $K \oplus M$ with M a reduced p-adic module is, by Theorem 2, cyclic projective over its endomorphism ring. On the other hand, $K \oplus Q \otimes K$ is cyclic but not projective.

It is interesting to observe, and perhaps surprising at first, that the abelian groups that are cyclic projective over their endomorphism rings form a class of groups structurally as complicated as those in the class of all abelian groups. That is, for any abelian group A whatever, $Z \oplus A$ is cyclic projective over its endomorphism ring and, since Z is cancellable, two such groups $Z \oplus A$ and $Z \oplus B$ are isomorphic if and only if A and B are isomorphic. In special cases, of course, for example the case of p-groups above, one can say significant things. For the case of torsion free groups of finite rank, see [1].

Our results have reduced the study of abelian groups finitely generated and projective over their endomorphism rings ro essentially two questions. The first is to find all the *E*-rings. This appears to be difficult—cf. [2, 6]. The second question is to determine, for a given *E*-ring R, all modules M over R satisfying (cf. Theorem 2) $\mathrm{Hom}(R/Z, M) = 0$. Such modules are called *R-groups* in [2]. This second question is closely related to the problem of determining the structure, or structural properties of, the abelian group R/Z (more precisely the quotient of R^+ by the subgroup of R generated by the identity of R). We content ourselves here with the following remarks, which at least help in visualizing examples in the general case.

DEFINITION. Let K be a commutative ring. Denote by $\mathscr{C}$ the class of K-modules M satisfying

$$\mathrm{Hom}_Z(K, M) = \mathrm{Hom}_K(K, M).$$

It is easily seen that $M \in \mathscr{C}$ if and only if $\mathrm{Hom}(K/Z, M) = 0$, and it is clear that K is an *E*-ring if and only if $K \in \mathscr{C}$. Thus if K is an *E*-ring then $\mathscr{C}$ consists of all K-modules such that $K \oplus M$ is cyclic projective over its ring of Z-endomorphisms. These of course are the same as the K-endomorphisms. If K is an *E*-ring then $\mathscr{C}$ consists of the K-groups, in the terminology of Bowshell and Schultz [2]. The following is an extension of a result in [2].

PROPOSITION 6. *Let K be any commutative ring, let $N \in \mathscr{C}$ and let M be any K-module. Then*

$$\mathrm{Hom}_Z(M, N) = \mathrm{Hom}_K(M, N).$$

Proof. Choose $\varphi \in \mathrm{Hom}_Z(M, N)$. Then for any $\alpha \in \mathrm{Hom}_Z(K, M)$, we have $\varphi\alpha \in \mathrm{Hom}_Z(K, N) = \mathrm{Hom}_K(K, N)$. Thus for k, $x \in K$,

$$\varphi k\alpha(x) = \varphi\alpha(kx) = k(\varphi\alpha)(x).$$

If $y \in M$ is arbitrary then there exists $\alpha \in \mathrm{Hom}_K(K, M)$ such that $\alpha(l) = y$. Hence for $k \in K$,

$$\varphi(ky) = \varphi k\alpha(1) = k\varphi\alpha(1) = k\varphi(y)$$

as above. Therefore φ is a K-map as required.

COROLLARY. *$\mathscr{C}$ is closed under products.*

Proof. If $N_i \in \mathscr{C}$, $i \in I$ then

$$\begin{aligned}\mathrm{Hom}_Z\left(K, \prod N_i\right) &= \prod \mathrm{Hom}_Z(K, N_i)\\ &= \prod \mathrm{Hom}_K(K, N_i)\\ &= \mathrm{Hom}_K\left(K, \prod N_i\right).\end{aligned}$$

COROLLARY. *If $N \in \mathscr{C}$ and M is any K-module then* $\mathrm{Hom}(M, N) \in \mathscr{C}$. *In particular $M^* = \mathrm{Hom}_K(M, K) \in \mathscr{C}$ if K is an E-ring.*

Proof. $\mathrm{Hom}(M, N)$ is a K-submodule of the product N^M. It is clear that $\mathscr{C}$ is closed under taking submodules.

COROLLARY. *If K is an E-ring, M is any K-module and $N = \bigcap_{\varphi \in M^*} \ker \varphi$, then $M/N \in \mathscr{C}$.*

Proof. M/N is isomorphic to a submodule of *a* product of ideals (submodules) of K.

Finally, since $\mathscr{C}$ is closed under taking submodules, we have

PROPOSITION 7. *Let $N \in \mathscr{C}$ and let $x \in N$. Let* $\mathrm{ann}(x)$ *be the annihilator ideal of x;*

$$\mathrm{ann}(x) = \{a \in K \mid ax = 0\}.$$

Then $K/\mathrm{ann}(x) \in \mathscr{C}$.

It is clear, however, that $\mathscr{C}$ is not in general closed under homomorphic images.

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