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On the inverse problem of identifying Lamé coefficients in linear elasticity

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This paper is dedicated to Professor Mary Hoeft

Abstract

An output least-squares type functional is employed to identify the Lamé parameters in linear elasticity. To be able to identify even the discontinuous Lamé parameters the regularization is performed by the BV-seminorm. Finite element discretization is used and convergence analysis is given. Numerical examples are given to show the feasibility of the approach.

Keywords: Lamé coefficients; Output least-squares; Regularization; Total variation; Linear elasticity; Inverse problem; Finite element method

1. Introduction

The following system describes the response of an isotropic membrane or body to a traction applied to its boundary:

$$-\nabla \cdot \sigma = f \quad \text{in } \Omega, \tag{1a}$$

$$\sigma = 2\mu\varepsilon(u) + \lambda \operatorname{div} u \, I \tag{1b}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{1c}$$

$$\sigma \cdot n = h \quad \text{on } \Gamma_2. \tag{1d}$$

The domain Ω is a subset of $\mathbf{R}^2$ in the case of a membrane or $\mathbf{R}^3$ in the case of a body, and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ is a partition of its boundary. In this paper, we consider the two-dimensional problem, with some comments about the extension to $\mathbf{R}^3$. The vector-valued function $u = u(x)$ represents the displacement of the elastic membrane, f is the body force, n is the unit outward normal, and $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the (linearized) strain tensor. The tensor σ is the resulting stress tensor, and the stress–strain law or constitutive equation (1b) is derived from the assumption that the elastic membrane is isotropic and that the displacement is small (so that a linear relationship holds approximately).

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The coefficients μ and λ are the Lamé moduli, which quantify the elastic properties of the material. They are constants if the material is homogeneous and otherwise depend on $x \in \Omega$.

The boundary conditions (1c) and (1d) indicate that the membrane is fixed on $\Gamma_1 \subset \partial\Omega$ and that a traction h is applied to the rest of the boundary, Γ_2. The *direct problem* for (1a) is to compute the displacement u, given that h and the coefficients μ and λ are known (see [1]). We study the *inverse problem*: estimating the nonconstant coefficients μ and λ given a measurement of u. For the sake of simplicity, the analysis below will be presented for the case when $\Gamma_1 = \partial\Omega$ and $\Gamma_2 = \emptyset$.

The inverse elasticity problem, as stated above, has been studied from the theoretical standpoint, for example in [2–9]. Recently, interesting applications, such as elasticity imaging, have sparked a new interest in these problems (see [10–13] and the cited references therein). The present contribution differs from these papers in a few respects. Firstly, in contrast to [12], where only the problem of recovering parameter μ was considered, we emphasis on the simultaneous recovery of both the parameters. Secondly, we incorporate BV-regularization, which has received much attention for recovering discontinuities in the following scalar inverse problem of identifying the coefficient a in the PDE

$$-\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega, \tag{2}$$

augmented with suitable boundary conditions (see [14–16] and the references therein). Our work has been heavily influenced by papers by Knowles [17] and Zou [18], particularly the latter.

2. Formulation of the inverse problem

We begin by fixing some notation. Throughout this work, the dot product of tensors A, B will be denoted by $A \cdot B$:

$$A \cdot B = A_{11}B_{11} + A_{12}B_{12} + A_{21}B_{21} + A_{22}B_{22}.$$

The L^2 norm of a tensor-valued function $A = A(x)$ is defined by

$$\|A\|_{L^2}^2 = \|A\|_{L^2(\Omega)}^2 = \int_\Omega A \cdot A = \int_\Omega A_{11}^2 + A_{12}^2 + A_{21}^2 + A_{22}^2.$$

Moreover, the L^2 norm of a vector-valued function $u = u(x)$ is defined by

$$\|u\|_{L^2}^2 = \|u\|_{L^2(\Omega)}^2 = \int_\Omega u_1^2 + u_2^2,$$

while the H^1 norm of u is defined by

$$\|u\|_{H^1}^2 = \|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2.$$

Using the Green's identity and the boundary condition, we obtain a weak form:

$$\int_\Omega 2\mu\varepsilon(u) \cdot \varepsilon(v) + \lambda \operatorname{div} u \operatorname{div} v = \int_\Omega f v \quad \text{for all } v \in V := H_0^1(\Omega) \times H_0^1(\Omega). \tag{3}$$

By Korn's inequality (see [1], for example), there exists a constant $C > 0$ such that

$$\|\varepsilon(v)\|_{L^2} \geq C\|v\|_{H^1} \quad \text{for all } v \in V.$$

C is a generic constant throughout the paper. The following inequality, which holds pointwise in Ω, is easy to establish[1]:

$$2\mu\varepsilon(v) \cdot \varepsilon(v) + \lambda|\operatorname{div} v|^2 \geq \min\{2\mu + 2\lambda, 2\mu\}\varepsilon(v) \cdot \varepsilon(v).$$

Combining the above two inequalities, we obtain

$$\int_\Omega 2\mu\varepsilon(v) \cdot \varepsilon(v) + \lambda|\operatorname{div} v|^2 \geq \alpha\|v\|_{H^1}^2 \quad \text{for all } v \in V,$$

[1] This inequality is specific to $\mathbf{R}^2$; in a three-dimensional problem, the inequality becomes $2\mu\varepsilon(v) \cdot \varepsilon(v) + \lambda|\operatorname{div} v|^2 \geq \min\{2\mu + 3\lambda, 2\mu\}.\varepsilon(v) \cdot \varepsilon(v)$, and therefore $\alpha = C^2 \min\{2\mu + 3\lambda, 2\mu\}$.

where $\alpha = C^2 \min\{2\mu + 2\lambda, 2\mu\}$. This establishes that the bilinear form defining (3) is V-elliptic if $\min\{2\mu + 2\lambda, 2\mu\} > 0$ in Ω.

For later use we also need to give the definition of the space of functions of bounded variation. Recall that the *total variation* of $f \in L^1(\Omega)$ is defined by

$$\mathrm{TV}(f) = \sup\left\{\int_\Omega f(\operatorname{div} g) \;:\; g \in (C_0^1(\Omega))^2,\ |g(x)| \le 1 \text{ for all } x \in \Omega\right\}$$

where $|\cdot|$ represents the Euclidean norm of a vector.[2] If $f \in L^1(\Omega)$ satisfies $\mathrm{TV}(f) < \infty$, then f is said to have *bounded variation*, and the space $\mathrm{BV}(\Omega)$ is defined by

$$\mathrm{BV}(\Omega) = \{f \in L^1(\Omega) \;:\; \mathrm{TV}(f) < \infty\}.$$

The norm on $\mathrm{BV}(\Omega)$ is $\|f\|_{\mathrm{BV}(\Omega)} = \|f\|_{L^1(\Omega)} + \mathrm{TV}(f)$, where $\mathrm{TV}(\cdot)$ is the BV-seminorm on $\mathrm{BV}(\Omega)$. It is known that $\mathrm{TV}(\cdot)$ is convex and lower-semicontinuous with respect to $\|\cdot\|_{L^1(\Omega)}$ (see [19–21]).

The following properties of $\mathrm{BV}(\Omega)$ and $L^\infty(\Omega)$ are crucial:

1. $L^\infty(\Omega)$ is continuously embedded in $L^1(\Omega)$.
2. $\mathrm{BV}(\Omega)$ is compactly embedded in $L^1(\Omega)$.

For given constants m_1, m_2 satisfying $m_2 > m_1 > 0$, we define the following subset of $L^\infty(\Omega) \times L^\infty(\Omega)$:

$$K = \{(\mu, \lambda) \;:\; \min\{2\mu + 2\lambda, 2\mu\} \ge m_1,\ \max\{\mu, \lambda\} \le m_2 \text{ in } \Omega,\ \mathrm{TV}(\mu) < \infty,\ \mathrm{TV}(\lambda) < \infty\}.$$

Elements of K will often be denoted by $\ell = (\mu, \lambda)$. It is easy to see that for each $(\mu, \lambda) \in K$, there exists a unique $u \in V$ satisfying (3).

We will henceforth assume that a (possibly noisy) measurement z of u^* is available, where u^* and $\ell^* = (\mu^*, \lambda^*) \in K$ together satisfy (3). The purpose of this paper is to propose and analyze a method for estimating ℓ^* from z. We define the functional $J_0 : K \to \mathbf{R}$ by

$$J_0(\ell) = \frac{1}{2}\int_\Omega 2\mu|\varepsilon(u[\ell] - z)|^2 + \lambda|\operatorname{div}(u[\ell] - z)|^2$$

where $u[\ell]$ is the unique solution of (3) corresponding to ℓ. The functional J_0 is related to output least-squares functionals considered by other authors (cf. [12,16,22–28]), but we point out that J_0 is based on the ℓ-dependent energy norm rather than on the L^2 or H^1 norms, as is more customary. The analogue to J_0, in case of the scalar problem (2), was suggested independently by Knowles [17] and Zou [18].

In principle, it is reasonable to estimate ℓ^* by minimizing J_0 over $\ell \in K$. However, since the inverse problem under consideration is ill-posed, it is necessary to regularize J_0. Therefore we consider the following minimization regularized problem to estimate ℓ^* from z. Find $\ell^* \in K$ by solving

$$\min_{\ell \in K} J(\ell) := \frac{1}{2}\int_\Omega \left\{2\mu|\varepsilon(u[\ell] - z)|^2 + \lambda|\operatorname{div}(u[\ell] - z)|^2\right\} + \rho[\mathrm{TV}(\mu) + \mathrm{TV}(\lambda)], \tag{4}$$

where $\rho > 0$ is the regularization parameter. Notice that the functional $J : K \to \mathbb{R}$ is a regularized analogue of J_0, where the regularization is performed using $\mathrm{TV}(\mu)$ and $\mathrm{TV}(\lambda)$, the total variations of μ and λ, respectively.

3. Finite element discretization and convergence analysis

We begin by defining and analyzing a finite element discretization on a family $\{\mathcal{T}_h\}$ of triangulations of Ω. We define V_h to be the space of all continuous linear polynomials relative to $\mathcal{T}_h$, subject to the constraints that the homogeneous Dirichlet boundary conditions on Γ_1 are satisfied. The discrete set K_h of admissible coefficients is modified to

$$K_h = \{(\mu_h, \lambda_h) \in \mathcal{A}_h \times \mathcal{A}_h \;:\; 0 < \alpha_1 \le \mu_h(x) \le \beta_1,\ 0 < \alpha_2 \le \lambda_h(x) \le \beta_2,\ \forall x \in \Omega\}.$$

For some of the auxiliary results given below we set $k_1 = \min\{\alpha_1, \alpha_2\}$ and $k_1 = \max\{\beta_1, \beta_2\}$.

[2] When f belongs to $W^{1,1}(\Omega)$, then it is easy to show (by integration by parts) that $\mathrm{TV}(f) = \int_\Omega |\nabla f|$.

The discretization of the direct problem then takes the form: Given $\ell_h \in K_h$, find $u_h \in V_h$ such that

$$\int_\Omega 2\mu_h \varepsilon(u_h) \cdot \varepsilon(v_h) + \lambda_h \operatorname{div} u_h \operatorname{div} v_h = \int_\Omega f v_h \text{ for all } v_h \in V_h. \tag{5}$$

For $\ell_h \in K_h$, the unique solution of (5) will be denoted by $u_h[\ell_h]$.

Finally we consider the following discrete minimization problem: Find $\ell_h^* \in K_h$ by solving

$$\min_{\ell_h \in K_h} J(\ell_h) := \frac{1}{2} \int_\Omega \{2\mu|\varepsilon(u[\ell_h] - z)|^2 + \lambda|\operatorname{div}(u[\ell_h] - z)|^2\} + \rho[\mathrm{TV}(\mu_h) + \mathrm{TV}(\lambda_h)], \tag{6}$$

where $u[\ell_h]$ is the solution to (5) that corresponds to ℓ_h.

In the following we assume that the minimization problems (4) and (6) have nonempty solution sets. In fact, a proof for the solvability of (4) can be extracted from [4] (see also [24]) where a more general problem is studied. The solvability of the finite-dimensional problem (6) is based on straightforward arguments (see [18]).

For the convergence analysis presented below, the following known results will be used.

Let $I_h : C(\bar{\Omega}) \to V_h$ be the standard nodal interpolant associated with finite element space V_h and let the elliptic projection operator $P_h : H_1(\Omega) \to V_h$ be defined by

$$\int_\Omega \nabla P_h v \cdot \nabla w = \int_\Omega \nabla v \cdot \nabla w \quad \text{for every } w \in V_h,\ v \in H^1(\Omega). \tag{7}$$

The following well-known results will also be used:

$$\|v - I_h v\|_{L^2(\Omega)} \le Ch^2, \qquad \|v - I_h v\|_{H^1(\Omega)} \le Ch \tag{8}$$

$$\|v - P_h v\|_{L^2(\Omega)} \le Ch^2, \qquad \|v - P_h v\|_{H^1(\Omega)} \le Ch \tag{9}$$

for every $v \in H^1(\Omega)$ with constant C independent of h.

We begin the convergence analysis with the following continuity result.

Lemma 3.1. *Let $\{\ell_h^k\} \subset K_h$ be a parameter sequence such that $\ell_h^k \to \ell_h$ as $k \to \infty$. Let $v[\ell_h^k]$ be the solution that corresponds to ℓ_h^k, and let $v[\ell_h]$ be the solution that corresponds to ℓ_h. Then the following continuity result holds*

$$v[\ell_h^k] \to v[\ell_h] \quad \text{in } V_h \text{ as } k \to \infty.$$

Proof. In view of the definitions of $v[\ell_h^k]$ and $v[\ell_h]$, we have

$$\int_\Omega 2\mu_h^k \varepsilon(v[\ell_h^k]) \cdot \varepsilon(w) + \lambda_h^k \operatorname{div} v[\ell_h^k] \operatorname{div} w = \int_\Omega f w \quad \forall\, w \in V_h. \tag{10}$$

$$\int_\Omega 2\mu_h \varepsilon(v[\ell_h]) \cdot \varepsilon(w) + \lambda_h \operatorname{div} v[\ell_h] \operatorname{div} w = \int_\Omega f w \quad \forall\, w \in V_h. \tag{11}$$

By setting $w = v[\ell_h^k]$ in (10), and using the fact that $\ell_h^k \in K_h$, we obtain

$$k_1 \int_\Omega |\varepsilon(v[\ell_h^k])|^2 \le \int_\Omega 2\mu_h^k |\varepsilon(v[\ell_h^k])|^2 + \lambda_h^k |\operatorname{div} v[\ell_h^k]|^2 = \int_\Omega f v[\ell_h^k].$$

Consequently

$$\begin{aligned} k_1 \|\varepsilon(v[\ell_h^k])\|_{L_2}^2 &\le \|f\|_{L_2} \|v[\ell_h^k]\|_{L_2} \\ &\le \|f\|_{L_2} \|\nabla v[\ell_h^k]\|_{L_2} \\ &\le C_0 K_1^{-1} \|\varepsilon(v[\ell_h^k])\|_{L_2} \end{aligned}$$

where we used Poincare's and Korn's inequalities. Therefore

$$\|\varepsilon(v[\ell_h^k])\|_{L_2} \le C := C_0 (k_1 K_1)^{-1}. \tag{12}$$

By subtracting (11) from (10) and rearranging the terms, we obtain

$$\int_\Omega 2\mu_h \varepsilon(v[\ell_h^k] - v[\ell_h]) \cdot \varepsilon(w) + \int_\Omega \lambda_h \mathrm{div}(v[\ell_h^k] - v[\ell_h])\mathrm{div}(w)$$
$$= \int_\Omega (\lambda_h - \lambda_h^k)\mathrm{div}\, v[\ell_h^k]\mathrm{div}\, w + \int_\Omega (2\mu_h - 2\mu_h^k)\varepsilon(v[\ell_h^k]) \cdot \varepsilon(w).$$

We set $w = v[\ell_h^k] - v[\ell_h]$ in the above equation to get

$$\begin{aligned} k_1 \|\varepsilon(v[\ell_h^k] - v[\ell_h])\|_{L_2}^2 &\le \int_\Omega 2\mu_h |\varepsilon(v[\ell_h^k] - v[\ell_h])|^2 + \lambda_h |\mathrm{div}(v[\ell_h^k] - v[\ell_h])|^2 \\ &\le \left| \int_\Omega (\lambda_h - \lambda_h^k)\mathrm{div}\, v[\ell_h^k]\, \mathrm{div}(v[\ell_h^k] - v[\ell_h]) \right| \\ &\quad + \left| \int_\Omega (2\mu_h - 2\mu_h^k)\varepsilon(v[\ell_h^k]) \cdot \varepsilon(v[\ell_h^k] - v[\ell_h]) \right| \\ &\le \max_{x\in\Omega} |2\mu_h - 2\mu_h^k|\ \|\varepsilon(v[\ell_h^k])\|_{L_2} \|\varepsilon(v[\ell_h^k] - v[\ell_h])\|_{L_2} \\ &\quad + \max_{x\in\Omega} |\lambda_h - \lambda_h^k|\ \|\mathrm{div}\, v[\ell_h^k]\|_{L_2} \|\mathrm{div}(v[\ell_h^k] - v[\ell_h])\|_{L_2}. \end{aligned}$$

Since

$$\|\mathrm{div}\, u\|_{L_2} \le C\|\varepsilon(u)\|_{L_2}, \quad \text{for all } u \in H^1,$$

we get

$$k_1 \|\varepsilon(v[\ell_h^k] - v[\ell_h])\|_{L_2} \le C\|\varepsilon(v[\ell_h^k])\|_{L_2} \left(\max_{x\in\Omega} |2\mu_h - 2\mu_h^k| + \max_{x\in\Omega} |\lambda_h - \lambda_h^k| \right).$$

The above inequality yields that $\varepsilon(v[\ell_h^k]) \to \varepsilon(v[\ell_h])$ as $k \to \infty$, from which we further conclude that $v[\ell_h^k] \to v[\ell_h]$. This completes the proof. □

Lemma 3.2. *If $\{\ell_h\} \subset K_h$ converges to some $\ell \in K$ as $h \to 0$ in $L_1 \times L_1$, then the sequence $v_h[\ell_h]$ converges weakly to $v[\ell]$ in $H_0^1 \times H_0^1$. Moreover, the following estimate holds*

$$\lim_{h\to 0} \int_\Omega 2\mu_h |\varepsilon(v_h[\ell_h] - z)|^2 + \lambda_h |\mathrm{div}\,(v_h[\ell_h] - z)|^2 = \int_\Omega 2\mu |\varepsilon(v[\ell] - z)|^2 + \lambda |\mathrm{div}\,(v[\ell] - z)|^2. \tag{13}$$

Proof. By the arguments used in Lemma 3.1, it follows that the sequence $\{\|\varepsilon(v_h[\ell_h])\|_{L_2}\}$, and hence $\{\|\nabla v_h[\ell_h]\|_{L_2}\}$) is bounded, independently of h. Therefore there exists a subsequence, still denoted by $\{v_h[\ell_h]\}$, such that $v_h[\ell_h]$ converges weakly to some $v^* \in V$ as $h \to 0$. Let $w \in V$ be arbitrary. Taking $w_h = P_h w$ in (5) (with P_h defined in (7) and applied componentwise to w), we obtain

$$\begin{aligned} \int_\Omega f w_h &= \int_\Omega 2\mu_h \varepsilon(v_h[\ell_h]) \cdot \varepsilon(w_h) + \lambda_h \mathrm{div}\, v_h[\ell_h] \mathrm{div}\, w_h \\ &= \int_\Omega 2\mu \varepsilon(v_h[\ell_h]) \cdot \varepsilon(w) + \lambda \mathrm{div}\, v_h[\ell_h] \mathrm{div}\, w \\ &\quad + \int_\Omega 2\mu_h \varepsilon(v_h[\ell_h]) \cdot \varepsilon(w_h - w) + \lambda_h \mathrm{div}\, v_h[\ell_h] \mathrm{div}(w_h - w) \\ &\quad + \int_\Omega (2\mu_h - 2\mu)\varepsilon(v_h[\ell_h]) \cdot \varepsilon(w) + (\lambda_h - \lambda)\mathrm{div}\, v_h[\ell_h] \mathrm{div}\, w \\ &=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned} \tag{14}$$

The weak convergence of $v_h[\ell_h]$ implies that

$$\mathrm{I}_1 := \int_\Omega 2\mu \varepsilon(v_h[\ell_h]) \cdot \varepsilon(w) + \lambda \mathrm{div}\, v_h[\ell_h] \mathrm{div}(w) \to \int_\Omega 2\mu \varepsilon(v^*) \cdot \varepsilon(w) + \lambda \mathrm{div}\, v^* \mathrm{div}\, w.$$

Moreover with the use of Hölder inequality and (9), we have

$$\begin{aligned}|\mathrm{I}_2| &= \left|\int_\Omega 2\mu_h\varepsilon(v_h[\ell_h])\cdot\varepsilon(w_h-w)+\lambda_h\mathrm{div}\,v_h[\ell_h]\mathrm{div}(w_h-w)\right|\\ &\le k_2\|\varepsilon(v_h[\ell_h])\|_{L_2}\|\varepsilon(w_h-w)\|_{L_2}+k_2\|\mathrm{div}\,v_h[\ell_h]\|_{L_2}\|\mathrm{div}(w_h-w)\|_{L_2}\to 0.\end{aligned}$$

Since $\ell_h\to\ell$ almost everywhere in Ω, we have

$$\begin{aligned}|\mathrm{I}_3| &= \left|\int_\Omega (2\mu_h-2\mu)\varepsilon(v_h[\ell_h])\cdot\varepsilon(w)+(\lambda_h-\lambda)\mathrm{div}\,v_h[\ell_h]\mathrm{div}\,w\right|\\ &\le \|\varepsilon(v_h[\ell_h])\|_{L_2}\|(2\mu_h-2\mu)\varepsilon(w)\|_{L_2}+\|\mathrm{div}\,v_h[\ell_h]\|_{L_2}\|(\lambda_h-\lambda)\mathrm{div}\,w\|_{L_2}\\ &= \|\varepsilon(v_h[\ell_h])\|_{L_2}\left(\int_\Omega|2\mu_h-2\mu|^2|\varepsilon(w)|^2\right)^{1/2}+\|\mathrm{div}\,v_h[\ell_h]\|_{L_2}\left(\int_\Omega|\lambda_h-\lambda|^2|\mathrm{div}(w)|^2\right)^{1/2}\to 0.\end{aligned}$$

Since the convergence $w_h=P_hw\to w$ as $h\to 0$ implies $\int_\Omega fw_h\to\int_\Omega fw$, from (14) we infer

$$\int_\Omega 2\mu\varepsilon(v^*)\cdot\varepsilon(w)+\lambda\mathrm{div}\,v^*\mathrm{div}\,w=\int_\Omega fw\quad\forall w\in H_0^1\times H_0^1.$$

Finally the uniqueness of the solution to the variational problem ensures that $v^*=v[\ell]$.

It remains to prove (13). Let $z_h=P_hz$. It follows from (3), that

$$\int_\Omega 2\mu_h\varepsilon(v_h[\ell_h]-z_h)\cdot\varepsilon(w)+\lambda_h\mathrm{div}(v_h[\ell_h]-z_h)\,\mathrm{div}\,w=\int_\Omega fw-2\mu_h\varepsilon(z_h)\cdot\varepsilon(w)-\lambda_h\mathrm{div}\,z_h\mathrm{div}\,w.$$

Choosing $w=v_h[\ell_h]-z_h$ we obtain

$$\begin{aligned}&\int_\Omega 2\mu_h\varepsilon(v_h[\ell_h]-z_h)\cdot\varepsilon(v_h[\ell_h]-z_h)+\lambda_h\mathrm{div}(v_h[\ell_h]-z_h)\,\mathrm{div}(v_h[\ell_h]-z_h)\\ &\quad=\int_\Omega f(v_h[\ell_h]-z_h)-2\mu_h\varepsilon(z_h)\cdot\varepsilon(v_h[\ell_h]-z_h)-\lambda_h\mathrm{div}(z_h)\mathrm{div}(v_h[\ell_h]-z_h).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_\Omega 2\mu_h|\varepsilon(v_h[\ell_h]-z_h)|^2+\lambda_h|\mathrm{div}(v_h[\ell_h]-z_h)|^2\\ &\quad=\int_\Omega[f(v_h[\ell_h]-z_h)-2\mu\varepsilon(z)\cdot\varepsilon(v_h[\ell_h]-z_h)-\lambda\mathrm{div}\,z\mathrm{div}(v_h[\ell_h]-z_h)]\\ &\qquad-\int_\Omega[2\mu\varepsilon(z_h-z)\cdot\varepsilon(v_h[\ell_h]-z_h)+\lambda\mathrm{div}(z_h-z)\,\mathrm{div}(v_h[\ell_h]-z_h)]\\ &\qquad-\int_\Omega[(2\mu_h-2\mu)\varepsilon(z)\cdot\varepsilon(v_h[\ell_h]-z_h)+(\lambda_h-\lambda)\mathrm{div}(z)\mathrm{div}(v_h[\ell_h]-z_h)]\\ &\qquad-\int_\Omega[(2\mu_h-2\mu)\varepsilon(z_h-z)\cdot\varepsilon(v_h[\ell_h]-z_h)+(\lambda_h-\lambda)\mathrm{div}(z_h-z)\,\mathrm{div}(v_h[\ell_h]-z_h)]\\ &\quad=:\mathrm{II}_1+\mathrm{II}_2+\mathrm{II}_3+\mathrm{II}_4.\end{aligned}$$

We now analyze the terms II_i for $i\in\{1,\dots,4\}$. By the weak convergence of $v_h[\ell_h]\to v[\ell]$ and the strong convergence of $z_h=P_hz\to z$, we have

$$\begin{aligned}\mathrm{II}_1 &= \int_\Omega f(v_h[\ell_h]-z_h)-\int_\Omega[2\mu\varepsilon(z)\cdot\varepsilon(v_h[\ell_h]-z_h)+\lambda\mathrm{div}\,z\,\mathrm{div}(v_h[\ell_h]-z_h)]\\ &\quad\to\int_\Omega f(v[\ell]-z)-\int_\Omega[2\mu\varepsilon(z)\cdot\varepsilon(v[\ell]-z)+\lambda\mathrm{div}\,z\,\mathrm{div}(v[\ell]-z)]\end{aligned}$$

$$\mathrm{II}_2=-\int_\Omega[2\mu\varepsilon(z_h-z)\cdot\varepsilon(v_h[\ell_h]-z_h)+\lambda\mathrm{div}(z_h-z)\mathrm{div}(v_h[\ell_h]-z_h)]\to 0.$$

$$\mathrm{II}_4=-\int_\Omega[(2\mu_h-2\mu)\varepsilon(z_h-z)\cdot\varepsilon(v_h[\ell_h]-z_h)+(\lambda_h-\lambda)\mathrm{div}(z_h-z)\mathrm{div}(v_h[\ell_h]-z_h)]\to 0.$$

Using the Cauchy–Schwarz inequality and the Lebesgue dominant convergence theorem, we get

$$\begin{aligned}|\mathrm{II}_3| &= \left|-\int_\Omega [(2\mu_h - 2\mu)\varepsilon(z)\cdot\varepsilon(v_h[\ell_h]-z_h) + (\lambda_h-\lambda)\operatorname{div} z \operatorname{div}(v_h[\ell_h]-z_h)]\right| \\ &\le \|\varepsilon(v_h[\ell_h]-z_h)\|_{L_2}\|(2\mu_h-2\mu)\varepsilon(z)\|_{L_2} + \|\operatorname{div}(v_h[\ell_h]-z_h)\|_{L^2}\,\|(\lambda_h-\lambda)\operatorname{div} z\|_{L^2} \\ &= \|\varepsilon(v_h[\ell_h]-z_h)\|_{L_2}\left[\left(\int_\Omega |2\mu_h-2\mu|^2|\varepsilon(z)|^2\right)^{1/2} + \left(\int_\Omega |\lambda_h-\lambda|^2|\operatorname{div} z|^2\right)^{1/2}\right] \to 0 \quad \text{as } h\to 0.\end{aligned}$$

In view of Lemma 3.1, $\|\varepsilon(v_h[\ell_h])\|_{L_2}$ is bounded independently of h for all $v_h[\ell_h]\in V_h$ and $\ell_h \in K_h$. Consequently, we have

$$\begin{aligned}&\lim_{h\to 0}\int_\Omega 2\mu_h|\varepsilon(v_h[\ell_h]-z_h)|^2 + \lambda_h|\operatorname{div}(v_h[\ell_h]-z_h)|^2 \\ &\quad= \int_\Omega f(v[\ell]-z) - \int_\Omega [2\mu\varepsilon(z)\varepsilon(v[\ell]-z) + \lambda \operatorname{div} z \operatorname{div}(v[\ell]-z)] \\ &\quad= \int_\Omega [2\mu\varepsilon(v[\ell])\varepsilon(v[\ell]-z) + \lambda \operatorname{div} v[\ell] \operatorname{div}(v[\ell]-z)] - \int_\Omega [2\mu\varepsilon(z)\varepsilon(v[\ell]-z) + \lambda \operatorname{div} z \operatorname{div}(v[\ell]-z)] \\ &\quad= \int_\Omega 2\mu|\varepsilon(v[\ell]-z)|^2 + \lambda|\operatorname{div}(v[\ell]-z)|^2,\end{aligned}$$

which gives (13). The proof is complete. □

We our next result, we modify the set of admissible constraints to be

$$K = \{(\mu,\lambda)\in L \mid \alpha_1 \le \mu(x) \le \beta_1,\ \alpha_2 \le \lambda(x) \le \beta_2 \text{ in } \Omega, TV(\mu) < \infty,\ TV(\lambda) < \infty\}.$$

The above preparation enables us to give the following main convergence result.

Theorem 3.1. *Assume that $\{\ell_h^*\}$ is a sequence of solutions to (6). Then every subsequence of $\{\ell_h^*\}$ has a convergent subsequence with limit point as a solution to (4).*

Proof. By choosing $\mu_h = k_1$, $\lambda_h = k_2$ and $v[\ell_h]$ as the corresponding solution we notice that the functional $J(\ell_h^*)$ is bounded above by a constant which is independent of h. This further implies that (ℓ_h^*) is a bounded sequence. Since the BV functions are relatively compact in $L_1(\Omega)$, we deduce the existence of a subsequence $\{\ell_h^*\}$ converging to $\ell^* \in \mathbb{K}$.

For any $(\mu,\lambda)\in K_h$ and for constants $\kappa_1 > 0$ and $\kappa_2 > 0$, there exists function $(\mu_{\kappa_1},\lambda_{\kappa_2}) \in C^\infty(\bar\Omega)\times C^\infty(\bar\Omega)$ such that (see [19, p. 127 and p. 172])

$$\begin{aligned}&\left\|\mu_{\kappa_1}-\mu\right\|_{L_1} \le \kappa_1, &&\left|TV(\mu_{\kappa_1}) - TV(\mu)\right| \le \kappa_1 \\ &\left\|\lambda_{\kappa_2}-\lambda\right\|_{L_1} \le \kappa_2, &&\left|TV(\lambda_{\kappa_2}) - TV(\lambda)\right| \le \kappa_2.\end{aligned}$$

Following the ideas of Zou [18], we define

$$\hat\mu_{\kappa_1}(x) = \begin{cases}\mu_{\kappa_1}(x), & \text{if } \alpha_1 \le \mu_{\kappa_1}(x) \le \beta_1;\\ \alpha_1, & \text{if } \mu_{\kappa_1}(x) < \alpha_1;\\ \beta_1, & \text{if } \mu_{\kappa_1}(x) > \beta_1;\end{cases} \quad\text{and}\quad \hat\lambda_{\kappa_2}(x) = \begin{cases}\lambda_{\kappa_2}(x), & \text{if } \alpha_2 \le \lambda_{\kappa_2}(x) \le \beta_2;\\ \alpha_2, & \text{if } \lambda_{\kappa_2}(x) < \alpha_2;\\ \beta_2, & \text{if } \lambda_{\kappa_2}(x) > \beta_2.\end{cases}$$

Then, we have $(\hat\mu_{\kappa_1},\hat\lambda_{\kappa_2}) \in W^{1,\infty}(\Omega)\times W^{1,\infty}(\Omega)$.

Moreover

$$\|\hat\mu_{\kappa_1}-\mu\|_{L^1(\Omega)} \le \|\mu_{\kappa_1}-\mu\|_{L^1(\Omega)} \le \kappa_1$$

and

$$\begin{aligned}\int_\Omega |\nabla\hat\mu_{\kappa_1}| &= \int_{\hat\mu_{\kappa_1}=\mu_{\kappa_1}} |\nabla\hat\mu_{\kappa_1}| = \int_{\hat\mu_{\kappa_1}=\mu_{\kappa_1}} |\nabla\mu_{\kappa_1}| \\ &\le \int_\Omega |\nabla\mu_{\kappa_1}| \le TV(\mu)+\kappa_1.\end{aligned}$$

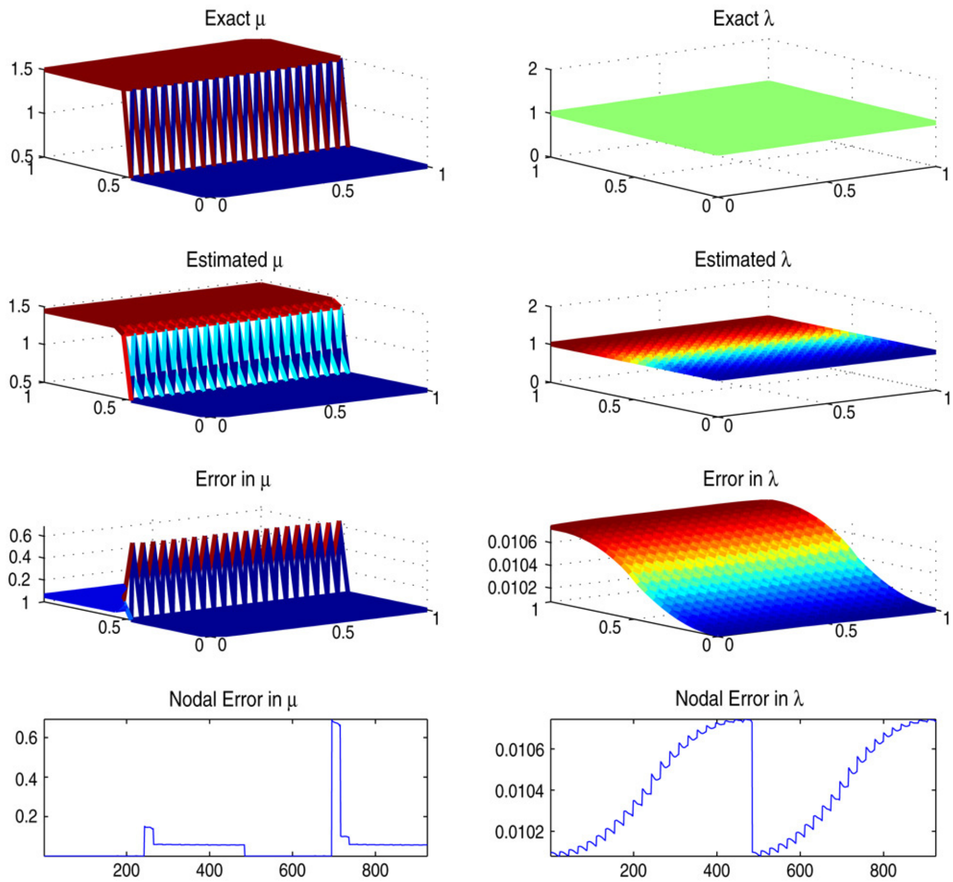

Fig. 1. Simulation results for Example 1: Identification of discontinuous Lamé parameters.

Similarly, we have $\|\hat{\mu}_{\kappa_2} - \mu\|_{L^1(\Omega)} \le \kappa_2$ and $\int_\Omega |\nabla \hat{\mu}_{\kappa_2}| \le TV(\lambda) + \kappa_2$.

We set $\mu_h = I_h \hat{\mu}_{\kappa_1}$ and $\lambda_h = I_h \hat{\lambda}_{\kappa_2}$, where $I_h(\cdot)$ is the nodal interpolant.

By using the lower-semicontinuity of the BV norm, we have

$$\begin{aligned} J(\ell^*) &= \frac{1}{2}\int_\Omega \left[2\mu^*|\varepsilon(v[\ell]-z)|^2 + \lambda^*|\mathrm{div}(v-z)|^2\right] + \rho\{TV(\mu^*) + TV(\lambda^*)\} \\ &\le \lim_{h\to 0} \frac{1}{2}\int_\Omega \left[2\mu^*|\varepsilon(v[\ell]-z)|^2 + \lambda^*|\mathrm{div}(v-z)|^2\right] + \rho \liminf_{h\to 0}\left\{TV_\epsilon(\bar{\mu}_h) + TV_\epsilon(\bar{\lambda}_h)\right\} \\ &\le \liminf_{h\to 0}\left\{\frac{1}{2}\int_\Omega \left[2\mu^*|\varepsilon(v[\ell]-z)|^2 + \lambda^*|\mathrm{div}(v-z)|^2\right] + \rho\left(TV_\epsilon(\bar{\mu}_h) + TV_\epsilon(\bar{\lambda}_h)\right)\right\} \\ &\le \liminf_{h\to 0} J_h(I_h\hat{\mu}_{\kappa_1}, I_h\hat{\lambda}_{\kappa_2}) \\ &= \frac{1}{2}\int_\Omega \left[2\hat{\mu}_{\kappa_1}(x)|\varepsilon(v[\ell_h]-z)|^2 + \hat{\lambda}_{\kappa_2}|\mathrm{div}(v[\ell_h]-z)|^2\right] + \rho\int_\Omega \left[\left|\nabla\hat{\mu}_{\kappa_1}\right| + \left|\nabla\hat{\lambda}_{\kappa_2}\right|\right] \\ &\le \frac{1}{2}\int_\Omega \left[\hat{\mu}_{\kappa_1}|\varepsilon(v-z)|^2 + \hat{\lambda}_{\kappa_2}|\mathrm{div}(v-z)|^2\right] + \rho\left[TV(\mu) + TV(\lambda)\right] + \rho(\kappa_1+\kappa_2). \end{aligned}$$

Now passing $\{\kappa_1, \kappa_2\} \to 0$ and using the previous results, we obtain

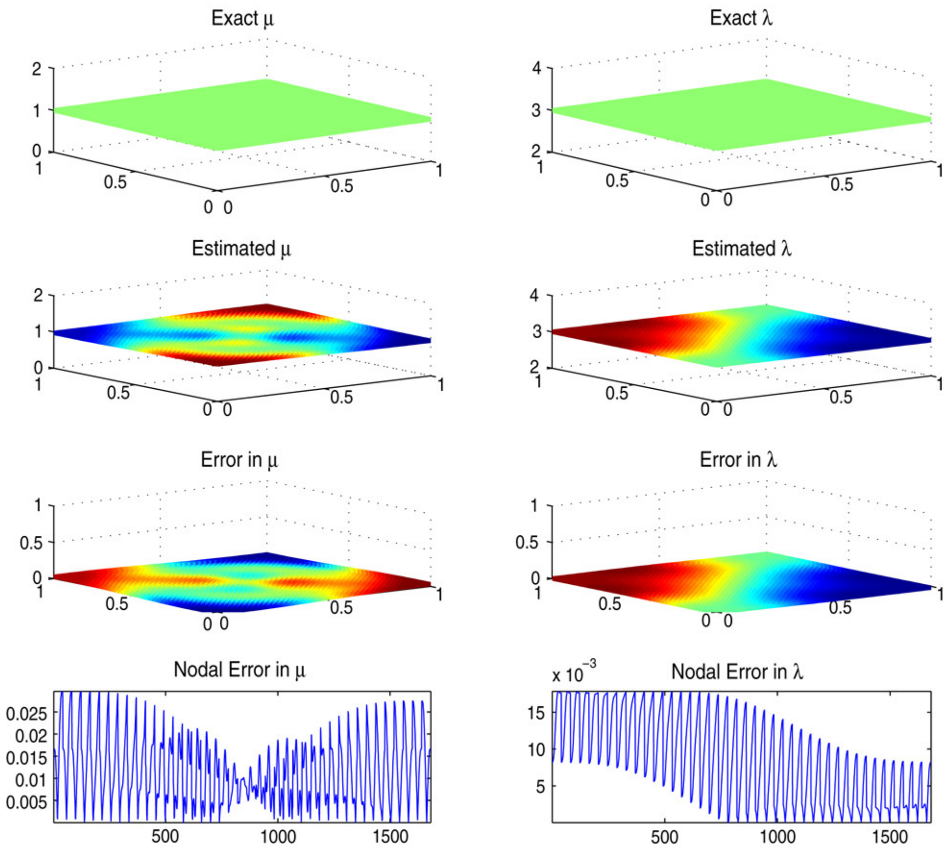

Fig. 2. Simulation results for Example 2: Identification of constant Lamé parameters.

$$J(\mu^*, \lambda^*) \le J(\mu, \lambda) \quad \text{for all } (\mu, \lambda) \in K.$$

Therefore $(\mu^*, \lambda^*) \in K$ solves (4). □

4. Numerical examples

In this section, we use the proposed output least-squares approach for the numerical identification of Lamé moduli in linear elasticity. We present three examples, one for nonsmooth Lamé parameters and two for smooth Lamé parameters.

4.1. Example 1

We consider an isotropic elastic membrane occupying the unit square $\Omega = (0, 1) \times (0, 1)$. The exact Lamé moduli are $\lambda = 1$ and $\mu = 1 + \chi_S$, where $S = \{(x, y) \in \Omega : \quad y \ge 0.5\}$ and χ_S is the characteristic function of S. In other words, μ is the discontinuous function whose value is 1.5 on S and 0.5 on the rest of Ω. We perform one "experiment" of stretching the membrane by a boundary traction h and measuring the resulting displacement u. The boundary traction chosen is

$$h = \frac{1}{10}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} n,$$

where n is the outward point unit normal to $\partial\Omega$. This traction is applied to the bottom, left, and right edges of the membrane, while the top edge ($y = 1$) is fixed by a Dirichlet condition. The results shown in Fig. 1. These results

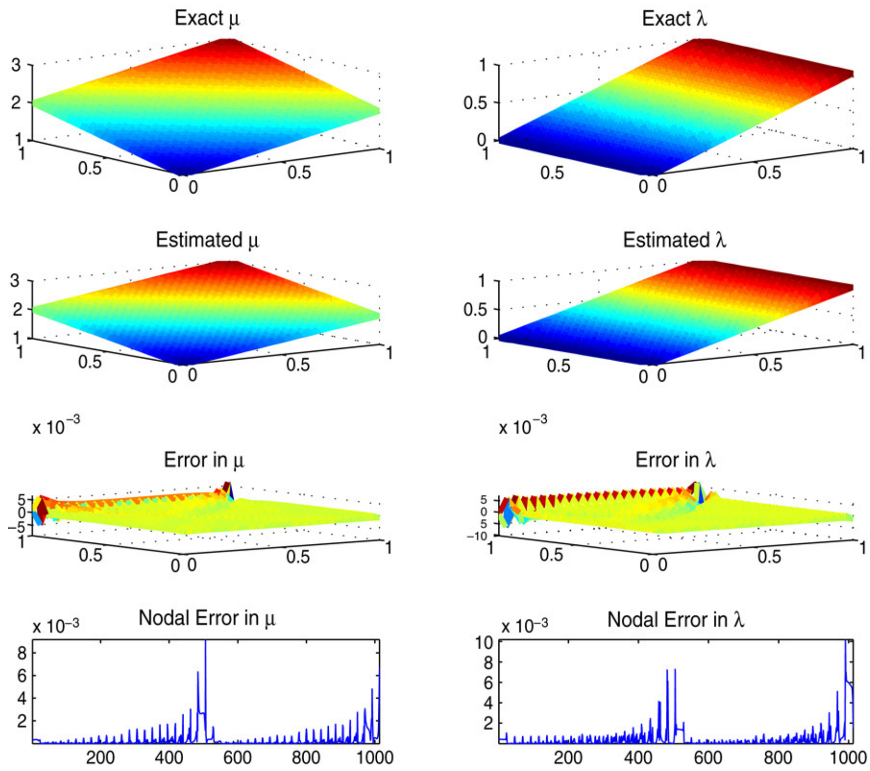

Fig. 3. Simulation results for Example 3: Identification of smooth Lamé parameters.

seem quite satisfactory. We remark that the discontinuous coefficient was identified quite accurately, except along the line of discontinuity.

4.2. Example 2

We consider the following example of a pure displacement problem which is borrowed from Brenner [29]. Given a unit square domain Ω and the force $f = (f_1, f_2)^\top$ defined by:

$$f_1(x, y) = \pi^2 \left(4 \sin 2\pi y(-1 + 2\cos 2\pi x) - \cos \pi(x + y) + \frac{2}{1 + \lambda} \sin \pi x \sin \pi y\right)$$

$$f_2(x, y) = \pi^2 \left(4 \sin 2\pi x(1 - 2\cos 2\pi y) - \cos \pi(x + y) + \frac{2}{1 + \lambda} \sin \pi x \sin \pi y\right).$$

We consider the following system

$$-\nabla \cdot \sigma = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{15a}$$

For $\mu = 1$ and $f(x, y)$ as above, the exact solution $u(x, y) = (u_1(x, y), u_2(x, y))^\top$ is given by:

$$u_1(x, y) = \sin 2\pi y(-1 + \cos 2\pi x) + \frac{1}{1 + \lambda} \sin \pi x \sin \pi y$$

$$u_2(x, y) = \sin 2\pi x(1 - \cos 2\pi x) + \frac{1}{1 + \lambda} \sin \pi x \sin \pi y.$$

We take $u(x, y)$ for a particular value of $\lambda = 3$ and consider the inverse problem of identifying the Lamé moduli μ and λ. The results are given in Fig. 2.

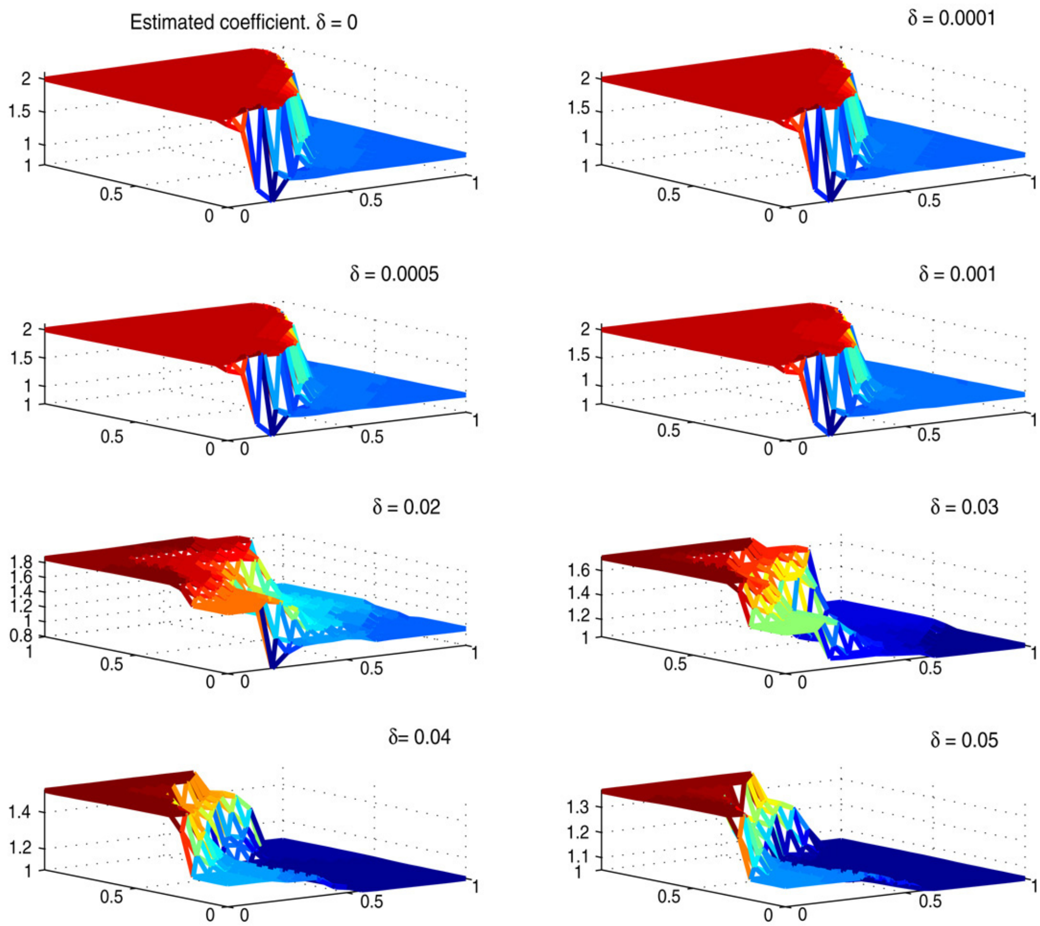

Fig. 4. Identification of discontinuous Lamé parameter μ with various levels of noise in the data.

4.3. Example 3

In this example the Lamé parameters are smooth functions. As previously we consider an isotropic elastic membrane occupying the unit square $\Omega = (0, 1) \times (0, 1)$ and the exact Lamé moduli are

$$\mu(x, y) = 1 + x + y$$
$$\lambda(x, y) = x.$$

We perform another experiment of stretching the membrane by a boundary traction h given by

$$h(x, y) = 0.02(6 + 6x, -6 - 10x)^T \quad \text{on the bottom edge}$$
$$h(x, y) = 0.02(8 + 2y, -12 - 6y)^T \quad \text{on the left edge}$$
$$h(x, y) = 0.02(-2 - 2y, 6 + 6y)^T \quad \text{on the right edge.}$$

The membrane is fixed on the top edge by a Dirichlet boundary condition. Then the exact displacement $u = (u_1, u_2)$ is given by

$$u_1(x, y) = 0.02(x - 6y)$$
$$u_2(x, y) = 0.06y.$$

The results are shown in Fig. 3.

4.4. Identification with noisy data

We again consider an isotropic elastic membrane occupying the unit square $\Omega = (0, 1) \times (0, 1)$. The exact Lamé moduli are $\lambda = 1$ and $\mu = 1 + \chi_S$, where $S = \{(x, y) \in \Omega : \quad y \geq x^2\}$, and χ_S is the characteristic function of S. Other details of the experiment are same as in Example 1.

Our main concern here is to see the behavior of the proposed method when the data are corrupted by some noise. To obtain a noisy data set we proceed as follows: We first compute an accurate solution z and then the noisy data are obtained by using the identity $z_\delta = z + \delta R_d \max_\Omega |z|$ where δ is a constant, R_d is a vector of uniformly distributed random numbers in $[-1, 1]$, $\dim(R_d) = \dim(z_\delta)$, and z is the accurate solution.

We collected m levels of noises, by varying δ, and n levels of regularization parameters. We identified the coefficients (μ, λ) for $n \times m$ possible combinations. The idea is to obtain the most suitable regularization parameter for a particular level of noise. Based on the experiments, we have chosen several levels of noise and corresponding regularization parameters for each noise-level. Then the Lamé coefficients were identified by using the proposed output least-squares. The results are shown in Fig. 4, where we only show the behavior of μ, which is certainly the more interesting one. As expected, the quality of the reconstruction diminishes as the noise increases. One possible remedy might be data smoothing which seemed to work well for the equation error approach (see [9]).

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