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MATHEMATICS

REMARKS ON RIGID STRUCTURES

BY

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This note contains the results of lunch-break discussions concerning the fact that for every model $\mathscr{A}$, each "reasonably" definable element of $\mathscr{A}$ must be left fixed by every automorphism of $\mathscr{A}$. To be more precise, let us introduce the following notation:

Let $\mathscr{A}$ be a structure and let L be a logic. In this note we will only be concerned with logics of the type $L_{\varkappa\lambda}$ and $L_{\infty\lambda}$. In the following ξ is an ordinal and i is a natural number such that $i<4$. Put $H_0^{(i)}(\mathscr{A}, L)=\mathscr{A}$ and suppose that the structures $H_\xi^{(i)}(\mathscr{A}, L)$ are given. Then define a subset $P_\xi^{(i)}$ of A as follows:

$P_\xi^{(0)}$ – is the set of all elements of $H_\xi^{(0)}(\mathscr{A}, L)$ which are definable by a single formula of L;

$P_\xi^{(1)}$ – is the set of all elements of $H_\xi^{(1)}(\mathscr{A}, L)$ which are definable by a set of formulas of L;

$P_\xi^{(2)}$ – is the set of all elements of $(H_\xi^{(2)}(\mathscr{A}, L), d)_{d\in R_\xi}$, where $R_\xi=\bigcup_{\eta<\xi} P_\eta^{(2)}$, which are definable by a single formula of L (i.e. the set of all elements which are definable by a formula of L with parameters from R_ξ), and

$P_\xi^{(3)}$ – is the set of all elements of $(H_\xi^{(3)}(\mathscr{A}, L), d)_{d\in S_\xi}$, where $S_\xi=\bigcup_{\eta<\xi} P_\eta^{(3)}$, which are definable by a set of formulas of L (i.e. the set of all elements which are definable by a set of formulas of L with parameters from S_ξ).

If we have defined $P_\xi^{(i)}$ we can define $H_{\xi+1}^{(i)}(\mathscr{A}, L)$ to be the expansion of $H_\xi^{(i)}(\mathscr{A}, L)$ by adding a new unary relation $P_\xi^{(i)}$, i.e. $H_{\xi+1}^{(i)}(\mathscr{A}, L)=(H_\xi^{(i)}(\mathscr{A}, L), P_\xi^{(i)})$. Finally, if δ is a limit ordinal, put $H_\delta^{(i)}(\mathscr{A}, L)=(\mathscr{A}, P_\delta^{(i)})_{\xi<\delta}$.

Notice, that the sets $P_\xi^{(i)}$, form an increasing sequence, thus there is an ordinal $\sigma(i)$ such that for all $\xi>\sigma(i)$ we have $P_\xi^{(i)}=P_{\sigma(i)}^{(i)}$. The difference $A-P_{\sigma(i)}^{(i)}$ is called the i-remainder of $\mathscr{A}$ with respect to the logic L. The following facts are easy to see.

PROPOSITION 1. For every logic L, every structure $\mathscr{A}$ and every automorphism π of $\mathscr{A}$:

(i) π is an automorphism of the structure $H_\xi^{(i)}(\mathscr{A}, L)$ (for every ordinal ξ and $i<4$)

(ii) π leaves each element of $P_\xi^{(i)}$ fixed, i.e. $\pi\ P_\xi^{(i)} = id\ P_\xi^{(i)}$ (for every ordinal ξ and $i<4$)

(iii) $\mathscr{A}$ is rigid if there exists $i<4$ such that the i-remainder of $\mathscr{A}$ with respect to L is empty.

In the rest of this note we shall try to find a possible converse to (iii). Let us start with some negative examples.

Example 1. Let A be the set of sequences of 0's and 1's which become constant after a finite number of steps. Let a and b be distinct elements not in A and define B to be $A \cup \{a, b\}$. For elements of A define the unitary relations P_n, $n \in \omega$, by $P_n(x)$ iff $x(n)=0$. Finally, define the binary relation R on B by $R(x, y)$ iff $x \in A$, $y \in B-A$ and either x is eventually 0 and $y=a$ or x is eventually 1 and $y=b$. Consider the structures

$$\mathscr{A} = \langle A, P_n \rangle_{n\in\omega} \text{ and } \mathscr{B} = \langle B, P_n, R \rangle_{n\in\omega}.$$

Then:

(1) both $\mathscr{A}$ and $\mathscr{B}$ are rigid;

(2) the 0-remainder of $\mathscr{A}$ with respect to $L_{\omega\omega}$ is just A, thus the 2-remainder of $\mathscr{A}$ with respect to $L_{\omega\omega}$ is A;

(3) the 0-remainder of $\mathscr{B}$ with respect to $L_{\omega\omega}$ is just B, thus also the 2-remainder of $\mathscr{B}$ with respect to $L_{\omega\omega}$ is B. Moreover, the 1-remainder of $\mathscr{B}$ with respect to $L_{\omega\omega}$ is $\{a, b\}$ and the 3-remainder of $\mathscr{B}$ with respect to $L_{\omega\omega}$ is empty.

This example suggests the following theorem.

Theorem 1. Let $i=2$ or 3. Suppose that the i-remainder of a structure $\mathscr{A}$ with respect to $L_{\omega\omega}$ is finite. Then $\mathscr{A}$ is rigid if and only if the i-remainder of $\mathscr{A}$ with respect to $L_{\omega\omega}$ is empty.

Proof. We shall give the proof in the case that $i=3$. The proof for the case $i=2$ is similar.

Suppose the 3-remainder of $\mathscr{A}$ with respect to $L_{\omega\omega}$ is non empty. To simplify the notation let $\sigma=\sigma(3)$, $P=P_{\sigma(3)}^{(3)}$ and $A=P \cup \{a_0, \ldots, a_n\}$ where $a_1, \ldots, a_n$ are the distinct elements of the remainder. We shall define an enumeration $\{b_0, \ldots, b_n\}$ of this remainder such that the function

$$id \restriction P \cup \{\langle a_k, b_k \rangle : k \leqslant n\}$$

will be a non-trivial automorphism of $\mathscr{A}$. More precisely we shall define $\{b_0, \ldots, b_n\}$ in such a way that for all $k \leqslant n$,

$$(H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, a_0, \ldots, a_k)_{d\in P} \equiv (H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, b_0, \ldots, b_k)_{d\in P}$$

Firstly, notice that there exists an element b_0 of $\{a_0, \ldots, a_n\}$ such that $(H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, a_0)_{d\in P} \equiv (H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, b_0)_{d\in P}$. If this were not so a_0 would have belonged to P. Suppose $b_0, \ldots, b_{k-1}$ have been defined for

$k \leqslant n$ and let p be the type of a_k in the structure

$$(H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, a_0, \ldots, a_{k-1})_{d \in P},$$

say $p = p(x, a_0, \ldots, a_{k-1})$. Consider the set of formulas $p' = p(x, b_0, \ldots, b_{k-1}) = \{\phi(x, b_0, \ldots, b_{k-1}) : \phi(x, a_0, \ldots, a_{k-1}) \in p\}$. Note that the formulas $x \neq b_j$, for $j < k$, and $\neg P(x)$ are in p'. We claim that one of the elements of $\{a_0, \ldots, a_n\}$ realizes p'. Suppose this were not so then for each a_j, $j \leqslant n$, we would have a formula $\phi_j(x, b_0, \ldots, b_{k-1}) \in p'$ such that

$$(H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d)_{d \in P} \models \neg \phi_j[a_j, b_0, \ldots, b_{k-1}].$$

Let

$$\psi = \bigwedge_{j \leqslant n} \phi_j(x, a_0, \ldots, a_{k-1}) \wedge \neg P(x).$$

Then $\psi \in p$, thus $(H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, a_0, \ldots, a_{k-1})_{d \in P} \models (\exists x)\psi$. On the other hand, by our choice of ϕ_j, for $j \leqslant n$, we see that no element of the remainder can satisfy $\psi(x, b_0, \ldots, b_{k-1})$. Hence

$$(H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, b_0, \ldots, b_{k-1})_{d \in P} \models \neg (\exists x)\psi.$$

So

$$(H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, a_0, \ldots, a_{k-1})_{d \in P} \not\equiv (H_\sigma^{(3)}(\mathscr{A}, L_{\omega\omega}), d, b_0, \ldots, b_{k-1})_{d \in P},$$

contradicting the induction assumption. Now let b_k be any element of $\{a_0, \ldots, a_n\}$ satisfying p'. In this way we can define the function

$$id \restriction P \cup \{\langle a_k, b_k \rangle : k \leqslant n\},$$

and since $a_0 \neq b_0$ it is a non-trivial automorphism of $\mathscr{A}$.

The converse implication is trivial by proposition 1.

For $L_{\omega\omega}$, this theorem is the best possible result as we can see from Example 1 and Example 2 (below). Before we offer our second example we prove the following proposition.

PROPOSITION 2. Let L be $L_{\omega\omega}$ or $L_{\infty\lambda}$ for some infinite λ. If there are two non-isomorphic structures $\mathscr{A}$ and $\mathscr{B}$ for L such that $\mathscr{A}$ and $\mathscr{B}$ are rigid and realize the same types w.r.t. L, then there exists a rigid structure without elements which are definable by a type of L.

PROOF. Let $\mathscr{A}$ and $\mathscr{B}$ be given with all properties as described above. We may assume L does not have function symbols and constants. Construct $\mathscr{C}$ as follows:

$\mathscr{C} = \langle C, R^{\mathscr{C}}, \sim \rangle_{R \in L}$ where

i) $C = A \times \{0\} \cup B \times \{1\}$

ii) $\langle a, a' \rangle \sim \langle b, b' \rangle$ iff $a' = b'$

iii) $R^{\mathscr{C}}(\langle a_0, a_0' \rangle, \ldots, \langle a_{n+1}, a_{n-1}' \rangle)$ iff

$$a_0' = a_1' = \ldots = a_{n-1}' = 0 \text{ and } R^{\mathscr{A}}(a_0, \ldots, a_{n-1})$$

or

$$a_0' = a_1' = \ldots = a_{n-1}' = 1 \text{ and } R^{\mathscr{B}}(a_0, \ldots, a_{n-1}).$$

Let us denote $\mathscr{C}$ by $\mathscr{A} \oplus \mathscr{B}$. *)

For $L_{\infty\lambda}$ realizing the same types is the same as being elementarily equivalent. Using theorem 5.2.7 in Dickmann [1] we can see that $\oplus$ preserves elementary equivalence with respect to $L_{\infty\lambda}$. Now we proceed as follows: for no formula ϕ of L: $\mathscr{A} \oplus \mathscr{A} \models (\exists!\, x)\phi$, hence for no formula ϕ of L: $\mathscr{A} \oplus \mathscr{B} \models (\exists!\, x)\phi$.

For $L_{\omega\omega}$ we use the Feferman-Vaught theorem to show that $\mathscr{A} \oplus \mathscr{A}$ and $\mathscr{A} \oplus \mathscr{B}$ realize the same types. Every type which is realized in $\mathscr{A} \oplus \mathscr{B}$ is realized in $\mathscr{A} \otimes \mathscr{A}$. Every type which is realized in $\mathscr{A} \oplus \mathscr{A}$ is realized by at least 2 elements. So the same holds for $\mathscr{A} \oplus \mathscr{B}$.

It is easy to see that $\mathscr{C}$ is rigid, because every automorphism $\pi: \mathscr{C} \to \mathscr{C}$ induces either an isomorphism $\pi': \mathscr{A} \to \mathscr{B}$, or isomorphisms $\pi_1: \mathscr{A} \to \mathscr{A}$ and $\pi_2: \mathscr{B} \to \mathscr{B}$. The first case is not possible and in the second case π_1 and π_2 are trivial, hence π is.

EXAMPLE 2. Let $L = L_{\omega\omega}$ and $\mathscr{C}$ be the following structure:

$$\mathscr{C} = \langle \omega^{\omega^{\omega}} \cdot (\omega^2 \cdot 2),\ \langle\ \rangle \oplus \langle \omega^{\omega^{\omega}} \cdot (\omega^2 \cdot 2 + \omega), \langle\ \rangle.$$

Both structures are rigid and realize the same types. Hence $\mathscr{C}$ is rigid and no element of C is definable by a type, i.e. $P_\xi^{(3)} = \emptyset$ for all ξ. (Hence $P_\xi^{(4)} = \emptyset$ for all ξ).

The situation is a bit better when we consider definability using logics which are stronger than $L_{\omega\omega}$.

THEOREM 2. Suppose the 3-remainder of a structure $\mathscr{A}$ with respect to $L_{\omega_1\omega}$ is countable.

Then $\mathscr{A}$ is rigid if and only if this remainder is empty.

PROOF. The proof is similar to the proof of Theorem 1. Suppose the 3-remainder of $\mathscr{A}$ with respect to $L_{\omega_1\omega}$ is non-empty.

To simplify the notation, let $\sigma(3) = \sigma$ and $P_{\sigma(3)} = P$. Then we can present the universe A in the form $A = P \cup \{a_i : i < \omega\}$, where a_i's are the distinct elements of the remainder. We are going to define inductively two enumerations of the remainder, say $\{b_i : i < \omega\}$ and $\{c_i : i < \omega\}$ such that $b_0 \neq c_0$ and for each $n < \omega$

$$(*)\quad (H_\sigma^{(3)}(\mathscr{A}, L_{\omega_1\omega}), d, b_0, \ldots, b_{n-1})_{d \in P} \equiv_{\omega_1\omega} (H_\sigma^{(3)}(\mathscr{A}, L_{\omega_1\omega}), d, c_0, \ldots, c_{n-1})_{d \in P}.$$

First of all, note that there is an element c_0 of the remainder such that $a_0 \neq c_0$ and

$$(H_\sigma^{(3)}(\mathscr{A}, L_{\omega_1\omega}), d, a_0)_{d \in P} \equiv_{\omega_1\omega} (H_\sigma^{(3)}(\mathscr{A}, L_{\omega_1\omega}), d, c_0)_{d \in P}.$$

*) This special kind of direct sum was also used in [4].

If not, then a_0 would be definable by a set of formulas and thus a_0 would be an element of P, which is impossible. Thus we can put $b_0 = a_0$ and we have $(*)$ for $n=1$.

Assume that we have defined $\{b_0, \ldots, b_n\}$ and $\{c_0, \ldots, c_n\}$ such that $(*)$ holds. We shall consider two cases.

Case I. $n=2k$. Let $m = \min\{k \in \omega : a_k \neq c_j, \text{ for } j \leqslant n\}$. Let p be the type of a_m in the structure $(H_\sigma^{(3)}(\mathscr{A}, L_{\omega_1\omega}), d, c_0, \ldots, c_n)_{d \in P}$ in the logic $L_{\omega_1\omega}$. Say, $p = p(x, c_0, \ldots, c_n)$. Consider the set, of formulas of $L_{\omega_1\omega}$,

$$p' = p(x, b_0, \ldots, b_n) = \{\phi(x, b_0, \ldots, b_n) : \phi(x, c_0, \ldots, c_n) \in p\}.$$

We claim that there is some element a_1 of the remainder, which realizes p'. Suppose not. Then for each $j \in \omega$, we could choose a formula

$$\phi_j(x_0, c_0, \ldots, c_n) \in p$$

such that

$$(H_\sigma^{(3)}(\mathscr{A}, L_{\omega_1\omega}), d)_{d \in P} \models \neg\, \phi_j[a_j, b_0, \ldots, b_n].$$

Let

$$\theta(x_0, c_0, \ldots, c_n) = \bigwedge_{j \in \omega} \phi_j(x_0, c_0, \ldots, c_n) \wedge \neg\, P(x_0),$$

then $\theta \in p$ thus

$$(H_\sigma^{(3)}(\mathscr{A}, L_{\omega_1\omega}), d, c_0, \ldots, c_n)_{d \in P} \models \theta[a_m].$$

But on the other hand, by our choice of the formulas ϕ_j we can see that $(H_\sigma^{(3)}(\mathscr{A}, L_{\omega_1\omega}), d, b_0, \ldots, b_n) \models \neg\, (\exists x_0)\theta$. This contradicts our inductive hypothesis. This contradiction proves our claim. Thus we can put $c_{n+1} = a_m$ and $b_{n+1} = a_1$.

Case II. $n = 2k+1$. We make the same construction with the rôles of b_n's and c_n's interchanged.

Now, having both enumerations of our remainder we can put

$$f = id \restriction P \cup \{\langle b_n, c_n \rangle : n \in \omega\}.$$

It is easy to see by the construction that f is a non-trivial automorphism of $\mathscr{A}$.

From this theorem we get the following corollary.

Corollary. If $\mathscr{A}$ is a countable structure then $\mathscr{A}$ is rigid iff the 3-remainder of $\mathscr{A}$ with respect to $L_{\omega_1\omega}$ is empty.

In the rest of this paper we shall show that the corollary above is in fact the best possible result. It is rather hard to get any characterization of uncountable rigid structures, even working with quite strong logics.

EXAMPLE 3. Let $\mathscr{B}$ be a rigid, atomless Bolean algebra of cardinality ω_1. (The existence of such algebras has been proved by Shelah [3]).

Notice that for any two elements $a, b \in \mathscr{B}$, if $0 \neq a \neq 1$ and $0 \neq b \neq 1$, we have $(\mathscr{B}, a) \equiv_{\infty\omega} (\mathscr{B}, b)$. Thus even the 3-remainder of $\mathscr{B}$ with respect to $L_{\infty\omega}$ is just $\mathscr{B}$-$\{0, 1\}$.

For our last example we use proposition 2.

EXAMPLE 4. For every regular cardinal numbers $\varkappa$ there is a rigid structure $\mathscr{A}$ of cardinality $\varkappa$ such that the 3-remainder of $\mathscr{A}$ with respect to the logic $L_{\infty\varkappa}$ is just A.

Indeed, Gregory [2] has proved that for every regular cardinal number $\varkappa$ there are two non-isomorphic rigid structures of cardinality $\varkappa$, which are elementary equivalent with respect to $L_{\infty\varkappa}$. Thus applying Proposition 2, we get a rigid structure $\mathscr{A}$ such that the 2-remainder with respect to $L_{\infty\varkappa}$ is A.

To see that the 3-remainder of $\mathscr{A}$ with respect to the logic $L_{\infty\varkappa}$ is also A it is sufficient to notice that for logics of the type $L_{\infty\varkappa}$ definability by a single formula coincides with definability by a set of formulas.

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